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Introduction

In this paper, we discuss some properties of intersection matrices, as they arise in the study of degenerating curves in algebraic geometry. In particular, we provide some information on the Smith normal form (row and column reduction) of such integer positive definite forms of rank $(n-1)$.

Before defining the notion of arithmetical graph, we state some notations used throughout this work.

G a connected non oriented graph.
n number of vertices of G.
C_i a vertex of G.
c_{ij} the number of edges linking C_i to C_j.
m number of edges of G.
β first Betti number of $G: \beta = m-(n-1)$.
d_i number of edges at the vertex C_i = degree of C_i.

Adjacency matrix	The matrix $A=(a_{ij})$ where $a_{ii}=0\ \forall i$ and $a_{ij}=c_{ij}$ for $i \neq j$.
d-matrix D	A diagonal matrix $(c_{11},\ldots,c_{nn})$ with $c_{ii}\in\mathbb{Z}_{\geqq 1}$. We denote by M the matrix $D-A$.
Multiplicity vector R	a vector ${}^tR=(r_1,\ldots,r_n)$ with $r_i\in\mathbb{Z}_{\geqq 1}$ and $\gcd(r_1,\ldots,r_n)=1$. We let ${}^tJ=(1,1,\ldots,1)$.

Definition. An arithmetical graph consists of the following data: a connected graph G, a d-matrix D and a multiplicity vector R such that $MR=0$, where $M:=D-A$.

We may also say that (M,R) defines an arithmetical structure on G. The arithmetical graph (G,M,R) is called "minimal" if $c_{ii}>1$ for all $i=1,\ldots,n$. It is said to be "reduced" if $R=J$ and "simple" if $\mathrm{sk}(G)=G$ (see Sect. 3).

Every connected graph has a natural structure of reduced arithmetical graph: take $D=\mathrm{diag}(d_1,\ldots,d_n)$ where d_i is the degree of the vertex C_i in G.

Let (G,M,R) be an arithmetical graph. The matrices M arise in algebraic geometry as intersection matrices of degenerating curves. Let $\mathrm{diag}(e_1,\ldots,e_{n-1}\, 0)$,

$e_1|e_2|\cdots|e_{n-1}$ be a matrix row and column euivalent (over $\mathbb{Z}$) to the matrix M. Raynaud has proven that the group $\Phi := \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1}\mathbb{Z}$ is the group of components of the Néron model of the jacobian associated to the generic curve. We refer the reader to [8] for the geometric motivations in studying arithmetical graphs. We discuss in this paper the properties of such a matrix M and its associated group Φ and we hope that by presenting here our results without any references to geometry, some non algebraic geometers will take interest in this subject and bring new techniques to the study of these matrices.

With the above notations, $2\beta - 2 = \sum_{i=1}^{n} (d_i - 2)$. For any arithmetical graph, we define its linear rank g_0 by $2g_0 - 2 = \sum_{i=1}^{n} r_i(d_i - 2)$. Since it might happen that $r_i > 1$ when $d_i = 1$, it is not clear from the definitions that $g_0 \geqq \beta$. We prove this fact in 4.7. Both integers g_0 and $g_0 - \beta$ can be interpreted geometrically [8]. The reader will find tables for arithmetical graphs of linear rank one and two in [9] and [11].

Let $l^{\mathrm{ord}_l(a)}$ denote the exact power of the prime l dividing the integer a. Our guess relative to the structure of Φ (for simple graphs) is the following: if $\beta \leqq n-1$, then $\psi := e_1 \cdot \cdots \cdot e_{n-1-\beta}$ satisfies $\sum_{l\,\mathrm{prime}} \mathrm{ord}_l(\psi)(l-1) \leqq 2g_0 - 2\beta$. This implies in particular that $e_1 \cdots e_{n-1-\beta} \leqq 2^{2g_0 - 2\beta}$. This can also be expressed by saying that Φ splits as a product $\Upsilon \times C_1 \times \cdots \times C_\beta$, where $C_1, \ldots, C_\beta$ are cyclic groups and $\Upsilon = \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1-\beta}\mathbb{Z}$ is bounded by an explicit constant depending on $g_0 - \beta$ only.

We prove this fact for a wide class of arithmetical graphs (6.2), including the cases where:

- G is a simple tree (3.5); $|\Phi| = \prod r_i^{d_i - 2}$ and $\sum_{l\,\mathrm{prime}} \mathrm{ord}_l(|\Phi|)(l-1) \leqq 2g_0 - 2\beta$. This theorem complements a result of Oort and Lenstra [6], where a bound for $\sum_{l\,\mathrm{prime}} \mathrm{ord}_l(|\Phi|)(l-1)$ is discussed for the first time.
- $R = J$ (6.2); in this case $|\Phi|$ equals κ, the number of spanning trees of G.
- For any arithmetical graph, we show that $v := \prod r_i^{d_i - 2}$ is an integer (4.6) and satisfies the bound $\sum_{l\,\mathrm{prime}} \mathrm{ord}_l(v)(l-1) \leqq 2g_0 - 2\beta$ (4.7). Moreover, we show that $|\Phi| \leqq v\kappa$ (3.5).

There exists only finitely many structures of arithmetical graph on any given graph G (1.6). For each such structure, we defined its volume v and its linear rank g_0. We do not know if these integers are related in any way to the standard numerical invariants associated to a graph.

1. The Group Φ

Proposition 1.1. *Let (G, M, R) be an arithmetical graph. The matrix M satisfies the following properties:*

- *It is symmetric and represents a positive semidefinite quadratic form of rank $n-1$. Its kernel is generated over $\mathbb{Q}$ by R.*

- *The adjoint of the matrix M is given by* $\mathrm{ad}(M) = \tilde{\phi}\cdot R({}^tR)$ *where $\tilde{\phi}$ is a positive integer.*

Proof. In order to prove the first statement, it is sufficient to show that the determinant of every principal minor of M is strictly positive. We shall prove this fact only for M^{nn}, the minor obtained by deleting the last row and the last column from M. The proof for a minor of any dimension is similar.

Let N be the matrix obtained from M^{nn} by multiplying its i^{th} column by r_i. Let N_i denote the i^{th} column of N. Then $\sum N_i$ is a vector with positive coefficients, and because G is connected, one of these coefficients is strictly positive. Suppose that $\det(N) = 0$. Then up to reordering, we may assume that there exists a set $(a_1 = 1, a_2, \ldots, a_{n-1})$ with $a_i \leqq 1$ for all i and such that $\sum a_i N_i = 0$. Since G is connected, this last condition on the N_is contradicts the previous one, and hence $\det(M^{nn}) \neq 0$. It is not hard to find a path in $GL_n(\mathbb{R})$ between M^{nn} and a matrix with positive determinant, so that $\det(M^{nn}) > 0$.

By definition of the adjoint, $M\cdot\mathrm{ad}(M) = \det M\cdot I_n = 0$. Since the kernel of M is generated by R, $\mathrm{ad}(M) = R\cdot{}^tS$ where ${}^tS$ is a vector with integer coefficients. Ad(M) is a symmetric matrix and hence $R\cdot{}^tS = S\cdot{}^tR$. In particular, $({}^tR\cdot R){}^tS = ({}^tR\cdot S){}^tR$. This implies that $S = \tilde{\phi}R$ for some (nonzero) rational number $\tilde{\phi}$. Since R and S have both integer coefficients and $\gcd(r_1,\ldots,r_n) = 1$, $\tilde{\phi}$ is also an integer. $\tilde{\phi}$ is a positive integer because the principal minors of M have positive determinants.

Remark 1.2. The matrices that we call intersection matrices in this work, i.e. the matrices M of arithmetical graphs (G, M, R), are called *M-matrices* by some authors: see for instance [3, chapter 6], where many equivalent properties of such matrices M are listed.

Corollary 1.3. *We denote by M^{ij} the $(n-1\times n-1)$ minor obtained from M by removing the i^{th} line and the j^{th} column. Then $\tilde{\phi} = \det(M^{ij})\cdot(r_i r_j)^{-1}$ for all $i, j \in \{1,\ldots,n\}$. In particular, $\tilde{\phi}$ equals the* gcd *of the determinants of all $(n-1\times n-1)$ minors of M.*

Keeping the terminology and notations used in algebraic geometry, we define the group of components of the arithmetical graph (G, M, R) to be the group $\Phi = \mathrm{Ker}({}^tR)/\mathrm{Im}(M)$, where $M: \mathbb{Z}^n \to \mathbb{Z}^n$ and ${}^tR: \mathbb{Z}^n \to \mathbb{Z}$.

