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Dual graphs of degenerating curves

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Introduction

Let X/K be a proper smooth geometrically connected curve of genus g, where K is a complete field with a discrete valuation v. Let $\mathcal{O}_K$ denote the ring of integers of v in K, with algebraically closed residue field k and $s := \operatorname{Spec} k$. Let L/K be the minimal extension such that the minimal model $\mathcal{Y}/\mathcal{O}_L$ of X_L/L is semi-stable. In this paper, we attempt to express mathematically the "feeling" that if the special fiber of $\mathcal{Y}$ is "rather simple", then the special fiber of a regular model $\mathcal{X}/\mathcal{O}_K$ should also be "not too complicated".

We succeed in doing so when the extension $[L:K]$ is tame. We treat the case where $\mathcal{Y}/\mathcal{O}_L$ is smooth (the graph of the special fiber is just one vertex) and the case where the Jacobian of the curve X_L/L has good reduction: the graph associated to the special fiber of $\mathcal{Y}$ is a connected tree.

Recall that as an effective divisor, the special fiber $\mathcal{X}_s/k$ of a regular model of X/K can be written as $\mathcal{X}_s = \sum_{i=1}^{n} r_i C_i$, where C_i denotes an irreducible component of $\mathcal{X}_s$ having multiplicity r_i. When $\gcd(r_1, \ldots, r_n) = 1$, we say that X/K is an *S-curve* (see [7], definition 1.1). When the irreducible components of $\mathcal{X}_s$ are smooth and the singularities of $\mathcal{X}_s^{\mathrm{red}}$ are formally isomorphic to the one of the union of the coordinate axis in an affine space $\mathbb{A}^2$, we say that the regular model $\mathcal{X}/\mathcal{O}_K$ is a SNC-*model*.

The *dual graph* G associated to the regular model $\mathcal{X}/\mathcal{O}_K$ has vertices C_i, $i = 1, \ldots, n$ and C_i is linked to C_j by exactly $c_{ij} := (C_i \cdot C_j)$ edges, where c_{ij} denotes the intersection number in $\mathcal{X}$ of the given curves. Recall also that a *node* of a graph is a vertex having three or more adjacent edges. We measure how complicated the special fiber of a regular model is by counting the nodes of the dual graph.

2.1. **Theorem.** *Let X/K be an S-curve of genus $g \geqq 1$. If it has tame potential good reduction, the graph of one of its regular* SNC-*model has at most one node.*

3.1. *Let X/K be an S-curve of genus $g \geqq 2$. If the Jacobian of X/K has tame potential good reduction, the graph of one of the regular* SNC-*models of X/K has at most $2g - 2$ nodes.*

4.3. *Let X/K be an S-curve of genus g, unipotent rank u and abelian rank a. The* *graph associated to any almost minimal regular model $\mathscr{X}/\mathcal{O}_K$ of X/K has at most $2g-2+u-\frac{1}{2}a$ nodes.*

In the first section, we discuss how to construct the prescribed regular SNC-models. Our main tool is the description of the resolution of quotient singularities as in Viehweg [15]. The hypothesis $p \nmid [L:K]$ is essential, the first result above is false when L/K is wild (2.2).

When the graph associated to an SNC-model $\mathscr{X}/\mathcal{O}_K$ is a tree, it is usually quite hard to find out if the special fiber of $\mathscr{Y}/\mathcal{O}_L$ is or is not a tree. For instance, there exists two curves of genus 2 with regular SNC-models having the same type over K and such that the Jacobian of one has potential good reduction but the Jacobian of the other does not. The theorem above gives a criterion to find out if the special fiber of $\mathscr{Y}$ has a cycle: this happens if the graph of one of the almost minimal SNC-model of X/K has "too many" nodes.

As an application of our theorem, we show that $[L:K] \leqq 2(2u+1)$ for a curve with tame potential good reduction (2.7). When X/K is a curve with potential good reduction such that its Jacobian is K-simple with complex multiplication defined over K, then $[L:K] = p^n lq$ or $p^n l^k$ with p, l, q distinct primes (2.5).

1. Quotient of semi-stable models

Let X/K be a smooth proper geometrically connected curve of genus g and L/K a tamely ramified field extension. In this section, we first describe a normal scheme $\mathscr{Z}/\mathcal{O}_K$, quotient of the minimal model $\mathscr{Y}/\mathcal{O}_L$ of X_L/L by the natural action of $\mathrm{Gal}(L/K)$. Then, under the further assumption that X_L/L has semi-stable reduction (see [3, p. 3]) and that L is minimal with this property, we describe a regular model $\mathscr{X}/\mathcal{O}_K$ of X/K obtained by resolving the singularities of $\mathscr{Z}/\mathcal{O}_K$.

We follow closely Viehweg's article [15] and we refer the reader to his work for more details. Though he states at the beginning of his paper that he considers only the equicharacteristic case, his proofs of the facts listed below are also correct in the mixed characteristic case.

Fact I. *Let σ be a generator of* $\mathrm{Gal}(L/K)$. *σ induces a canonical morphism of the generic fiber of $\mathscr{Y}$ and hence a birational proper map $\mathscr{Y} \to \mathscr{Y} \times_{\mathcal{O}_L} \mathcal{O}_L$. By the universal property of a minimal model [2, page 310], this map extends to a morphism from $\mathscr{Y}$ to $\mathscr{Y} \times_{\mathcal{O}_L} \mathcal{O}_L$. Since $\mathscr{Y}$ is reduced and separated, there exists then a unique automorphism τ of $\mathscr{Y}$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\tau} & \mathscr{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_L & \xrightarrow{\sigma} & \operatorname{Spec} \mathcal{O}_L . \end{array}$$

We assume that $\operatorname{ord}(\tau_s) = \operatorname{ord}(\sigma) = r$ *and we let* $G = \langle \tau \rangle$.

Fact II. *Since $\mathscr{Y}/\mathcal{O}_L$ is projective, the quotient $\mathscr{Z} = \mathscr{Y}/G$ can be constructed in the usual way by glueing together the rings of invariants of G-invariant affine open sets* *of* $\mathcal{Y}$. $\mathcal{Z}/\mathcal{O}_K$ *is a normal scheme and hence its singular points are closed points of its special fiber. We let* $f:\mathcal{Y}\to\mathcal{Z}$ *denote the quotient map.*

Fact III [15, p. 303]. *Let* $\tau_s:\mathcal{Y}_s\to\mathcal{Y}_s$ *and* $\tau_s^{\mathrm{red}}:\mathcal{Y}_s^{\mathrm{red}}\to\mathcal{Y}_s^{\mathrm{red}}$ *the natural morphisms induced by* τ. *Then the natural map*

$$\mathcal{Y}_s^{\mathrm{red}}/\langle\tau_s^{\mathrm{red}}\rangle\to\mathcal{Z}_s^{\mathrm{red}}=(\mathcal{Y}/\langle\tau\rangle)_s^{\mathrm{red}}$$

is an isomorphism over s.

For any irreducible component $Y_i\subset\mathcal{Y}_s$, let $D_i=\{\mu\in G\mid\mu(Y_i)=Y_i\}$ and $I_i=\{\mu\in G\mid\mu_{|Y_i}=\mathrm{id}\}$.

Fact IV [15, p. 303]. *Let* m_i *be the multiplicity if* Y_i *in* $\mathcal{Y}_s$ *and let* $Z_j=f(Y_i)$. *The multiplicity* r_j *of* Z_j *in* $\mathcal{Z}_s$ *is equal to* $m_i\cdot r/|I_i|$.