Theorem 1.4. *Let (G, M, R) be an arithmetical graph and* $\mathrm{diag}(e_1,\ldots,e_{n-1},0)$ *be an integer matrix row and column equivalent to M over $\mathbb{Z}$. Then $\Phi \cong \mathbb{Z}/e_1\mathbb{Z}\times\cdots\times\mathbb{Z}/e_{n-1}\mathbb{Z}$ and $\tilde{\phi}(M)$ equals the order ϕ of the group Φ.*

Proof. There exists two matrices A, B in $GL(n,\mathbb{Z})$ such that:

$$AMB = \mathrm{diag}(e_1,\ldots,e_{n-1},0) \quad e_i \in \mathbb{Z} \quad \forall i = 1,\ldots,n-1.$$

It is easy to check that $\mathrm{Ker}({}^tR{}^tB^{-1})/\mathrm{Im}({}^tBM{}^tA) \cong \mathrm{Ker}({}^tR)/\mathrm{Im}(M)$. Since

$$MB = \begin{pmatrix} & 0 \\ * & \vdots \\ & 0 \end{pmatrix}$$

and because $\operatorname{Ker} M$ is generated by R, the last column of B is a multiple of R. In fact, since B is a integer matrix and $\gcd(r_1, \ldots, r_n) = 1$, the last column of B is up to a sign equal to R. Since $B^{-1} \cdot B = \mathrm{Id}$, $B^{-1} \cdot R = \pm E_n$ where ${}^tE_n = (0, \ldots, 0, 1)$. It follows that $\operatorname{Ker}({}^tR^tB^{-1})$ has the first $(n-1)$ standard vectors as an obvious basis and hence

$$\Phi \cong \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1}\mathbb{Z} \quad \text{and} \quad |\Phi| = |e_1 \cdots e_{n-1}|.$$

To show that $\tilde{\phi} = \phi$, we note (see [5, p. 179]) that if M is equivalent to M', the $\gcd \Delta_i(M)$ of the determinants of all $(i \times i)$ minors of M equals, up to a sign, the $\gcd \Delta_i(M')$ of the determinants of all $(i \times i)$ minors of M'. It is clear then that $\phi = \Delta_{n-1}(AMB)$ and by 1.3, $\tilde{\phi} = \Delta_{n-1}(M)$.

The following "Theorem of the Elementary Divisors" gives a computable way to determine the structure of the group Φ. For its proof, we refer the reader to the book of Jacobson [5, p. 179].

Theorem 1.5. *Let M be a $(n \times n)$ matrix with integer coefficients. We let Δ_i denote the* gcd *of the determinants of all $(i \times i)$ minors in M. Then M is equivalent over $\mathbb{Z}$ to a diagonal matrix* $\operatorname{diag}(f_1, \ldots, f_r, 0, \ldots, 0)$ *where r is the (determinantal) rank of M and where, up to a sign, $f_1 = \Delta_1, f_2 = \Delta_2/\Delta_1, \ldots, f_r = \Delta_r/\Delta_{r-1}$.*

Let $h = h(M)$ be the minimal integer such that $\Delta_{h+1} \neq 1$ and $\bar{h} = \operatorname{rank}(M) - h$. If $\operatorname{diag}(e_1, \ldots, e_r, 0, \ldots, 0)$ is any diagonal matrix equivalent to M over $\mathbb{Z}$, then $e_i = 1$ for at most $h = h(M)$ distinct i's. In particular, when M is an intersection matrix, Φ can be written as a product of $\bar{h}$ cyclic groups and $\bar{h}$ is minimal for this property.

Lemma 1.6. *There exists only finitely many structures of arithmetical graph on any given graph.*

Proof. Since all principal minors of M have positive determinants, intersection matrices can be characterized by the following property: $M + X$ is non singular for all matrices $X = \operatorname{diag}(x_1, \ldots, x_n)$ with $x_i \geqq 0$ for all $i = 1, \ldots, n$ and $X \neq 0$ (see [3, Chap. 6]). Suppose that there exists an infinite sequence of pairwise distinct arithmetical structures $(M_k = D_k - A, R_k)$ on a given graph G. As was pointed out to the author by H. Lenstra, we can then extract from the sequence $(D_k)_{k=1}^{\infty}$ an infinite subsequence $(D_j)_{j=1}^{\infty}$ with the following properties: let $D_j = \operatorname{diag}(c_{j1}, \ldots, c_{jn})$; each "coordinates sequence" $(c_{ji})_{j=1}^{\infty}$ is either a strictly increasing sequence of positive integers or a constant sequence. This contradicts the fact that the matrices M_js are singular.

Remark 1.7. Let (G, M, R) be an arithmetical graph such that the diagonal coefficients of the matrix M are all equal to 2. The graphs G having such a structure arise in connection to Lie algebras or to elliptic curves and have been classified. Such a structure on G is minimal and is in fact the only one having this property: as quoted in the above lemma, any matrix of the form $M + X$ with X positive is non singular.

1.8. Let (G, M, R) be an arithmetical graph and ${}^tQ = (q_1, \ldots, q_n)$ be an integer vector such that $x = \sum_{i=1}^{n} q_i r_i \neq 0$. We consider the matrices

$$M_Q = \begin{pmatrix} M + Q^t Q & -Q \\ -{}^tQ & 1 \end{pmatrix} \quad \text{and} \quad M_{Q^+} = \begin{pmatrix} M - Q^t Q & Q \\ -{}^tQ & 1 \end{pmatrix}$$

It is very easy to check that both matrices are equivalent to $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$. We note that $({}^tR \quad x) \cdot M_Q = (0 \cdots 0) = (-{}^tR \quad x) \cdot M_{Q^+}$.

The matrix M_Q is symmetric. When it defines an arithmetical graph, we may call this new graph the *blow-up of G with respect to Q* in analogy to the geometrical situation. In particular, the blow-up has the same group of components as the graph G. The fact that M_Q defines an arithmetical graph depends only on the matrix A, not on the structure (D, R). Using the same kind of arguments as in 1.1, it is easy to check that

1) $\operatorname{ad}(M_Q) = \pm \phi(M) \begin{pmatrix} R \\ x \end{pmatrix} ({}^tR \quad x)$

2) $\operatorname{ad}(M_{Q^+}) = \pm \phi(M) \begin{pmatrix} R \\ x \end{pmatrix} ({}^tR \quad x)$.

In particular, we have obtained the following proposition, which generalizes a well-known theorem of Temperley (see [4, p. 35]).

Proposition 1.9. *Let (G, M, G) be an arithmetical graph and $Q = (q_1, \ldots, q_n)$ an integer vector such that $x = \sum_{i=1}^{n} r_i q_i \neq 0$. Then $\phi = x^{-2} |\det(M - Q^t Q)| = x^{-2} |\det(M + Q^t Q)|$.*

Example 1.10. Every multiplicity vector R defines an arithmetical graph $(G(R), M(R), R)$ where $M(R) = rI_n - (R \cdot {}^tR)$ and $r = r_1^2 + \cdots + r_n^2$. The group Φ associated to this arithmetical graph is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{n-2}$. Note that $G(J)$ is the complete graph on n vertices and its group Φ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{n-2}$.

Proof. Using the previous proposition, we easily check that $\phi = r^{n-2}$. To determine the group structure, we compute the elementary divisors (see 1.5) of $M(R)$. It is easy to check that $\Delta_1 = 1$ because $\gcd(r_1, \ldots, r_n) = 1$. The reader may also check that all determinants of (2×2) minors are either 0 or divisible by r, so that $r | f_2 = \Delta_2 / \Delta_1$. Since $f_2 | f_3 | \cdots | f_{n-1}$ and the product of the f_is equals r^{n-2}, we must have $f_2 = \cdots = f_{n-1} = r$.

Remark/Definition 1.11. Let (G, M, R) be an arithmetical graph. The *associated reduced structure* $(\tilde{G}, \tilde{M}, J)$ of an arithmetical graph (G, M, R) is the arithmetical graph corresponding to the matrix $\tilde{M} = \tilde{R} M \tilde{R}$, where $\tilde{R} = \operatorname{diag}(r_1, \ldots, r_n)$. We claim that $\phi(\tilde{G}) = (r_1 \cdots r_n)^2 \phi(G)$. By corollary 1.3, we know that $\phi(M) = \det(M^{11}) r_1^{-2}$ and that $\phi(\tilde{M}) = \det(\tilde{M}^{11})$. Since $\tilde{R}$ is a diagonal matrix, $\det(\tilde{M}^{11}) = (r_2 \cdots r_n) \det(M^{11})(r_2 \cdots r_n)$ and hence our claim is true.

Proposition 1.12. *The following exact sequences of abelian groups relate the structure of Φ to the structure of $\tilde{\Phi}$.*

$$0 \to \prod \mathbb{Z}/r_i\mathbb{Z} \to E \to \Phi \to 0$$
$$0 \to E \to \tilde{\Phi} \to \prod \mathbb{Z}/r_i\mathbb{Z} \to 0$$

where $E \cong \operatorname{Ker}({}^tR)/\operatorname{Im}(M\tilde{R})$.

Proof. We first note that $\operatorname{Ker}({}^tR)/\operatorname{Im}(M\tilde{R}) \cong \tilde{R}\operatorname{Ker}({}^tR)/\tilde{R}(\operatorname{Im}(M\tilde{R}))$ because $\tilde{R}$ is injective. Then the inclusions

$$\operatorname{Im}(M\tilde{R}) \subset \operatorname{Im}(M) \subset \operatorname{Ker}({}^tR)$$
$$\tilde{R}(\operatorname{Im}(M\tilde{R})) \subset \tilde{R}(\operatorname{Ker}{}^tR) \subset \operatorname{Ker}({}^tJ)$$

and the following lemma lead to the desired exact sequences.

Lemma 1.13. *Let* (G, M, R) *be an arithmetical graph. Then* $\operatorname{Im}(M)/\operatorname{Im}(M\tilde{R}) \cong \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_n\mathbb{Z} \cong \operatorname{Ker}({}^tJ)/\tilde{R}\operatorname{Ker}({}^tR)$.

Proof. Let $C = \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_n\mathbb{Z}$. The first isomorphism is proven by considering the following diagram of groups with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}(M)\cap\tilde{R}(\mathbb{Z}^n) & \longrightarrow & \operatorname{Ker}(M) & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{R}(\mathbb{Z}^n) & \longrightarrow & \mathbb{Z}^n & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow M & & \downarrow M & & \downarrow & & \\
0 & \longrightarrow & M\tilde{R}(\mathbb{Z}^n) & \longrightarrow & M(\mathbb{Z}^n) & \longrightarrow & \operatorname{Im}(M)/\operatorname{Im}(M\tilde{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$\operatorname{Ker}(M)$ is generated by R. Since $\tilde{R}(\mathbb{Z}^n)$ contains $\tilde{R}(J) = R$, $\operatorname{Ker}(M) \subset \tilde{R}(\mathbb{Z}^n)$ and $K = 0$. In order to prove the second isomorphism, we consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Ker}({}^tR) & \longrightarrow & \mathbb{Z}^n & \xrightarrow{{}^tR} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow \tilde{R} & & \| \\
0 & \longrightarrow & \operatorname{Ker}({}^tJ) & \longrightarrow & \mathbb{Z}^n & \xrightarrow{{}^tJ} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Ker}({}^tJ)/\tilde{R}\operatorname{Ker}({}^tR) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

2. The case of a tree

We say that a vertex C_m of a graph G is a *terminal vertex* if it has degree equal to one. A *node* of G is a vertex C_i of degree $d_i \geqq 3$. A *terminal chain* at C_i is a connected component of $G - C_i$ which contains a terminal vertex but no nodes.

For any graph G, we define the *skeleton of* G to be the graph $\mathrm{sk}(G)$ which has the same set of vertices as G but where C_i is linked by exactly one edge to C_j iff $c_{ij} > 0$. When $c_{ij} \neq 0$, we denote by (i, j) the unique edge in $\mathrm{sk}(G)$ linking C_i and C_j. When $G = \mathrm{sk}(G)$, we say that G is a *simple* graph.

Given a graph G, let $G^{(1)}$ be the graph obtained from G by removing all vertices (and all their adjacent edges) of the terminal branches of $\mathrm{sk}(G)$. We write $G^{(i)} = (G^{(i-1)})^{(1)}$. The sequence $G \supseteq G^{(1)} \supseteq \cdots \supseteq G^{(i)} \cdots$ is stationary for i large enough. We let

$$\mathrm{di}(G) = \bigcap_{i=1}^{\infty} G^{(i)}$$

$\mathrm{Sk}(\mathrm{di}(G)) = \mathrm{di}(\mathrm{sk}(G))$ is the maximal subgraph of $\mathrm{sk}(G)$ that does not have any terminal chains. Each one of its vertices has degree bigger or equal to 2. When G is a tree, $\mathrm{di}(G)$ is the trivial graph with one vertex and no edges. We set the group of components of the trivial graph to be the trivial group.

Theorem 2.1. *Let (G, M, J) be a reduced arithmetical graph. Then*

$$\Phi(G) \cong \prod_{C_i \text{ or } C_j \notin \mathrm{di}(G)} \mathbb{Z}/c_{ij}\mathbb{Z} \times \Phi(\mathrm{di}(G))$$

Proof. We shall perform rows and column operations and abbreviate "add x_i times the i^{th} column and x_j times the j^{th} column to the k^{th} column" by "$x_i \mathrm{Col}_i + x_j \mathrm{Col}_j + \mathrm{Col}_k$", and similarely for the rows. Let TC be a terminal chain of $\mathrm{sk}(G)$ and G' be the graph obtained from G by removing all the vertices $C_1, \ldots, C_{q-1}$ of TC. We denote by C_q the node by which TC is attached to G'. We claim that $\Phi(G) \cong \prod_{i=1}^{q-1} \mathbb{Z}/c_{i,i+1}\mathbb{Z} \times \Phi(G')$. The matrix M has the following form:

$$\begin{pmatrix} c_{11} & -c_{12} & & & \\ -c_{12} & c_{22} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & c_{q-1,q-1} & -c_{q-1,q} \\ & & \ddots & -c_{q-1,q} & \\ & & & & N \end{pmatrix}$$

We note that M is not quite the direct sum of two blocks. Perform "$\mathrm{Col}_1 + \cdots + \mathrm{Col}_q$" and "$\mathrm{Row}_1 + \cdots + \mathrm{Row}_q$" to get a new matrix which is the direct sum of two blocks; the lower one is the matrix M' of the graph G'. In order to get the other block in the desired diagonal form, it is sufficient to do "$\mathrm{Row}_1 + \mathrm{Row}_2$", "$\mathrm{Row}_2 + \mathrm{Row}_3$", …, "$\mathrm{Row}_{q-2} + \mathrm{Row}_{q-1}$" and "$\mathrm{Col}_1 + \mathrm{Col}_2$", …, "$\mathrm{Col}_{q-2} + \mathrm{Col}_{q-1}$".

Corollary 2.2. *Let (G, M, J) be a reduced arithmetical graph such that* $\mathrm{sk}(G)$ *is a* *tree. Then*

$$\Phi \cong \prod_{c_{ij} \neq 0, i<j} \mathbb{Z}/c_{ij}\mathbb{Z}$$

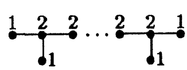
Corollary 2.3. *Let (G, M, R) be an arithmetical graph. Suppose that* sk(G) *is a tree and let $\tilde{d}_i$ be the degree of C_i in* sk(G). *Then*

$$\phi = \Bigg(\prod_{c_{ij} \neq 0, i<j} c_{ij}\Bigg)\Bigg(\prod_{i=1}^{n} r_i^{\tilde{d}_i - 2}\Bigg)$$

and Φ is killed by L, where $L = \operatorname{lcm}(c_{ij} r_i r_j,\ c_{ij} \neq 0)$.

Proof. To show that $\phi = \Big(\prod_{c_{ij} \neq 0, i<j} c_{ij}\Big)\Big(\prod_{i=1}^{n} r_i^{\tilde{d}_i}\Big)(r_1 \cdots r_n)^{-2}$, apply the previous corollary to the reduced structure of G and then use 1.11. We shall give a different proof of this fact in 3.5. Sk$(\tilde{G})$ is also a tree and we can apply 2.2 to find the group structure of $\tilde{\Phi}$. It is clear that L kills $\tilde{\Phi}$. But then, by 1.12, L also kills Φ.

Example 2.4. Following the notations introduced by Kodaira for the reductions of elliptic curves [11], we denote the graph below by I_v^*, where $v \geqq 1$ is the number of vertices of multiplicity 2 in the graph minus one.



When v is even, $\Phi(I_v^*) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and when v is odd, $\Phi(I_v^*) \cong \mathbb{Z}/4\mathbb{Z}$. We see that in this last case, the bound obtained in the corollary is achieved.