We assume now that $\mathcal{Y}/\mathcal{O}_L$ is a semi-stable model of X_L/L. In particular, $\mathcal{Y}_s=\bigcup Y_i$ is reduced and is a divisor with normal crossings. Each irreducible component Y_i has at worst ordinary double points as singularities.

Fact V [15, Sect. 6]. *There exists a regular model* $\mathcal{X}/\mathcal{O}_K$ *and a proper birational morphism* $\pi:\mathcal{X}\to\mathcal{Z}$ *such that* π *induces an isomorphism between* $\mathcal{X}-\{\pi^{-1}(\mathcal{Z}_{\mathrm{sing}})\}$ *and* $\mathcal{Z}-\{\mathcal{Z}_{\mathrm{sing}}\}$ *and such that, for any* $z\in\mathcal{Z}_{\mathrm{sing}}$, $\pi^{-1}(z)$ *is a connected chain of rational curves. By a chain of rational curves, we mean:*

1. $\pi^{-1}(z)=\bigcup_{i=1}^{q}E_i, E_i$ *smooth and rational curves on* $\mathcal{X}$.
2. $(E_i\cdot E_{i+1})=1$ *for all* $i=1,\dots,q-1$ *and* $(E_i\cdot E_j)=0$ *for all* $j\neq i+1$. *Moreover,* $(E_i\cdot E_i)\leqq-2$ *for all i.*

In order to describe the intersection of this chain with the rest of the fiber, we set $\delta(z)=1$ *if z belongs to exactly one irreducible component of* $\mathcal{Z}_s$ *and* $\delta(z)=2$ *in the other case. If z is a singular point of* $\mathcal{Z}$, $\pi^{-1}(z)$ *intersects the* $\overline{\textit{rest of the special fiber}}$ *normally in exactly* $\delta(z)$ *points. If* $\delta(z)=1$, *we have* $E_1\cap\overline{\mathcal{X}_s-\{\pi^{-1}(z)\}}\neq\varnothing$; *if* $\delta(z)=2$, *the set* $E_1\cap\overline{\mathcal{X}_s-\{\pi^{-1}(z)\}}$ *and the set* $E_q\cap\overline{\mathcal{X}_s-\{\pi^{-1}(z)\}}$ *are both non empty.*

Viehweg states in 8.1.d on p. 306 of [15] that the model $\mathcal{X}$, obtained by taking the quotient of the semi-stable model and then resolving the singularities, has normal crossings.

We note that the multiplicity of every irreducible components of $\mathcal{Z}$ is prime to the residual characteristic.

1.1. **Corollary.** *Let* X/K *be a smooth proper curve of genus* $g\geqq 2$. *Let* L/K *be the minimal extension such that* X_L/L *has semi-stable reduction over* $\mathcal{O}_L$. *The condition 1)* $\gcd(p,[L:K])=1$ *implies that* 2) *there exists a regular* SNC-*model* $\mathcal{X}/\mathcal{O}_K$ *of* X/K *with the following property: let* $\mathcal{X}_s=\sum r_iX_i$; *if* $p|r_i$, *then* X_i *is a smooth rational curve, and* $(X_i\cdot\overline{\mathcal{X}_s^{\mathrm{red}}-X_i})=2$; *moreover,* X_i *intersects exactly two other irreducible components of the special fiber.*

Saito shows in [13, 3.11] that condition 1 is in fact equivalent to condition 2.

1.2. We do not want to attempt to sketch a proof of Fact V. However, we would like to stress one of the key point in the proof of the resolution of quotient singularities. Let $y \in \mathscr{Y}$ be a smooth closed point of $\mathscr{Y}/\mathcal{O}_L$ so that $\widehat{\mathcal{O}_{\mathscr{Y},y}} \cong \mathcal{O}_L[[x]]$. Let μ be the generator of the decomposition group of y. The action of μ can be linearized in the following way:

Let $(R, \mathscr{M})$ be a local ring and $\mu: R[[X]] \to R[[X]]$ an automorphism having the properties 1) $\mu(R) = R$ and 2) $\operatorname{ord}(\mu) = r$ is a unit in R.

1.3. **Lemma.** *Let $X \xrightarrow{\mu} f(X) = a_0 + a_1 X + a_2 X^2 + \cdots$. Assume that R is separated and complete for the I-adic topology, where $I := (a_0, \mu(a_0), \ldots, \mu^{r-1}(a_0))$. Then there exists an $s \in I$ such that $\mu(X - s) = b_1(X - s) + b_2(X - s)^2 + b_3(X - s)^3 + \cdots$.*

Proof. We need only to show that there exists an $s \in I$ such that $f(s) - \mu(s) = 0$. We construct by induction a Cauchy sequence $(s_n)_{n=1}^{\infty}$ for the I-adic topology, with $(s_{n+1} - s_n) \in I^n$, and such that $f(s_n) - \mu(s_n) \in I^n$ for all $n \in \mathbb{N}$. We claim that $s := \lim (s_n)$ has the desired property. In fact, $s = s_n + a$, with $a \in I^n$ by construction. Hence $f(s) = f(s_n) + b$ with $b \in I^n$ so that $f(s) - \mu(s) = [f(s_n) - \mu(s_n)] + [b - \mu(a)] \in I^n$ for all $n \in \mathbb{N}$.

Set $s_1 = 0$ and suppose that $s_{n-1} \in I$ is already constructed with $c := f(s_{n-1}) - \mu(s_{n-1}) \in I^{n-1}$. Let $Z := (X - s_{n-1})$ and write $Z \xrightarrow{\mu} g(Z) = c + c_1 Z + c_2 Z^2 + \cdots$. Since I is μ-invariant, we have $J := (c, \mu(c), \ldots, \mu^{r-1}(c)) \subseteq I^{n-1}$. Let $\delta := 1/r \cdot \sum_{l=0}^{r-1} \mu^{-l}(c^{(l)})$, where $\mu^l(Z) = c^{(l)} + c_1^{(l)} Z + \cdots$. We claim that $s_n := s_{n-1} + \delta$ has the desired properties.

From the relation $\mu^{(l+1)}(Z) = \mu^{(l)}(\mu(Z))$, we get that $c^{(l+1)} = \mu^{(l)}(c) + \mu^{(l)}(c_1) c^{(l)}$ modulo J^2. In particular, $c^{(l)} \in J$ for all l and $\delta \in J$. Using the relation $\mu^{-l}(c^{(l+1)}) = c + c_1 \mu^{-l}(c^{(l)})$ modulo J^2, it easy to check that $g(\delta) - \mu(\delta) \in J^2$. Hence $\mu(X - s_n)$ can be written as a power series in $(X - s_n)$ with constant term $f(s_n) - \mu(s_n)$ in $J^2 \subseteq I^n$.

1.4. **Lemma.** *Let $X \xrightarrow{\mu} g(X) = aX + a_2 X^2 + a_3 X^3 + \cdots$. Then there exists $Y \in (X)$, $Y = X \pmod{X^2}$ such that $\mu(Y) = aY$.*

Proof. Write $\mu^l(X) = a^{(l)} X + a_2^{(l)} X^2 + \cdots$, where $a^{(l)} = \mu^{l-1}(a) \mu^{l-2}(a) \cdots \mu(a) a$ and $a^{(r)} = 1$. We let $a^{(r)} = b_l a^{(l)}$ with the relation $\mu(b_l) = a b_{l+1}$. We conclude the proof by setting $Y = 1/r \cdot \sum_{l=0}^{r-1} b_l \mu^{(l)}(X)$.