We note (C, r) a vertex C with multiplicity r. When (G, M, R) is a simple arithmetical graph, we can bound the exponent of Φ in a slightly different way because of the following properties:

1. Let (C_i, r_i), $i = 1, 2, 3$ be three consecutive vertices on a chain (i.e. C_2 cannot be a node). Then $\gcd(r_1, r_2) = \gcd(r_2, r_3)$ because r_2 divides $r_1 + r_3$.
2. Let (C_i, r_i), $i = 1, 2, m$ be three vertices on a terminal chain such that C_1 and C_2 are consecutive and that C_m is the terminal vertex. Then $r_m = \gcd(r_1, r_2)$.

For an edge (i, j) linking (C_i, r_j) and (C_j, r_j), we let $f_{ij} = \gcd(r_i, r_j)$ and we note that if (i, j) and (k, l) are on the same chain, then $f_{ij} = f_{kl}$.

Corollary 2.5. *The group of components Φ of a simple arithmetical tree (G, M, R) is killed by $K = \operatorname{lcm}(r_1, \ldots, r_n) \cdot \operatorname{lcm}(f_{ij})$ and has order equal to $\prod_{i=1}^{n} r_i^{d_i - 2}$, where d_i denotes the degree of the vertex C_i in G.*

Proof. By the previous corollary, we know that Φ is killed by $L = \operatorname{lcm}(r_i r_j)$, where $c_{ij} \neq 0$. We can write $r_i r_j = \operatorname{lcm}(r_i, r_j) \cdot f_{ij}$. But $\operatorname{lcm}(a_i b_i, i = 1, \ldots, n)$ divide $\operatorname{lcm}(a_i) \cdot \operatorname{lcm}(b_i)$ and $\operatorname{lcm}(\operatorname{lcm}(a_i, a_j)) = \operatorname{lcm}(a_1, \ldots, a_n)$. This imply that Φ is killed by $\operatorname{lcm}(r_1, \ldots, r_n) \cdot \operatorname{lcm}(f_{ij})$.

We do not know if the bound K can be replaced by $K' = \operatorname{lcm}(r_1, \ldots, r_n) \cdot \operatorname{lcm}(f_{kl}, (k, l)$

not on a terminal chain). In the special case where the graph is a tree with exactly one node, one can get a more precise result; we omit the proof of the next proposition (see [7]).

Proposition 2.6. *Let (G, M, R) be a simple arithmetical tree with exactly one node. Let r be the multiplicity of the unique node. Then the group Φ is killed by r.*

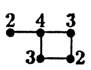
3. Volume and Linear Rank

Definition 3.1. A spanning subgraph of a graph G is a subgraph H of G such that its vertices are exactly the vertices of G. A spanning tree is a spanning subgraph which is a tree. We let $\kappa(G)$ denote the number of spanning trees of G.

Theorem 3.2. *Let (G, M, J) be a reduced arithmetical graph. Then $\phi = \kappa(G)$.*

This theorem is proven in the books of Berge [2] or Biggs [4].

Remark 3.3. When (G, M, J) is a reduced arithmetical graph, the previous theorem shows that Φ is trivial if and only if G is a tree. This is not true for non reduced arithmetical graphs in general, as it can be seen by the following example:



$\phi = 1$: compute the number of spanning trees of the associated reduced graph first and then divide by $\prod r_i^2$.

Given a graph G, we denote the edge of $\mathrm{sk}(G)$ linking C_i and C_j by (i, j) and the pair (i, j) represents an edge iff $c_{ij} \neq 0$. We describe a spanning tree $T = [(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})]$ using the $n-1$ edges of $\mathrm{sk}(G)$ that belong to T.

Corollary 3.4. *Let (G, M, J) be a reduced arithmetical graph. If $\mathrm{sk}(G)$ is a tree, then $\phi = \prod_{c_{ij} \neq 0, i<j} c_{ij}$. In the general case, $\phi = \sum_{\text{span.trees } T \text{ of } \mathrm{sk}(G)} \left(\prod_{(i,j) \in T} c_{ij} \right)$.*

Proof. By 3.2, we only need to compute $\kappa(G)$. In order to completely define a spanning tree of G, we must choose, for each $(i, j) \in \mathrm{sk}(G)$, one edge among the c_{ij} edges linking C_i and C_j. Hence $\kappa(G) = \prod_{c_{ij} \neq 0, i<j} c_{ij}$. Note that when $\mathrm{sk}(G)$ is a tree, this corollary follows also from 2.2.

Corollary 3.5. *Let (G, M, R) be an arithmetical graph and $\tilde{d}_i$ the degree of the vertex C_i in $\mathrm{sk}(G)$. Then*

$$\phi \leq \left(\prod_{i=1}^{n} r_i^{\tilde{d}_i - 2} \right) \cdot \kappa(G) \quad \text{and} \quad \kappa(G) \leq \left(\prod_{c_{ij} \neq 0, i<j} c_{ij} \right) \cdot \kappa(\mathrm{sk}(G))$$

Equality occurs in the first case iff G is reduced or $\mathrm{sk}(G)$ is a tree and it holds in the second case iff $\mathrm{sk}(G)$ is a tree or $G = \mathrm{sk}(G)$.

Proof. In order to prove the first inequality, we compute $\kappa(\tilde{G})$, noting, by 1.11 and 3.2, that $\phi(G)=(r_1\cdots r_n)^{-2}\kappa(\tilde{G})$.

$$\kappa(\tilde{G})=\sum_{\text{span.trees } T \text{ of sk}(G)}\left(\prod_{(i,j)\in T} c_{ij}r_ir_j\right)$$
$$=(\prod r_i^{\tilde{d}_i})\left[\sum_{\text{span.trees } T \text{ of sk}(G)}\left(\prod_{(i,j)\in T} c_{ij}\right)\left(\prod_{(i,j)\notin T} r_ir_j\right)^{-1}\right]$$

where $\left(\prod_{(i,j)\notin T} r_ir_j\right)^{-1}=1$ if $T=\text{sk}(G)$. Sinee $\left(\prod_{(i,j)\notin T} r_ir_j\right)^{-1}\leqq 1$, we get that

$$\kappa(\tilde{G})\leqq(\prod r_i^{\tilde{d}_i})\left(\sum_{\text{span.trees } T \text{ of sk}(G)}\left(\prod_{(i,j)\in T} c_{ij}\right)\right)=\prod r_i^{\tilde{d}_i}\cdot\kappa(G).$$

Equality occurs iff $\left(\prod_{(i,j)\notin T} r_ir_j\right)^{-1}=1$ for all T, and this occurs if only if all r_is are equal to 1 or sk(G) is a tree. We now compute $\kappa(G)$ in a similar way:

$$\kappa(G)=\sum_{\text{span.trees } T \text{ of sk}(G)}\left(\prod_{(i,j)\in T} c_{ij}\right)$$
$$=(\prod c_{ij})\cdot\left[\sum_{\text{span.trees } T \text{ of sk}(G)}\left(\prod_{(i,j)\notin T} c_{ij}\right)^{-1}\right].$$

Since $\left(\prod_{(i,j)\notin T} c_{ij}\right)^{-1}\leqq 1$, we get that

$$\kappa(G)\leqq\left(\prod_{c_{ij}\neq 0,i<j} c_{ij}\right)\cdot\kappa(\text{sk}(G))$$

Definition 3.6. Let (G,M,R) be an arithmetical graph. Let d_i be the degree of the vertex C_i in G. The volume v of the arithmetical graph (G,M,R) is the rational number $v=\prod_{q=1}^{n} r_q^{d_q-2}$.

It is very easy to check that $2\beta-2=\sum_{q=1}^{n}(d_q-2)$. To generalize this notion, we define the linear rank g_0 of the arithmetical graph (G,M,R) by the formula: $2g_0-2=\sum_{q=1}^{n} r_q(d_q-2)=\sum_{q=1}^{n} r_q(c_{qq}-2)$.

The linear rank of an arithmetical graph is an integer: since ${}^tRMR=0$, $\sum_{i=1}^{n} c_{ii}r_i^2=2\sum_{i<j} c_{ij}r_ir_j$. g_0 is then an integer because $\sum_{i=1}^{n} c_{ii}r_i^2\equiv\sum_{i=1}^{n} c_{ii}r_i \pmod 2$. If the graph is minimal (i.e. if $c_{qq}\geqq 2\ \forall q$), then $g_0\geqq 1$.

Let $2\gamma:=2g_0-2\beta=\sum_{q=1}^{n}(r_q-1)(d_q-2)$. It follows that $g_0=\beta(G)$ when the arithmetical graph G is reduced. When the graph is minimal, the converse is also true, namely: $g_0=\beta$ implies that (G,M,R) is reduced (see 4.13).