Lemmas 1.3 and 1.4 above are Lemmas 5.1 and 5.2 in [15]. However, the proof of 5.1 in [15] is false as stated and we do not know if the complete statement of 5.1 holds in the case of several variables. P. Vojta suggested to use Viehweg's argument to show that a slightly weaker statement holds, as in 1.3 above.

The reader may consult [11] and [15] to find out how to resolve the possible singularities of $(R[[x]])^{\langle \mu \rangle}$.

2. Curves with tame potential good reduction

We keep the notations and hypothesis of the previous section and we apply the quotient construction to the case where the minimal model $\mathscr{Y}/\mathcal{O}_L$ is smooth and L is minimal with this property. We say that such a curve X/K has tame potential good reduction.

The scheme $\mathscr{X}=\mathscr{Y}/\langle\tau\rangle$ has an irreducible special fiber. $\mathscr{X}_s^{\text{red}}$ is obtained as the quotient of $\mathscr{Y}_s$ by the action of $\langle\tau_s\rangle$ and is then a smooth and proper curve. The multiplicity of $\mathscr{X}_s$ in $\mathscr{X}$ equals $r/I(\mathscr{Y}_s)$. Because $\mathscr{Y}/I$ is a smooth model of X_{L^I}/L^I and L/K is assumed to be minimal with this property, we must have $I=\{id\}$; $\mathscr{X}_s$ has then multiplicity $r=[L:K]$. We let g_1 denote the genus of the curve $\mathscr{X}_s^{\text{red}}$. The regular model $\mathcal{X}/\mathcal{O}_K$ described in Fact V has the properties listed in the next theorem.

2.1. **Theorem.** *Let X/K be a curve of genus $g\geqq 1$ having tame potential good reduction over L. Then there exists a regular* SNC*-model $\mathcal{X}/\mathcal{O}_K$ of X/K such that the graph G associated to its special fiber is a tree with at most one node. All irreducible components of the special fiber are smooth and rational, with self-intersection smaller than or equal to -2, except possibly the component of multiplicity $r=[L:K]$.*

2.2. *Remark.* This theorem is false if the curve has wild potential reduction. In characteristic $p=2$, there are many elliptic curves with potential good reduction but with reduction type I_v^* (i.e. with 2 nodes).

2.3. *Remark.* Suppose that the graph G in 2.1 has a unique node C, with multiplicity r. Let $r_1,\dots,r_d$ be the multiplicities of the vertices adjacent to C. The simple graph G is completely determined by the set $(r,r_1,\dots,r_d)$ and the condition $r\mid\sum r_i$ [6, Remark 4.2]; the terminal vertex on the i^{th} terminal chain has multiplicity x_i equal to $\gcd(r,r_i)$ and Fact VI below gives a geometric interpretation for the terminal multiplicities x_i. We note that the set $(r,x_i,\dots,x_d)$ does not completely determine G.

When $\gcd(x_1,\dots,x_d)=1$, we say that the curve X/K is an S-curve. We computed the order $\phi=|\Phi|$ of the group of components of the Néron model of $\text{Jac}(X)/K$ as $\phi=r^{d-2}/x_1\cdots x_2$ [6, Corollary 2.3]. In the same section, we also proved in an elementary way that given a simple tree with exactly one node C, the multiplicity r of this node kills the group of components Φ.

We note now that given an S-curve with tame potential reduction, the model $\mathcal{X}/\mathcal{O}_K$ described in Fact V is such that either its associated graph is a simple chain, in which case its group of component is trivial, or its associated graph is a tree with exactly one node and this node has multiplicity $[L:K]$. In [8], McCallum proves that $[L:K]$ kills the canonical subgroup Ψ of Φ (see also [7, Proposition 3.7]). Since X/K has potential good reduction, $\Psi=\Phi$, and hence we obtain a new proof that the multiplicity of the node of the special fiber kills the group of components.

Fact VI [15, Sect. 6]. *Let $z_1,\dots,z_d$ be the closed points of $\mathscr{X}$ (or of $\mathscr{X}_s$) that are ramification points of the morphism $f_s:\mathscr{Y}_s\to\mathscr{X}_s^{\text{red}}$. Then $\{z_1,\dots,z_d\}$ is the set of singular points of $\mathscr{X}$. Moreover, if $\pi:\mathcal{X}\to\mathscr{X}$ is the desingularization of $\mathscr{X}$ described in Fact V, then the multiplicity x_i of the terminal vertex on the terminal chain $\pi^{-1}(z_i)$ is equal to the number of closed points in the fiber $f_s^{-1}(z_i)$.*

2.4. *Remark.* We can compute the genus g of the curve X/K in two different ways:

1) We can use the "adjunction formula" for the special fiber of $\mathcal{X}_s=\sum m_iC_i$: let $p(C_i)$ be the arithmetical genus of the irreducible component C_i and d_i its degree in the graph G associated to the special fiber $\mathcal{X}_s$. The genus g of X is expressed

as $2g-2=\sum m_i(-(C_i\cdot C_i)-2+2p(C_i))$. Letting $\alpha=\sum m_i p(C_i)$ as in [7, Definition 2.1], we can write $2g-2=2g_o-2+2\alpha$ where $2g_o-2=\sum m_i(d_i-2)$. The integer g_o was called the linear rank of $\mathscr{X}$. In the present case, the vertices with degree 1 have multiplicity $x_i,\dots,x_d$. The only possible vertex with degree bigger than or equal to 3 is the node and its has multiplicity $r=[L:K]$. All curves in $\mathscr{X}_s$ are rational except possibly the node which has genus equal to g_1. The adjunction formula becomes:

$$2g-2=[r(d-2)-\sum x_i]+2rg_1.$$

2) We can apply the Riemann–Hurwitz formula to the *tame* cover $\mathscr{Y}_s\to\mathscr{X}_s^{\text{red}}$. The ramification points are $z_1,\dots,z_2$ and $|f_s^{-1}(z_i)|=x_i$. Hence

$$2g-2=r(2g_1-2)+\sum x_i(r/x_i-1)=2rg_1+r(d-2)-\sum x_i.$$

In the remainder of this section, we discuss some consequences of the arithmetical properties of this formula for the genus.

2.5. **Proposition.** *Let X/K be a curve with potential good reduction such that its Jacobian is K-simple with complex multiplication defined over K. Then $[L:K]=p^n lq$ or $p^n l^k$ with p,l,q distinct primes and $n,k\geqq 0$.*

Proof. The Galois group $\mathrm{Gal}(L/K)$ is abelian [14, Corollary 2, p. 502]. We can then reduce the situation to the case where the extension L/K is tame by making the base extension M/K with $\mathrm{Gal}(M/K)$ isomorphic to the p-part of $\mathrm{Gal}(L/K)$. A curve whose Jacobian is K-simple and has complex multiplication over K with $K\subset L$ has abelian rank equal to zero for all subfields $K\subseteqq F\subset L$ (see [10, Lemma 2.4]). In the terminology of 2.4, the abelian rank of X/K is equal to g_1. We are then reduced to study curves with the following properties:

- For any field $L\supset F\supseteqq K$, the irreducible components of the special fiber of the minimal model of X_F/F are all *rational* curves.
- X/K has *tame potential good* reduction.

For short, we shall call a curve with these properties a *rtpg-curve.* Our claim follows now from the next proposition.