We note that since some of the d_is might be equal to 1, the volume v is a priori only a rational number. For the same reason, it is also not at all clear that $g_0 - \beta$ is a positive integer. In the next section, we shall first prove these facts for simple graphs (4.6, 4.7), and then (4.10) we shall reduce the general case to the case where the graph G is simple.

4. Bound for the volume

The function $l(x) = \sum_{p \text{ prime}} \operatorname{ord}_p(x)(p-1)$, defined for any integer x, was introduced by Lenstra and Oort in [6]. It satisfies $l(xy) = l(x) + l(y)$ and $ln_2(x) \leqq l(x) \leqq x - 1$. In particular, $x \leqq 2^{l(x)}$. Our aim in this section is to show that, for any arithmetical graph, the volume v is an integer and that

$$0 \leqq l(v) = l\left(\prod_{i=1}^{n} r_i^{d_i - 2}\right) \leqq \sum_{i=1}^{n} (r_i - 1)(d_i - 2) = 2g_0 - 2\beta.$$

These inequalities are trivial in the special case where all the terminal multiplicities of the graph are equal to one.

Proposition 4.1. *Let (G, M, R) be a simple arithmetical tree ($v = \phi$ in this case) with exactly one node. Then $l(\phi) \leqq 2g_0$. In particular g_0 is a positive integer and when $g_0 = 0$, $\phi = 1$.*

Proof. Let r denote the multiplicity of the node and $r_1, \ldots, r_d$ the multiplicities of the vertices adjacent to the node. Let $x_i = \gcd(r, r_i)$. We remarked already in 2.6 that x_i is the multiplicity of the terminal vertex on the chain started by (C_i, r_i). We also know that $\phi = \dfrac{r^{d-2}}{x_1 \cdots x_d}$ is an integer and that $2g_0 = (d-2)r - \sum_{i=1}^{d} x_i + 2$. Suppose first that $x_{d-1} = x_d = 1$:

$$l(\phi) = \sum_{i=1}^{d-2} l\left(\frac{r}{x_i}\right) \leqq \sum_{i=1}^{d-2} \left(\frac{r}{x_i} - 1\right) \leqq \sum_{i=1}^{d-2} (r - x_i) = (d-2)r - \sum_{i=1}^{d} x_i + 2 = 2g_0.$$

It is then sufficient to show how to find a set $(y_1, \ldots, y_{d-2}, 1, 1)$ of integers satisfying the same hypothesis as the set $(x_1, \ldots, x_d)$, i.e. $y_i | r$, $\forall i$ and $\phi = r^{d-2}/y_1 \cdots y_{d-2}$, with the extra condition that $\sum_{i=1}^{d} y_i \geqq \sum_{i=1}^{d} x_i$. Then

$$l(\phi) \leqq (d-2)r - \sum_{i=1}^{d} y_i + 2 \leqq (d-2)r - \sum_{i=1}^{d} x_i + 2 = 2g_0.$$

We assume that $x_1 \geqq \cdots \geqq x_d$ and write $\phi = \dfrac{r}{x_1} \cdot \cdots \cdot \dfrac{r}{x_{d-2}} \cdot \dfrac{1}{x_{d-1} x_d}$. Since ϕ is an integer, any prime p dividing $x_{d-1} x_d$ (say p divides x_d) must divide one of the $\dfrac{r}{x_i}$s $\left(\text{say } p \text{ divides } \dfrac{r}{x_1}\right)$. The new set $\left(px_1, x_2, \ldots, x_{d-1}, \dfrac{x_d}{p}\right)$ satisfies

$$\left(px_1 + x_2 \cdots + x_{d-1} + \frac{x_d}{p}\right) - \left(\sum_{i=1}^{d} x_i\right) = \left(x_1 - \frac{x_d}{p}\right)(p-1) \geqq 0.$$

We continue this process until we have used all the primes dividing $x_{d-1}x_d$.

Remark 4.2. The following facts are easy but fundamental properties of a simple arithmetical graph (G,M,R) We use them for instance to show that the volume $v=\prod_{i=1}^{n} r_i^{d_i-2}$ of G is an integer. However, even when it is known that v is an integer, it is not obvious to factor it into a product of integers (the terminal vertices have $d_i=1$). To overcome this difficulty, we break the graph G with k nodes into k simple arithmetical trees with one node only, as explained below.

A *connecting chain* of G is a set of vertices $C_0,\ldots,C_{q+1}$ such that 1) $c_{i,i+1}=1$ for all $i=0,\ldots,q$, 2) if $q\neq 0$, the degree of C_i in G equals 2 for all $i=1,\ldots,q$, and 3) C_0 and C_{q+1} are nodes of G. We define a *terminal chain* of G in the same way, except that we require that C_{q+1} be a terminal vertex of G. We denote by l_i the number of connecting chains attached at a vertex C_i. Let (C,r) denote a vertex C with multiplicity r.

1. Let (C_i,r_i), $i=1,2,3$ be three consecutive vertices on a chain (i.e. C_2 cannot be a node). Then $\gcd(r_1,r_2)=\gcd(r_2,r_3)$ because r_2 divides r_1+r_3.

2. Let (C_i,r_i), $i=1,2,m$ be three vertices on a terminal chain such that C_1 and C_2 are consecutive and that C_m is the terminal vertex. Then $r_m=\gcd(r_1,r_2)$.

3. Given two integers r,r_1, we can construct a terminal chain $(C,r),(C_1,r_1),\ldots,(C_m,r_m)$ of a simple arithmetical graph, with multiplicities r_i and "self-intersections" c_{ii} defined using Euclid's algorithm:

$$\begin{array}{llll}
(i)\ r\geqq r_1 & r=c_{11}r_1-r_2 & \text{with} & r_2<r_1\\
 & r_1=c_{22}r_2-r_3 & \text{with} & r_3<r_2\\
 & \vdots & \vdots & \\
 & r_{m-1}=c_{mm}r_m & \text{with} & r_m=\gcd(r,r_1)\\
(ii)\ r<r_1 & r=r_1-(r_1-r) & \text{with} & (r_1-r)<r_1\\
 & \text{and case } (i) & \text{with} & r_1,(r_1-r)
\end{array}$$

4. Given integers $r,r_1,\ldots,r_d, s=\gcd(r_1,\ldots,r_d)$ such that r divides $r_1+\cdots+r_d$, we can construct a simple arithmetical graph with exactly one node $(C,r/s)$ and d terminal chains such that the vertices $C_1,\ldots,C_d$ adjacent to the node have multiplicities $r_1/s,\ldots,r_d/s$ and such that the terminal vertex on the chain started by C_i has multiplicity $\gcd(r,r_i)/s$.

5. Given a connecting chain $(C_0,r_0),\ldots,(C_{q+1},r_{q+1})$ of G, we can talk about the "gcd" of this chain:$=\gcd(r_i,r_{i+1})$ for all $0\leqq i\leqq q$. By *break a connecting chain of G at an edge (i, j) and complete*, we mean the following construction: 1) Remove the edge (i,j) linking (C_i,r_i) and (C_j,r_j) 2) Construct as in 3. a terminal chain T_i with the pair (r_i,r_j) and a second terminal chain T_j with the pair (r_j,r_i) 3) Attach T_i at C_i and T_j at C_j to obtain a new arithmetical graph G'. Note that the terminal multiplicity of T_i and T_j are both equal to $\gcd(r_i,r_j)$.

6. Each node of a simple arithmetical graph defines, as in 4, a simple arithmetical tree with one node only.

Definition 4.3. Let (C,r) be a node of a simple arithmetical graph, $(C_1,r_1),\ldots,(C_d,r_d)$

the vertices adjacent to the node and $s(C) = \gcd(r, r_1, \ldots, r_d)$. The order of the node (C, r) is the rational number:

$$\phi(C) = \frac{r^{d-2}}{\gcd(r, r_1) \cdot \cdots \cdot \gcd(r, r_d)}$$

The local rank of the node is the integer:

$$2g(C) = (d-2)(r-1) - \sum_{i=1}^{d} (\gcd(r, r_i) - 1)$$

As noted above, each node of a simple arithmetical graph defines a simple arithmetical tree with exactly one node. We can apply the previous proposition to this new arithmetical tree and get that $s(C)^2 \cdot \phi(C)$ is an integer and $l(s^2 \phi(C)) \leqq 2g_s(C)$, where $g_s(C)$ is defined as:

$$g_s(C) = (d-2)\left(\frac{r}{s} - 1\right) - \sum_{i=1}^{d} \left(\frac{1}{s} \gcd(r, r_i) - 1\right)$$

We also have the relation: $(g(C) - 1) = s \cdot (g_s(C) - 1)$.

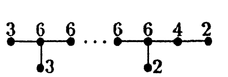
Corollary 4.4. *With the same notations as in the definition:*

$$\textit{When } g(C) \geqq 1, \quad l(\phi(C)) \leqq l(s^2 \phi(C)) \leqq 2g_s(C) \leqq 2g(C)$$

$$\textit{When } g(C) \leqq 0, \quad g_s(C) = 0, \quad g(C) = -(s-1), \quad \phi(C) = \frac{1}{s^2}$$

In particular, if $g(C) \geqq 0$, $l(\phi(C)) \leqq 2g(C)$.

Example 4.5. In the case of the last type of linear rank 2 in Namikawa and Ueno's classification [9], g_s equal 0 for both nodes.



$$\phi = \frac{6}{3 \cdot 3} \cdot \frac{6}{2 \cdot 2}$$

Given a spanning tree $T = [(i_1, j_1), \ldots, (i_{n-\beta}, j_{n-\beta})]$ of G, (notations as in 3.3), we let $f(T) = \prod_{(i,j) \notin T} f_{ij}$, where $f_{ij} = \gcd(r_i, r_j)$ when (C_i, r_i) is linked to (C_j, r_j) by (i, j).

Let $f = \operatorname{lcm}_{T \in G}(f(T))$ and when G is a tree, set $f = 1$.

Proposition 4.6. *The volume v of a simple arithmetical graph (G, M, R) is an integer divisible by f^2. In particular, $l(v) \geqq 0$.*

Proof. When G is a tree, $v(G) = \phi(G)$ is an integer by 3.5 and $f = 1$ by definition. Suppose that G is not a tree and let T be one of its spanning trees. Use 4.2 to break the connecting chains at all edges $(i, j) \notin T$ and to complete T to get a new arithmetical tree T_{ar}. In particular, $v(T_{ar})$ is an integer. T_{ar} has the same set of node as G. To each $(i, j) \notin T$ corresponds two terminal chains in T_{ar}, both with terminal multiplicity equal to f_{ij}. It is very easy to check that $v(G) = v(T_{ar}) \cdot f^2(T_{ar})$.

Theorem 4.7. *Let (G,M,R) be a simple arithmetical graph of volume v. Then*

$$0 \leqq l(v) \leqq 2g_0 - 2\beta.$$

In particular, the linear rank g_0 is a positive integer.

Proof. We deduce simultaneously the inequalities $2g_0 - 2\beta \geqq 0$ and $2g_0 - 2\beta \geqq l(v)$ without using the previous proposition. However, this method does not enable us to prove directly that $l(v) \geqq 0$ or that v is an integer.

For any node (C_i, r_i), we denote by $(C_{i1}, r_{i1}), \ldots, (C_{il_i}, r_{il_i})$ the vertices adjacent to C_i that are on a terminal chain. The others are denoted by $(C_{i,l_i+1}, r_{i,l_i+1}), \ldots, (C_{i,d_i}, r_{id_i})$. Remember that $\gcd(r_i, r_{ij})$ is the multiplicity of the terminal vertex of the j^{th} terminal chain at C_i. By 3.6, we get:

$$\begin{aligned}
2g_0 - 2\beta &= \sum_{q=1}^{n} (r_q - 1)(d_q - 2) \\
&= \sum_{i=1}^{k\,\text{nodes}} \left[(d_i - 2)(r_i - 1) - \sum_{j=1}^{l_i} (\gcd(r_i, r_{ij}) - 1) \right] \\
&= \sum_{i=1}^{k} \left[(d_i - 2)(r_i - 1) - \sum_{j=1}^{d_i} (\gcd(r_i, r_{ij}) - 1) + \sum_{j=l_i+1}^{d_i} (\gcd(r_i, r_{ij}) - 1) \right] \\
&= \sum_{i=1}^{k} \left[2g(C_i) + \sum_{j=l_i+1}^{d_i} (\gcd(r_i, r_{ij}) - 1) \right] = \sum_{i=1}^{k} \mu(C_i)
\end{aligned}$$

where $\mu(C_i) := 2g(C_i) + \sum_{j=l_i+1}^{d_i} (\gcd(r_i, r_{ij}) - 1)$. We defined the volume as:

$$\begin{aligned}
v &= \prod_{q=1}^{n} r_i^{d_i - 2} = \prod_{i=1}^{k\,\text{nodes}} \frac{r_i^{d_i - 2}}{\gcd(r_i, r_{i1}) \cdot \cdots \cdot \gcd(r_i, r_{l_i})} \\
&= \prod_{i=1}^{k} (\phi(C_i) \cdot \gcd(r_i, r_{l_i+1}) \cdot \cdots \cdot \gcd(r_i, r_{d_i}))
\end{aligned}$$

and in particular

$$l(v) = \sum_{i=1}^{k} \left[l(\phi(C_i)) + \sum_{j=l_i+1}^{d_i} l(\gcd(r_i, r_{ij})) \right] = \sum_{i=1}^{k} \nu(C_i)$$

where $\nu(C_i) := l(\phi(C_i)) + \sum_{j=l_i+1}^{d_i} l(\gcd(r_i, r_{ij}))$. We see that in order to complete the proof of the theorem, we need to study the nodes C_i with $\mu(C_i) < 0$ and the nodes C_j with $\nu(C_j) > \mu(C_j)$. We call such nodes "special" nodes.

Let C be such a node. If $\mu(C) < 0$, then $g(C)$ must obviously be strictly smaller than zero. If $\nu(C) > \mu(C)$, then 4.4 implies that $g(C)$ is also strictly negative. Hence in both cases, $g(C) = -(s(C) - 1)$ and $s(C) = s > 1$ by 4.4. We also note that this node C cannot have more than one connecting chain. In fact, suppose that C has two of them, with gcd's sa and sb:

$$\mu(C) = -2(s-1) + (sa - 1) + (sb - 1) + \sum_{j=l_i+3}^{d_i} (\gcd(r_i, r_{ij}) - 1)$$

$$\geqq (a-1)+(b-1)+(\text{positive terms})$$
$$\geqq 0$$

$$v(C) = -2l(s) + l(sa) + l(sb) + \sum_{j=l_i+3}^{d_i} l(\gcd(r_i, r_{ij}))$$
$$= l(a) + l(b) + (\cdots)$$
$$\leqq (a-1)+(b-1)+\sum_{j=l_i+3}^{d_i} (\gcd(r_i, r_{ij})-1)$$
$$\leqq \mu(C).$$

Let r_d denote the multiplicity of the vertex on the connecting chain linked to (C, r). We are going to show that $\gcd(r, r_d) = s(C)$. Suppose that $\gcd(r, r_d) = sa$, $a > 1$. We would have:

$$\mu(C) = -2(s-1) + (sa-1) = (s-1)(a-2) + (a-1) \geqq 1$$
$$v(C) = -2l(s) + l(sa) = -l(s) + l(a) \leqq l(a) \leqq (a-1) \leqq \mu(c).$$

We are now ready to finish the proof of this theorem by induction on the number of nodes of G. If G has one node only, then $s(C) = 1$ and this node cannot be special. Suppose the theorem true for any arithmetical graph having $k-1$ nodes and let G be a graph with k nodes. If G has no special nodes, the theorem is true. If G has a special node C, we are going to construct a new arithmetical graph G' with $k-1$ nodes, such that G and G' have the same invariants: $\beta(G') = \beta(G)$, $v(G') = v(G)$ and $g_0(G') = g_0(G)$.

We proceed as follow: the node C has only one connecting chain T. Let D be the other node on T. We break T at any edge and complete to get two new graphs. Let G' be the graph that contains D. This graph has $k-1$ nodes and it is clear that $\beta(G') = \beta(G)$. The degree in G' of a node of G' equals its degree in G. The number of terminal chains in G' of a node of G' equals its number of terminal chains in G, except for the node D which has one more terminal chain in G' than in G. This new terminal chain has terminal multiplicity equal to $s(C)$.

We remark at this point that G' is an arithmetical graph because the gcd of the multiplicities of its vertices is equal to one. Using the fact that $\phi(C) = s(C)^{-2}$, it is not hard to check that $v(G') = v(G)$. Similarly, using the fact that $g(C) = -(s(C) - 1)$, one checks that $g_0(G') = g_0(G)$.

Corollary 4.8. *Let (G, M, R) be an arithmetical graph with $\beta(G) = 0$. Then $l(\phi) \leqq 2g_0$.*

Proof. Since $\beta = 0$, G is a simple arithmetical tree and $\phi = v$.

Corollary 4.9. *Let (G, M, R) be an arithmetical graph. If $g_0 = 0$, then G is a non minimal simple tree.*

Proof. Since $g_0 = \beta = 0$, G is a simple tree. From the formula $2g_0 - 2 = \sum r_i(c_{ii} - 2) = -2$ we get that at least one of the c_{ii}s equal 1, which, by definition, means that G is not minimal.