Note that the assumption that $Jac(X)$ has complex multiplication defined over K is essential for X to be an rtpg-curve. Let us consider for instance a tame Fermat quotient X, defined over $K=\mathbb{Q}_p^{\text{unr}}$ as the smooth proper model of the plane curve $y^p=x^s(1-x)$. It has good reduction over a cyclic extension L/K of degree $2(p-1)$ (see [1, 4.6]). The complex multiplication is defined only after adjoining to K the p^{th}-roots of 1. X/K is not an rtpg-curve, even though its Jacobian has complex multiplication over some extension of K. The special fiber of the smooth regular model of X_L/L over $\mathcal{O}_L$ is isomorphic to the smooth proper model of the curve Y given by $v^2=u^p-u$ [1, 3.4]. The automorphism τ in Fact I induces an automorphism μ of Y of the form $(u,v)\to(au,\sqrt{a}v)$, with $a\in\mathbb{F}_p^*$ of order $p-1$. When $6|(p-1)$, let $\nu=\mu^{2(p-1)/3}$. It is not hard to check that $Y/\langle\nu\rangle$ has genus $(p-1)/6\geqq 1$.

2.6. **Proposition.** *Let X/K be a rtpg-curve of genus $g\geqq 1$ and let $r=[L:K]$. Denote by $\mathscr{X}/\mathcal{O}_K$ the regular model described in 2.1. The only possible values for r are:*

1. *$r = lq, l \neq q$ prime numbers. Then there are only two possible values for the genus g:*
(a) *$2g = (l-1)(q-1)$; in this case, the terminal vertices of $\mathcal{X}_s$ have multiplicities $(l, q, 1)$ and $\phi = 1$. After a field extension of degree l, the new model has $l+1$ terminal branches, all with terminal multiplicities equal to 1 and $\phi = q^{l-1}$.*
(b) *$g = (l-1)(q-1)$; in this case, the terminal multiplicities are (l, l, q, q) and $\phi = 1$. After an extension of degree l, the new model has $2l$ terminal branches, all with multiplicity equal to 1 and $\phi = q^{2(l-1)}$.*
2. *$r = l^k, l$ prime. Then $2g = Al^{k-1}(l-1)$ for some integer $A, \phi = l^A$ and the model has $A+2$ terminal branches, two of them with terminal multiplicity equal to one and A with terminal multiplicity equal to l^{k-1}. After a field extension of degree l, the new model has $lA+2$ terminal multiplicities, two of them equal one and A of them equal l^{k-2}. In this case, $\phi = l^{lA}$.*

Proof. Given L/K with Galois group $\langle \sigma \rangle$, any non trivial extension L/M, where $L \supset M \supseteq K$, has Galois group $\langle \sigma^a \rangle$ for some $a|r, a \neq r$ (we abbreviate this condition by $a|_{\neq}r$). For each $a|_{\neq}r$, we have a morphism $f_a: \mathcal{Y} \to \mathcal{X}_a := \mathcal{Y}/\langle \tau^a \rangle$ which induces the morphism $f_{a,s}: \mathcal{Y}_s \to \mathcal{X}_{a,s}^{\text{red}} \cong \mathcal{Y}_s/\langle (\tau_s)^a \rangle$. To simplify the notations, we let $Y := \mathcal{Y}_s$ and $Z_a := \mathcal{X}_{a,s}^{\text{red}}$. We let g_a be the genus of Z_a. The Riemann–Hurwitz formula applied to the *tame* cover $Y \to Z_1$ reads:

$$2g - 2 = r(2g_1 - 2) + \sum_{x|r, x \neq r} A(x)x(r/x - 1)$$

where $A(x)$ is the number of orbits of $\langle \tau_s \rangle$ containing exactly x closed points of Y and g, g_1 are respectively the genus of Y and Z_1. For each $x|_{\neq}r$, the model $\mathcal{X}/\mathcal{O}_K$ described in Fact VI has $A(x)$ terminal chains with terminal multiplicity x. In particular, the number of terminal chains is equal to $d := \sum_{x|_{\neq}r} A(x)$. The Riemann–Hurwitz formula applied to the *tame* cover $Y \to Z_a$ reads:

$$2g - 2 = \frac{r}{a}(2g_a - 2) + \sum_{x|r, x \neq r} A(x)(a, x)\frac{x}{(a, x)}\left(\frac{r/a}{x/(a, x)} - 1\right).$$

In particular, if $b = \gcd(x, A(x) \neq 0)$, we obtain an etale morphism $Z_b \to Z_1$ of degree b such that $b(g_1 - 1) = g_b - 1$. If $g_1 = 0$, then $b = 1$.

By definition, a rtpg-curve has $g_a = 0$ for all $a|_{\neq}r$. We prove now that r cannot be divisible by l^2q, with l, q distinct primes. Suppose that $r = l^2q$. We can write the integer $2g - 2$ in five different ways, using the R–H formula for all divisors of l^2q.

$$\begin{array}{ll}
a = 1 & -2l^2q + A(1)(l^2q - 1) + A(l)l(lq - 1) + A(l^2)l^2(q - 1) \\
& + A(lq)lq(l - 1) + A(q)q(l^2 - 1) \\
a = l & -2lq + A(1)(lq - 1) + A(l)l(lq - 1) + A(l^2)l^2(q - 1) \\
& + A(lq)lq(l - 1) + A(q)q(l - 1) \\
a = q & -2l^2 + A(1)(l^2 - 1) + A(l)l(l - 1) + A(lq)lq(l - 1) + A(q)q(l^2 - 1) \\
a = lq & -2l + A(1)(l - 1) + A(l)l(l - 1) + A(lq)lq(l - 1) + A(q)q(l - 1) \\
a = l^2 & -2q + A(1)(q - 1) + A(l)l(q - 1) + A(l^2)l^2(q - 1).
\end{array}$$

From the first two lines above, we get that $A(1) + A(q) = 2$. From the next two, we get that $A(1) + qA(q) = 2$, so that $A(1) = 2$ and $A(q) = 0$. By substracting the last two lines to the second, we get a contradiction. We prove now the first assertion of the proposition and leave the others to the reader. Suppose that $r = lq$.

$$a=1 \quad 2g-2 = -2lq + A(1)(lq-1) + A(l)l(q-1) + A(q)q(l-1)$$
$$a=l \quad 2g-2 = -2q + A(1)(q-1) + A(l)l(q-1)$$
$$a=q \quad 2g-2 = -2l + A(1)(l-1) + A(q)q(l-1).$$

We obtain that: $A(1)+A(q)-2=0$ and $A(1)+A(l)-2=0$. The case $A(1)=2$ cannot happen because $g \geqq 1$. Finally, when $A(l)=A(q)=1, 2g=(l-1)(q-1)$ and when $A(l)=A(q)=2, 2g=2(l-1)(q-1)$.

2.7. **Proposition.** *Let X/K be an S-curve of genus $g \geqq 1$ with tame potential good reduction. Let u be the unipotent rank of the Néron model of* $\mathrm{Jac}(X)/K$. *Then $[L:K] \leqq 2(2u+1)$. If $[L:K] > u$, then either $u = g$ or $u+1 = g = [L:K]$. Moreover, if a prime $l > u$ divides $[L:K]$, then either*

1) *$u = g$ and $l = g+1$ or $l = 2g+1$*
2) *$u = g-1$ and $l = [L:K] = g$.*

2.8. *Remark.* In [8] (or [7, Proposition 3.3]), $r = q_1^{a_1} \cdots q_k^{a_k}$ is bounded as $\sum q_i^{a_i - 1}(q_i - 1) \leqq 2u$ when $r \equiv 0 \bmod 4$ or r odd. This bound implies directly that if a prime l divide r, then $l \leqq 2u+1$. When $2g+1$ is prime and divides $[L:K]$, the special fiber of the smooth model of X_L/L is well understood [5, Theorem 2.9].

Proof. We keep the notations used in 2.3 and 2.4. In particular, $r = [L:K]$. Given an S-curve with tame potential good reduction, the model described in 2.1 and 2.3 gives rise to a set $(r, r_1, \ldots, r_d)$ of positive integers with the following properties (compare with [4, p. 17–18]):

$S1$) $r_i < r$ for all $i = 1, \ldots, d$

$S2$) r divides $\sum_{i=1}^{d} r_i$

$S3$) $(d-2)r - \sum_{i=1}^{d} x_i$ is an even integer, where $x_i = \gcd(r, r_i)$

$S4$) $\gcd(r, r_1, \ldots, r_d) = 1$.

We note that, by $S2$, d is always bigger than or equal to 2. As in 2.4, g_0 denotes the integer defined by $2g_o - 2 = (d-2)r - \sum_{i=1}^{d} x_i$. We proved that g_o is positive in [6, Proposition 4.1]. By definition, $g_o + rg_1 = g$ and by Raynaud's Theorem [12] (see also [7, Theorem 1.3]), $u = g - g_1$. If $g_o = 0, r = g/g_1 \leqq u+1$. When $g_o \neq 0$, we bound r in terms of $g_o = g - rg_1 \leqq u$ by using the following facts:

1. $d=3$ implies $r \leqq 2(2g_o+1)$
2. $d=4$ implies $r \leqq 2(g_o+1)$
3. $d \geqq 5$ implies $r \leqq 2(g_o-1)$
4. $d=2$ is equivalent to $g_o = 0$.

These facts are well-known. Their proofs are tedious and consist of examining all possible subcases. Let us discuss now the primes $l > u$ that divide r. If $r-1 \geqq u$, we have $u := g - g_1 = g_o + (r-1)g_1 \geqq g_o + ug_1$. In particular, if $g_1 \neq 0$, then $g_1 = 1$, $g_o = 0, u = g-1$ and hence $g = r$ so that if $l > u$, then $l = g = r$.

We suppose now that $g_1 = 0$; then $g_o = g \geqq 1$ and we can assume that $d \geqq 3$. We show that $l > u = g = g_o$ can happen only in the case $l = 2g_o + 1$ and $d = 3$ or $l = g_o + 1$ and $d = 3$ or $d = 4$.

If $d \geqq 5, r \leqq 2(g_o - 1)$. In particular, if $l \geqq g_o + 1$ and $l | r$, then $l = r$. By the formula for the linear rank, we have $2g_o = (d-2)(l-1)$ and we obtain a contradiction.

If $d = 4, r \leqq 2(g_o + 1)$. Again, if $l > g_o + 1$, we must have $r = l$ and this leads to a contradiction in a similar way.

If $d = 3$, $r \leqq 2(2g_o + 1)$. In this case, $r = l$ implies $l = 2g_o + 1$. If we assume that $g_o + 1 \leqq l < 2g_o + 1$, the remaining possibilities for r are $r = 2l$ or $r = 3l$. It is easy to check that none of them occurs unless $l = g_o + 1$.

3. Curves whose Jacobian has tame potential good reduction

As a second application of the quotient/desingularization construction described in the first section, we prove the following theorem:

3.1. **Theorem.** *Let X/K be a curve of genus $g \geqq 2$ whose Jacobian has tame potential good reduction. There exists a regular SNC-model $\mathscr{X}/\mathcal{O}_K$ of X/K whose associated graph is a tree with at most $2g-2$ nodes.*

Proof. We keep our previous notations. In particular, $f: \mathscr{Y} \to \mathscr{Z}$ denotes the quotient map and $\pi: \mathscr{X} \to \mathscr{Z}$ denotes the resolution of singularities. Let $\mathscr{Y}_s = \mathscr{Y}_s^{\text{red}} = \cup Y_i$ denote the special fiber of the minimal semi-stable model $\mathscr{Y}/\mathcal{O}_L$ and let $G(Y)$ be its associated graph. Let $\mathscr{Z}_s^{\text{red}} = \cup Z_j$ be the special fiber of the scheme $\mathscr{Z}/\mathcal{O}_K$ and let $G(Z)$ be the following graph: its vertices are the irreducible components Z_j and two vertices Z_i, Z_j are linked in $G(Z)$ by exactly $|Z_i \cap Z_j|$ edges, where $|Z_i \cap Z_j|$ denotes the number of closed points in $Z_i \cap Z_j$.

Since Jac (X_L/L) has good reduction by hypothesis, all irreducible components of $\mathscr{Y}_s$ are smooth and the graph $G(Y)$ is a tree [7, Corollary 1.4]. We claim that the graph $G(Z)$ is also a tree. This follows easily from the following fact, which implies that a cycle of $G(Z)$ can always be lifted to a cycle in $G(Y)$: Let $P \in Z_i \cap Z_j$ with $i \neq j$. For any $Y_i \in f^{-1}(Z_i)$, there exists $Y_j \in f^{-1}(Z_j)$ and $Q \in Y_i \cap Y_j$ such that $f(Q) = P$. The proof is almost obvious: let $Q \in Y_i$ with $f(Q) = P$. Given $Y_k \in f^{-1}(Z_j)$, we find $Q_k \in Y_k$ with $f(Q_k) = P$, or equivalently, with $\mu(Q_k) = Q$ for some $\mu \in G$. Hence $Q \in Y_j = \mu(Y_k)$.

Let $\mathscr{X}/\mathcal{O}_K$ be the regular model of X/K described in Fact V. Let $\mathscr{X}_s^{\text{red}} = \cup X_k$ be its reduced special fiber and let $G(X)$ denote its associated graph.

According to Fact V, one can obtain the graph $G(X)$ from the graph $G(Z)$ by doing the following operations: for each *singular* point (of $\mathscr{Z}$) $P \in Z_i \cap Z_j$ with $i \neq j$, replace the corresponding edge of $G(Z)$ by a chain connecting Z_i to Z_j. The length of the chain is determined by P. For each singular point $P \in Z_i$ and $P \notin Z_j$ for all $j \neq i$, attach a terminal chain to Z_i. Again, the length of the chain is determined by P.

In particular, a node of $G(X)$ corresponds to a vertex of $G(Z)$. More precisely,

X_k is a node of the graph $G(X)$ if $X_k \to \pi(X_k)$ is birational and if one of the following conditions is satisfied:

Case A. $\pi(X_k)$ is a node of $G(Z)$ or equivalently $f^{-1}(\pi(X_k))$ contains only nodes of $G(Y)$.

Case B. For some $Y_i \in f^{-1}(\pi(X_k))$, $f_{|Y_i}: Y_i \to \pi(X_k)$ is not etale and $g(Y_i) > 0$.

Case C. For some $Y_i \in f^{-1}(\pi(X_k))$, $f_{|Y_i}: Y_i \to \pi(X_k)$ is not etale and $g(Y_i) = 0$.

We note that a priori,

- There are at most $g-2$ curves satisfying case A.

By minimality of the (reduced) model $\mathscr{Y}$, each terminal vertex in $G(Y)$ represents a curve of strictly positive genus. By the next lemma, we know that the number of nodes of the graph is bounded by the number of terminal vertices minus 2. We conclude then that the tree $G(Y)$ associated to the special fiber has at most $g-2$ nodes.