Theorem 4.10. *Let (G, M, R) be an arithmetical graph. Its volume v is an integer and $l(v) \leqq 2g_0 - 2\beta$.*

Proof. We use Proposition 4.11 below to reduce to the case of a simple graph and then we apply 4.6 and 4.7.

Proposition 4.11. *Let (G, M, R) be an arithmetical graph. There exists a simple arithmetical graph H, obtained from G by a sequence of elementary blow-ups, such that $\Phi(G) \cong \Phi(H)$, $v(G) = v(H)$, $g_0(G) = g_0(H)$ and $\beta(G) = \beta(H)$.*

Proof. In 1.8, we defined the blow up of G with respect of a vector Q. When Q is of the from $Q_{ij} = E_i + E_j$, where E_k denotes the k^{th} column vector of Id_n, we say that the blow up is *elementary*. Geometrically, it corresponds to blowing up the intersection point of two irreducible components that intersect normally.

Let (C_i, r_i) and (C_j, r_j) be two vertices of G linked by c_{ij} edges. If $c_{ij} > 1$, we blow up G with respect to Q_{ij} to get a new graph G_1 with one more vertex and one more edge than G. G_1 is in fact obtained by "dividing in two" one edge of G linking C_i and C_j: replace this edges by a vertex E, of multiplicity $r_i + r_j$, linked exactly once to C_i and C_j.

It is trivial that $\beta(G) = \beta(G_1)$, $v(G) = v(G_1)$, $g_0(G) = g_0(G_1)$. Moreover, since G_1 is obtained as a blow-up, $\Phi(G) \cong \Phi(G_1)$ by 1.8. It is clear that after finitely many such elementary blow-ups, we will obtain a simple graph (i.e. a graph such that $c_{ij} = 1$ for all $i \neq j$).

Remark 4.12. Let d_i denote the degree of the vertex (C_i, r_i) in the graph G and let $\tilde{d}_i$ be its degree in $\mathrm{sk}(G)$. We could define in a similar way $\tilde{v} = \prod r_i^{\tilde{d}_i - 2}$, $\tilde{\beta} = \beta(\mathrm{sk}(G))$, $2\tilde{g}_0 - 2 = \sum r_i(\tilde{d}_i - 2)$. We note that $\tilde{v}$ and $\tilde{g}_0$ are not always integers, as it can be seen on the following example: $M\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$.

Theorem 4.13. *Let (G, M, R) be an arithmetical graph. If $g_0 = \beta$, the multiplicity of all nodes and all terminal vertices in G equals one. Moreover, if G is minimal, G is reduced if and only if $g_0 = \beta$.*

We sketched a proof of this theorem in [7]. Once again, this is a non trivial result due to the fact that some of the d_is in the formula $2g_0 - 2\beta = \sum (r_i - 1)(d_i - 2)$ might be equal to one. This theorem is also proven by Saito in [10], 2.4.

For each prime p, define g_p by the formula $2g_p - 2 = \sum r_i/p^{\mathrm{ord}_p(r_i)}(d_i - 2)$. It is easy to show that g_p, as well as g_0, is an integer. It might be possible to deduce the statement $2g_p - 2\beta \geqq l(v^{(p)}) \geqq 0$, where $v^{(p)} = vp^{-\mathrm{ord}_p(v)}$, from the method used in this section. Saito has classified the graphs for which the equality $g_0 = g_p$ holds ([10], 2.4). We conjecture that the inequalities $g_0 \geqq g_p \geqq \beta$ always hold. Unlike the statement $g_0 \geqq \beta$, these two inequalities cannot be proven by geometric methods, due to the fact that an analogue to Winters' Existence Theorem [12], for curves degenerating in characteristic $p > 0$, is not known.

5. The integer $\bar{h}$

In this section, we discuss the minimal number $\bar{h}$ of generators of the finite abelian group Φ. We prove that if the arithmetical graph (G, M, J) is reduced, $\bar{h} \leqq \beta(G)$.

Given an intersection matrix M, let $\mathrm{diag}(e_1,\ldots,e_{n-1},0)$ be a diagonal matrix row and column equivalent to M. Then $\Phi \cong \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1}\mathbb{Z}$ by 1.4. It is clear that $\bar{h} \leqq n-1$. As noted in 1.5, the integer $\bar{h}$ can be computed using the Theorem of the Elementary Divisors. In particular, $\bar{h}=n-1$ iff the greatest common divisor of the coefficients of M is not equal to 1; hence if the graph is simple, $\bar{h} \leqq n-2$. Note also that $\bar{h}=0$ iff the group Φ is trivial.

We discuss now a construction needed to prove the main result of this section. Let (G, M, J) be a reduced arithmetical graph and (i, j) an edge of $\mathrm{sk}(G)$. The number of spanning trees κ of G equals $\kappa_1 + c_{ij}\kappa_2$, where κ_1 is the number of its spanning trees not passing through any of the c_{ij} edges linking C_i and C_j and κ_2 is the number of its spanning trees passing through one (fixed) of these edges.

In fact, κ_1 is the number of spanning trees of the graph G_1 obtained by removing from G these c_{ij} edges. When G_1 is not connected, $\kappa_1 = 0$. κ_2 is the number of spanning trees of the graph G_2 obtained from G_1 by identifying the two vertices C_i and C_j. We note that $\beta(G_1) = \beta(G) - c_{ij}$ and $\beta(G_2) = \beta(G_1) + 1$. This construction can be generalized to arithmetical graphs. We shall say that (i, j) is an *r-edge* of (G, M, R) if:

1. Its endpoints (C_i, r_i) and (C_j, r_j) are such that $r_i = r_j = r$.
2. $\mathrm{sk}(G) - \{(i, j)\}$ is connected.

We renumber the vertices of G such that $(i, j) = (1,2)$ and write $c = c_{12}$. Let M_1 be the matrix obtained by adding the matrix $\begin{pmatrix} -c & c \\ c & -c \end{pmatrix}$ to the upperleft corner of M. If $\begin{pmatrix} x & -c \\ -c & y \end{pmatrix}$ is the (2×2) upperleft minor of M, the corresponding one in M_1 is $\begin{pmatrix} x-c & 0 \\ 0 & y-c \end{pmatrix}$. Since $\mathrm{sk}(G) - \{(1,2)\}$ is connected and $M \cdot R = 0$, $(x-c)r$ and $(y-c)r$ are strictly positive integers. Hence (M_1, R) defines an arithmetical graph G_1.

Perform "$\mathrm{Row}_1 + \mathrm{Row}_2$" and "$\mathrm{Col}_1 + \mathrm{Col}_2$" on the matrix M (or on the matrix M_1) to get a new matrix N and let $M_2 = (N)^{11}$. Let R_2 be the projection of R on the last $(n-1)$ coordinates. The reader will check that $M_2 \cdot R_2 = 0$ and that G_2 is connected.

5.1. *Let* $(1,2)$ *be an r-edge of an arithmetical graph* (G, M, R). *The pairs* (M_1, R) *and* (M_2, R_2) *constructed above define two arithmetical graphs* (G_1, M_1, R) *and* (G_2, M_2, R_2). *Moreover,* $\phi(G) = \phi(G_1) + c\phi(G_2) > 1$ *and if* $\gcd(\phi(G_1), \phi(G_2)) = \gcd(r, \phi(G)) = 1$, *then the group* $\Phi(G)$ *is cyclic.*

Proof. By 1.3, we know that $\phi(G) = r^{-2}\det(M^{12})$, $\phi(G_1) = r^{-2}\det(M_1^{12})$ and $\phi(G_2) = r^{-2}\det(M_2^{11}) = r^{-2}\det((M^{11})^{11})$. This shows our claim because $\det(\mathrm{M}^{12}) = \det(M_1^{12}) + c_{12}\det((M^{11})^{11})$. In order to show that Φ is cyclic, it is sufficient to show that $\Delta_{n-2}(M) = 1$. The matrices M, M_2 both contain the minor M_2^{11}. In particular, $\Delta_{n-2}(M)$ divides the determinant of M_2^{11}, which is equal to $r^2\phi(G_2)$. On the other hand, $\Delta_{n-2}(M)$ divides $\Delta_{n-1}(M) = \phi(G)$. Hence, if $\gcd(\phi(G), r^2\phi(G_2)) = 1$, Φ is cyclic.