3.2. **Lemma.** *Let G be a connected graph having β independent cycles. Given a vertex C_i, let d_i denote its degree in G. Then $2\beta - 2 = \sum_{q=1}^{n} (d_q - 2)$. In particular, the number of nodes of G is at most equal to the number of terminal vertices of G plus $(2\beta - 2)$.*

Proof. Let m and n denote respectively the number of edges and the number of vertices of G. By definition, $\beta = m - (n-1)$ and hence $2\beta - 2 = 2m - 2n$. It is trivial to check that $\sum_{q=1}^{n} d_q = 2m$. The second statement follows since $\sum_{j=1}^{n} (d_j - 2) = \sum_{i=1}^{k \text{ nodes}} (d_i - 2) + 0 + \sum_{\text{all term.vert.}} (-1)$ and $\sum_{i=1}^{k \text{ nodes}} (d_i - 2)$ is bigger than or equal to the number of nodes of the graph.

- There are at most g curves satisfying case B because $g = \sum g(Y_i)$, where $g(Y_i)$ denotes the geometric genus of the smooth component Y_i.

Hence, if all irreducible components that satisfy case C also satisfy case A, the theorem is proven. We study now the irreducible components of $G(X)$ satisfying case C but not case A. Let X_0 be such a curve and let $Y_0 \in f^{-1}(\pi(X_0))$ be an irreducible component of $G(Y)$ of genus 0 which is not a node of $G(Y)$ but is such that $Y_0 \to f(Y_0) = \pi(X_0)$ is not etale. In particular the inertia group I_0 is not equal to the decomposition group D_0. Let μ be the generator of D_0, acting on the smooth rational curve Y_0. μ has either zero or two fixed points.

When the degree in $G(Y)$ of Y_0 equals 2 and the generator μ of D_0 fixes the intersection points, the morphism $p_{|Y_0}: Y_0 \to \pi(X_0)$ is ramified only at the intersection points and X_0 is not a node of $G(X)$.

Suppose now that the degree of Y_0 equals 2 and that the generator μ of D_0 permutes the intersection points. There exists then two fixed points which are not intersection points; since $\mu^2_{|Y_0}$ has four fixed points, $\mu^2_{|Y_0} = id$, and $D_0/I_0 \cong \mathbb{Z}/2\mathbb{Z}$. Let $Y_{-k}, \dots, Y_{-1}, Y_0, Y_1, \dots, Y_l$ be the vertices on the chain corresponding to Y_0, i.e. Y_i intersects Y_{i+1} exactly once and the only vertices Y_i with degree different than 2 are the vertices Y_{-k} and Y_l. The case $Y_{-k} = Y_l$ cannot occur because $G(Y)$ is a tree. We note that $\mu(Y_{-j}) = Y_j$ and hence $k = l$.

Suppose first that $G(Y)$ is a chain, i.e. that Y_{-k} and Y_k are terminal vertices. Then by minimality, $g(Y_{-k}) = g(Y_k) > 0$ which implies that $g \geqq 2$. Note that $\mu^2(Y_i) = Y_i$ for all i. Except for $f(Y_0)$, all eventual nodes of $G(X)$ would have to satisfy Case B. Since $\mu(Y_i) = Y_{-i}$, the number of such nodes is bounded by $g/2$. Hence $G(X)$ has at most $g/2 + 1 \leqq 2g - 2$ nodes.

Suppose now that $G(Y)$ is not a chain. Each connecting chain on this graph has at most one vertex Y_0 as above. Remove one edge from each connecting chain of $G(Y)$ that does not contain such a Y_0. Suppose that we remove in this way b edges: we obtain then $b + 1$ disjoint subtrees of $G(Y)$; our discussion above shows that the nodes of $G(Y)$ in each of these subtrees are identified in $G(Z)$. Hence the number of nodes in case A is at most $b + 1$.

Note that in a tree, the number of nodes equals the number of connecting chains plus one. Since the number of nodes in case C is at most equal to the sum of the number of connecting chains in each of the subtrees defined above, we can conclude that the number of nodes in case A plus the number of nodes in case C is at most equal to the number of nodes of $G(Y)$. We already noted that $G(Y)$ has at most $g - 2$ nodes and that there are at most g nodes satisfying case B. Hence the number of nodes of $G(X)$ is at most $2g - 2$.

4. The maximum number of nodes of a type

Our aim in this section is to bound the number of nodes of the graph G associated to a regular model of an S-curve X/K in terms of the unipotent, toric and abelian ranks u, t, a of the Jacobian of X/K. We shall first bound the number of nodes of G in terms of the integers γ, β, α introduced in [7, Definition 2.1]. Recall that given a type $T = (n, M, R, P, G)$, $\alpha := \sum r_i p(C_i)$ and $\gamma := g(T) - \alpha - \beta$, where β denotes the first Betti number of the graph G. When T is associated to a special fiber $\mathscr{X}_s = \sum r_i C_i$, we have $g(T) = g(X), \gamma \leqq u$, and $\alpha \geqq a$; moreover, $\beta = t$ when $\mathscr{X}/\mathcal{O}_K$ is a regular SNC-model of X/K.

It should be noted that given a curve X/K and an integer N, it is always possible to find a regular (SNC-)model of X/K such that the number of nodes of its associated type is bigger than N. In fact, blowing up three distinct regular points lying on the same irreducible component will turn this component into a node of the new type.

4.1. **Definition.** We say that a regular model or a type is *almost minimal* if the only possible vertices C with arithmetical genus $p(C) = 0$ and self-intersection $(C \cdot C) = -1$ are the nodes of the associated graph G or the vertices of degree 2 on the connecting chains.

It is clear that the minimal regular model of X/K is almost minimal. The following lemma shows that the number of nodes may increase after a blow up.

4.2. **Lemma.** *Let E be an exceptional curve on a regular model $\mathscr{Y}$. Let $C_1, \ldots, C_n$ denote the irreducible components of the special fiber that intersect E. Let $D_i = \pi(C_i)$,* *where $\pi:\mathcal{Y}\to\mathcal{Y}_E$ is the contraction of E. If $(E\cdot C_i)=1\ \forall i=1,\ldots,n$, then $(D_i\cdot D_j)=1$ for all $i\neq j$ and $p(C_i)=p(D_i)$ for all $i=1,\ldots,n$.*

Proof. Let $-c_i=(C_i\cdot C_i)$, $-d_i=(D_i\cdot D_i)$ and $a_{ij}=(D_i\cdot D_j)$ $i\neq j$. Let r_i denote the multiplicity of the curves C_i and D_i in $\mathcal{Y}_s$ and $\mathcal{Y}_{Es}$. From the intersection matrices of the special fibers, we get

$$-c_ir_i+\sum_j r_j=-d_ir_i+\sum_{j\neq i} a_{ij}r_j \quad \text{for all} \quad i=1,\ldots,n$$

or, equivalently

$$\sum_{j\neq i}(a_{ij}-1)r_j=(d_i+1-c_i)r_i.$$

In particular, $a_{ij}=1\ \forall i\neq j$ if and only if $d_i+1-c_i=0$ for all $i=1,\ldots,n$. The above equalities also imply that $d_i+1-c_i\geqq 0$. The lemma cannot follow only from considerations on the intersection matrices. We show now that $d_i+1-c_i=0$ using the fact that the genus of the special fiber is preserved under blowups. From the formula $2g-2=X_s\cdot(X_s-K)$ applied to $\mathcal{Y}_s$ and $\mathcal{Y}_{Es}$, we get that

$$\sum r_i(c_i-2)-\sum r_i+2\sum r_ip(C_i)=\sum r_i(d_i-2)+2\sum r_ip(D_i).$$

Since $p(C_i)\leqq p(D_i)$ and $d_i+1-c_i\geqq 0$, we must have $p(C_i)=p(D_i)$ and $d_i+1-c_i=0$.

4.3. **Theorem.** *Let X/K be an S-curve of genus $g\geqq 2$. Let $\mathcal{X}/\mathcal{O}_K$ be any almost minimal regular model of X. The number of nodes of the graph associated to $\mathcal{X}_s$ is smaller than or equal to $2g-2+u-\frac{1}{2}a$. This bound is sharp when $t=a=0$.*

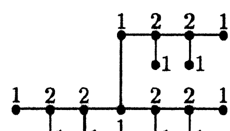
The proof of this theorem is postponed to the end of this section. It is purely algebraic and follows from our study of types.

4.4. *Remark.* Let X/K be a curve of genus $g\geqq 2$. For any M/L, let $\mathcal{Y}_M$ denote the minimal model of X_M/M. The number of nodes of $G(\mathcal{Y}_M)$ is constant for all such $M\neq L$ and is bounded by $2g-2-a_L$.

Proof. Let M_1/M be a field extension. The model $\mathcal{Y}_{M_1}$ is obtained by resolving the singularities of the model $(\mathcal{Y}_M)_{\mathcal{O}_{M_1}}$. The number of nodes is constant because the resolution of such ordinary double points does not introduce new nodes in the graph (see [3, Proposition 2.2]).

By 3.2, we know that the number of nodes k of any type equals at most $\sum_{\text{nodes}} l_i+2\beta-2$, where l_i denotes the number of terminal chains of the node C_i. When T is reduced, all terminal vertices have self-intersection (-1); when the type is also almost minimal, each terminal vertex has $p(C)>0$. It follows that $\sum_{\text{nodes}} l_i\leqq\alpha$. In the semi-stable case, $\beta=t$ by [7, Corollary 1.4] and all irreducible components are smooth. Hence $\alpha\leqq a$ and $k\leqq 2t+a-2=2g-2-a$.

4.5. *Example.* In the following example with $t=a=0$, $u=g$, the number of nodes is equal to $7=3g-2$.



We note that by 3.1, this graph cannot be associated to a curve whose Jacobian has potential good reduction.

Before proving our main result on the number of nodes of an almost minimal type, we summarize in the following lemmas a few properties of some special nodes. As usual, a node C is denoted by $(r, r_1, \dots, r_d)$ where r is its multiplicity, $r_1, \dots, r_l$ are the multiplicities of its adjacent vertices that are on a terminal chain and $r_{l+1}, \dots, r_d$ are the multiplicities of its other adjacent vertices. Let $s(C) = \gcd(r, r_1, \dots, r_d)$. We made the following definitions in [6, Theorem 4.7]:

$$2g(C) = (d-2)(r-1) - \sum_{i=1}^{d} (x_i - 1), \quad \text{where} \quad x_i = \gcd(r, r_i)$$

$$\mu(C) = 2g(C) + \sum_{l+1}^{d} (x_i - 1).$$

We say that a type $T = (n, M, R, P, G)$ is a *NC-type* if $(C_i \cdot C_j) \leqq 1$ when $i \neq j$. A NC-type corresponds to a simple arithmetical graph $(G, -M, R)$ [6]. Proposition 4.11 in [6] allows us to reduce most problems on types to the study of NC-types only.

4.6. **Lemma.** *Let (C, r) be a node of a NC-type.*

- *Suppose that $g(C) \leqq 0$; if $r > s(C)$, then exactly two branches have $x_i = s(C)$, the others have $x_i = r$.*
- *Suppose that $\mu(C) < 0$; such a node has exactly one connecting chain and if (C_i, r_i) belongs to it, then $x_i = s(C)$. In particular, $\mu(C) = 1 - s(C)$.*

Proof. Possibly replacing r_i by $r - r_i$, we can assume that $1 \leqq r_i \leqq r$ and $r \mid \sum r_i$. Let $s = \gcd(r, r_1, \dots, r_d)$. Replacing $(r, r_1, \dots, r_d)$ by $(r/s, r_1/s, \dots)$, we may assume that $s = 1$. Construct with $(r/s, r_1/s, \dots)$ a simple tree T with exactly one node, as in [6, Remark 4.2]. By [6, Corollary 4.9], the fact that $g_o(C) = 0$ implies that one of the self intersections has to equal (-1). We claim that the self intersection of the node cannot equal (-1). Otherwise, $r = \sum r_i$ and $2g_o - 2 = -2 = (d-2)r - \sum \gcd(r, r_i) \geqq (d-2)r - \sum r_i = (d-3)r$ leads to a contradiction. If $r > r_i$, then by construction all self intersections on the i^{th} terminal chain of T are smaller than or equal to (-2). Hence one of the r_is must be equal to r. Without loss of generality, we can assume that $r_d = r$. We can then repeat the same argument with the set $(r, r_1, \dots, r_{d-1})$ because $r \,\Big|\, \sum_{i=1}^{d-1} r_i$. Finally, we obtain that $r_3 = \cdots = r_d = r$. Using once more the definition of the linear rank, $2g_o - 2 = -2 = (d-2)r - (d-2)r - \gcd(r, r_1) - \gcd(r, r_2)$, we get that $\gcd(r, r_1) = \gcd(r, r_2) = 1$. We proved the second part of this lemma in [6, Theorem 4.7].

Let $\eta(C) = \mu(C) + \alpha(C)$, where $\alpha(C) = \sum r_k p(C_k)$, the sum being taken over all curves (C_k, r_k) having self-intersection equal to (-1) and belonging to a terminal chain of C. Note that if $\alpha(C) \neq 0$, then $\alpha(C) \geqq s(C)$. Let $\alpha' = \sum_{\text{nodes } C} \alpha(C) \leqq \alpha$.

4.7. **Lemma.** *Let (C, r) be a node of an almost minimal NC-type.*

- *Suppose that $\eta(C) \leqq 0$; such a node has $r = 1$ and $l = 0$. In particular, $\eta(C) = 0$. We shall say that such a node is a slim node.*
- *Suppose that $\eta(C) = 1$; there are 3 different kinds of such nodes:*

1. $r=2$, *exactly three connecting branches with* $x_i=2$ *and no terminal branches.* $\mu(C)=1, \alpha(C)=0$.

2. $r=2$, *exactly three branches with* $x_i=2,1,1$ *and the branch with* $x_i=2$ *is connecting.* $\mu(C)=1, \alpha(C)=0$.

3. $r=1$, *exactly one terminal branch, whose terminal vertex has* $p(C)=1$. $\mu(C)=0$, $\alpha(C)=1$.

Proof. Suppose that $\eta(C)\leqq 0$. The case $\alpha(C)>0$ cannot happen: this is clear if $\mu(C)=0$; in case $\mu(C)<0$ and $\alpha(C)\geqq s(C)>0$, then $\eta(C)\geqq -(s-1)+s\geqq 1$, which is a contradiction. We remark that $\alpha(C)=0$ implies that no curves on any terminal chains of C have self-intersection equal to (-1). In particular, no terminal multiplicities of the node C can equal r. Hence, if $r=s(C)$, $l=0$. It is easy to check that in this case, $\eta(C)\leqq 0$ implies $r=1$. Finally, the case $r>s(C)$ cannot happen. Since $g(C)\leqq 0$, C has two chains with gcd equal to $s(C)$; the others have gcd equal to r. It is easy to check that $\eta(C)\leqq 0$ cannot happen in this case.

Suppose that $\eta(C)=1$. The case $\alpha(C)>1$ cannot happen because otherwise $\mu(C)=-(s-1)<0$ leads to a contradiction. If $\alpha(C)=1$, then $s(C)=1$ and $g(C)=0$. Moreover, no terminal chain can have gcd equal to $r>1$. We leave it to the reader to check that only case 3 can happen if $\alpha(C)=1$. If $\alpha(C)=0$, then $g(C)\leqq 0$ and none of the $d-2$ chains with gcd equal to r can be terminal. We leave it to the reader to check that only case 1 and case 2 can happen if $\alpha(C)=0$.

4.8. **Proposition.** *Let* $T=(n,M,R,P,s=1)$ *be an almost minimal type such that the multiplicity of each of its nodes is bigger than or equal to* 3. *Then* T *has at most* $\gamma+\frac{1}{2}\alpha'$ *nodes.*

Proof. We first reduce to the case of an almost minimal NC-type, obtained by blow-ups, using Proposition 4.11 in [6]. According to this proposition, we can consider an almost minimal NC-type with same nodes and same invariants γ,β,α' as G. For any NC-type, we have the formulas ([6], Theorem 4.7):

$$2g_o-2\beta=\sum_{\text{nodes } C}\mu(C) \quad\text{and}\quad 2g_o-2\beta+\alpha'=\sum_{\text{nodes } C}\eta(C).$$

By our assumption and the previous lemma, none of the nodes of T has $\eta(C)\leqq 1$, so that $\sum_{\text{nodes } C}\eta(C)\geqq 2k$. Hence the conclusion follows.

4.9. **Theorem.** *Let* $T=(n,M,R,P,s=1)$ *be an almost minimal type whose graph has* k *nodes. If* G *does contain a slim node (see* 4.7) *then* $k\leqq 3(\gamma+\frac{1}{2}\alpha')+2\beta-2$. *Otherwise,* $k\leqq 2(\gamma+\frac{1}{2}\alpha')$.

Proof. As in the previous proposition, we first reduce to the case of an almost minimal NC-type. If the type has no slim nodes, then by definition all its nodes have $\eta(C)\geqq 1$. We use the formulas quoted in the previous proposition to conclude that $k\leqq 2(\gamma+\frac{1}{2}\alpha')$. When T has at least one slim node, our theorem follows from the next lemma:

4.10. **Lemma.** *Let* T *be an almost minimal* NC*-type with at least one slim node. Then the number of slim nodes of* T *is bounded by* $\gamma+\frac{1}{2}\alpha'+2\beta-2$.

Proof. Let N be the set of nodes of G. Let $\mathscr{E}$ be the set of slim nodes in N. We say that two nodes are connected if there exists a connecting chain between them. Let $\mathscr{D}$ be the set of nodes in $N-\mathscr{E}$ connected to two or more nodes of $\mathscr{E}$. Let $\mathscr{C}$ be the set of nodes in $N-\mathscr{E}-\mathscr{D}$ connected to exactly one node of $\mathscr{E}$.

We want to construct a new graph $\mathscr{G}$ such that $\mathscr{E}$ is included in the set of nodes of $\mathscr{G}$ and then apply 3.2 to bound the number of nodes of $\mathscr{G}$. But first we construct a subgraph $\mathscr{G}'$ of $\mathscr{G}$, whose vertices are the elements of $\mathscr{S}=\mathscr{E}\coprod\mathscr{D}\coprod\mathscr{C}$. Two vertices C and C' of $\mathscr{S}$ are linked in $\mathscr{G}'$ if there exists a connecting chain in G linking the corresponding nodes. $\mathscr{E}$ is included in the set of nodes of this subgraph by 4.7. Note that by construction, only vertices in $\mathscr{C}$ can be terminal vertices of this subgraph.

Consider a vertex C in $\mathscr{C}$ with $\eta(C)=1$. According to our description in 4.7 of nodes with $\eta(C)=1$, a node C in $\mathscr{C}$ cannot be a node with only one connecting chain. This is clear if C is in case 1 or 3. In case 2, the chain with $x_i=2$ is connecting. Since $C\in\mathscr{C}$, it is connected to a slim node, which has $r=1$. Hence a chain with $x_i=1$ must also be a connecting chain. It follows that each vertex C in $\mathscr{C}$ with $\eta(C)=1$ has a connecting chain (in G) to a node v_C of G not in $\mathscr{E}$. We can now construct the graph $\mathscr{G}$ in the following way. For each node $v_C\in\mathscr{D}\coprod\mathscr{C}$, we introduce a new edge that links in $\mathscr{G}\supset\mathscr{G}'$ the vertices C and v_C. Let $\mathscr{F}=\{v_1,\dots,v_b\}$ be the set of nodes v_C not in $\mathscr{D}\coprod\mathscr{C}$; we introduce b new vertices, noted again by $v_1,\dots,v_b$, and we link in $\mathscr{G}$ the vertex v_i to each vertex C_j in $\mathscr{G}'$ such that $v_{C_j}=v_i$.

The terminal vertices of $\mathscr{G}$ are either in $\mathscr{C}$ or $\mathscr{F}$. For each terminal vertex C of $\mathscr{G}$ in $\mathscr{C}$, $\eta(C)\geqq 2$ by construction. Also by construction, there exists a bijection between the set of terminal vertices of $\mathscr{G}$ in $\mathscr{F}$ and a set of vertices of $\mathscr{G}'$ in $\mathscr{C}$ with $\eta(C)=1$.

$$\begin{aligned}
2g_o-2\beta+\alpha' &= \sum_{\text{nodes } C}\eta(C)\\
&\geqq \sum_{C \text{ in } \mathscr{G}}\eta(C)\\
&\geqq \sum_{C\in\mathscr{C}\text{ with }\eta(C)\geqq 2}[\eta(C)]+\sum_{v\in\mathscr{F},C\text{ corresp.in }\mathscr{C}}[\eta(v)+\eta(C)]\\
&\geqq 2\cdot|\mathscr{C}\textstyle\coprod\mathscr{F}|\\
&\geqq 2\cdot(\text{number of terminal vertices in }\mathscr{G}).
\end{aligned}$$

Hence the number of terminal chains of $\mathscr{G}$ is smaller than or equal to $g_o-\beta+\frac{1}{2}\alpha'$. We note that the first Betti number of $\mathscr{G}$ is smaller than or equal to β so that, by 3.2:

$$\begin{aligned}
|\mathscr{E}| &\leqq \text{number of nodes of }\mathscr{G}\\
&\leqq 2\beta-2+(\text{number of terminal chains of }\mathscr{G})\\
&\leqq 2\beta-2+g_o-\beta+\tfrac{1}{2}\alpha'\\
&=\gamma+2\beta+\tfrac{1}{2}\alpha'-2.
\end{aligned}$$

Proof of 4.3. As noted at the beginning of this section and proven in [7], we have $\alpha(\mathscr{X})\geqq a$ and $\gamma(\mathscr{X})\leqq u$. In particular, $\gamma(\mathscr{X})-\frac{1}{2}\alpha(\mathscr{X})\leqq u-\frac{1}{2}a$, so that $k\leqq 3(\gamma+\frac{1}{2}\alpha')+2\beta-2\leqq 2g-2+\gamma-\frac{1}{2}\alpha\leqq 2g-2+u-\frac{1}{2}a$.

By the previous theorem, the number of nodes of $G(\mathscr{X}_s)$ equals at most the maximum of $k_1 = 2g - 2 + \gamma - \frac{1}{2}\alpha$ and $k_2 = 2g - \alpha - 2\beta$. $k_1 \geqq k_2$ except when $\gamma + \frac{1}{2}\alpha + 2\beta \leqq 2$. This could happen if $g = 1$ but we assumed that $g \geqq 2$. This could also happen if $\gamma = \alpha = 1$ and $g = 2$. It is easy to check with the tables of [9] that when $g = 2$, the number of nodes is bounded by k_1.

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