Proposition 5.2. *Let (G, M, J) be a reduced arithmetical graph with $\beta(G)$ independent cycles. Then its group of components Φ can be minimally at most generated by $\beta(G)$ elements.*

Proof. We know that when G is a tree, Φ is trivial ($\bar{h}=0$). We prove our claim by induction on $\beta(G)$. Let (i, j) be an edge of sk(G) such that $\beta(G_1)<\beta(G)$. (If such an edge does not exist, sk(G) is a tree and the result follows from 2.2.) The claim is true for G_1 by induction hypothesis: $\bar{h}(G_1) \leqq \beta(G_1)$. The claim is true for G if we show that $\bar{h}(G) \leqq \bar{h}(G_1)+1$. This is done in the following lemma.

Lemma 5.3. *Let (G, M, R) be an arithmetical graph. Let (G_1, M_1, R) and (G_2, M_2, R_2) be the arithmetical graphs associated to an r-edge of G. Then*

$$|\bar{h}(G_1)-\bar{h}(G_2)| \leqq 1, \quad |\bar{h}(G)-\bar{h}(G_1)| \leqq 1 \quad \textit{and} \quad |\bar{h}(G)-\bar{h}(G_2)| \leqq 1$$

Proof. It is clear that there should be some relations between $\bar{h}=\bar{h}(G)$, $\bar{h}_1=\bar{h}(G_1)$ and $\bar{h}_2=\bar{h}(G_2)$ because M, M_1, M_2 have many minors in common. Our main tool is the Theorem of the Elementary Divisors. Our claim can be rephrased in the following way, where $h_i=h(G_i)=\operatorname{rank}(M_i)-\bar{h}_i$

$$h_1-1 \leqq h \leqq h_1+1 \quad \text{and} \quad h_2 \leqq h \leqq h_2+2$$

Case 1. M and M_1.

Let $k \geqq 2$. The determinant of every k-minor of M is a linear combination of determinants of k-minors of M_1 and of determinants of $(k-1)$-square matrices which are simultaneously $(k-1)$ minors of M and M_1. No $(k-2)$-minors are needed because $\det(C)=0$. The same is true when M and M_1 are interchanged. We have then:

$$\Delta_{k-1}(M) \neq 1 \Rightarrow \Delta_k(M_1) \neq 1$$
$$\Delta_{k-1}(M_1) \neq 1 \Rightarrow \Delta_k(M) \neq 1$$

and in particular

$$\text{If} \quad h \leqq n-3, \quad \Delta_{h+1}(M) \neq 1 \Rightarrow \Delta_{h+2}(M_1)>1 \Rightarrow h_1 \leqq h+1$$
$$\text{If} \quad h_1 \leqq n-3, \quad \Delta_{h_1+1}(M_1) \neq 1 \Rightarrow \Delta_{h_1+2}(M)>1 \Rightarrow h \leqq h_1+1.$$

Since $h, h_1 \leqq n-1$, the inequality $h_1 \leqq h+1$ also holds for $h=(n-2)$, $(n-1)$ and similarely for the inequality $h \leqq h_1+1$.

Case 2. M and M_2 (the case M_1 and M_2 is similar).

The elementary divisors of M and N are equal. Let $k \geqq 3$. The determinant of every k-minor of N is a linear combination of determinants of $(k-2)$-square matrices that are simultaneously $(k-2)$-minors of N and M_2. On the other hand, every $(k-2)$-minor of M_2 is a $(k-2)$-minor of N. We have then:

$$\Delta_{k-2}(N)>1 \Rightarrow \Delta_k(M_2) \neq 1$$
$$\Delta_{k-2}(M_2)>1 \Rightarrow \Delta_k(N) \neq 1$$

and in particular

$$\text{If} \quad h \leqq n-4, \qquad \Delta_{h+1}(N) \neq 1 \Rightarrow \Delta_{h+3}(M_2) > 1 \Rightarrow h_2 \leqq h+2$$

$$\text{If} \quad h_2 \leqq n-3, \quad \Delta_{h_2+1}(M_2) \neq 1 \Rightarrow \Delta_{h_2+1}(N) > 1 \Rightarrow h \leqq h_2 .$$

Since G has an r-edge, $\phi(G) = \Delta_{n-1}(M) > 1$ and $h \leqq n-2$. This shows that the inequalities hold for all possible values of h and h_2.

Remark 5.4. The graphs $I_\nu^*, \nu \geqq 1$ defined in 2.4 show that the second inequality is sharp. When ν is even, $G = G(I_\nu^*) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $G_2 = G(I_{\nu-1}^*) = \mathbb{Z}/4\mathbb{Z}$.

6. A splitting

Let (G, M, R) be a simple minimal arithmetical graph. We proved already the following facts:

	3.5	$\beta + \gamma = g_0$	$\phi \leqq v\kappa(G)$	with	$l(v) \leqq 2\gamma$
	4.7	If $\beta = 0$	$\phi = v$	with	$l(v) \leqq 2\gamma$
4.13	5.3	If $\gamma = 0$	$\phi = \kappa(G)$	and	$\bar{h}(\Phi) \leqq \beta$.

The cases $\beta = 0$ and $\gamma = 0$ being understood, it is natural to wonder if the general case is a combination of these two special cases. We might ask whether there exists an exact sequence $0 \to \Psi \to \Phi \to C \to 0$ such that

1. The order of Ψ is l-bounded by 2γ.
2. The minimal number of generators $\bar{h}(C)$ of C is bounded by β.

Let $\Phi = \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1}\mathbb{Z}$ with $e_1 | \cdots | e_{n-1}$. If $\beta(G) < n-1$, define $\Upsilon := \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1-\beta}$. It follows from Proposition 1.14 in Artin/Winters [1] that the existence of such an exact sequence is equivalent to having $|\Upsilon|$ l-bounded by 2γ. We show below that Υ satisfies the expected bound for many simple graphs. Let us first generalize our statement to non simple graphs.

Let $\tilde{\Upsilon} := \mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{n-1-\tilde{\beta}}$ if $\tilde{\beta} = \beta(\mathrm{sk}\,(G)) \leqq n-1$. In the remainder of this section, we show that $|\tilde{\Upsilon}|$ is indeed l-bounded by $2g_0 - \beta - \tilde{\beta}$ for a large family of graphs.

Remark 6.1. Artin and Winters have defined an integer $\bar{\beta} \leqq \tilde{\beta}$ (see the proof of 1.16, top of page 378 in [1]) and have shown that $e_1 \cdot \cdots \cdot e_{n-1-\bar{\beta}}$ divides a constant c depending only on the linear rank g_0. Their constant is not effective.

Let us consider now the following class $\mathscr{G}$ of arithmetical graphs (G, M, R):

1. G is not a tree.
2. G contains a spanning tree T such that each of the $\tilde{\beta}$ edges (i, j) in the complement of T in $\mathrm{sk}\,(G)$ is an r-edge for some integer r.

$\mathscr{G}$ is "very large" in the following sense: given a simple arithmetical graph (G, M, R) of linear rank g_0 which is not a tree, we can construct infinitely many distinct arithmetical graphs with same linear rank and belonging to $\mathscr{G}$.

Construction: Take an edge (i, j) in the complement of a spanning tree T and break the connecting chain at (i, j) as in 4.2. We get two new terminal chains with terminal vertices D_1 and D_2 having both multiplicity f_{ij}. For any integer $m \geqq 1$, we can link D_1 and D_2 by a simple chain of m vertices: the coefficient c_{ii} of each vertex is equal to 2 and each vertex has multiplicity f_{ij}. Repeat this construction for each edge in the complement of T.

It is worth noting that given an integer $g_0 \geqq 1$, Artin and Winters have shown in [1] that there are only finitely many arithmetical graphs "modulo chains of vertices with $c_{ii} = 2$" that have linear rank equal to the given g_0.

Theorem 6.2. *Let (G, M, R) be an arithmetical graph belonging to $\mathscr{G}$. Then $|\tilde{\Upsilon}|$ is l-bounded by $2g_0 - \beta - \tilde{\beta}$.*

Proof. Let T be a spanning tree of G such that its complement S in $\mathrm{sk}(G)$ consists of r-edges only, say with multiplicities $r_1, \ldots, r_{\tilde{\beta}}$. We assume for simplicity that these $\tilde{\beta}$ r-edges are $(1,2), (3,4), \ldots, (2\tilde{\beta} - 1, 2\tilde{\beta})$. We also let $C_{i,i+1} = \begin{pmatrix} -c_{i,i+1} & c_{i,i+1} \\ c_{i,i+1} & -c_{i,i+1} \end{pmatrix}$ for $i = 1, \ldots, 2\tilde{\beta} - 1$. We want to compare the matrices $M(G) = M$ and $M(T) = N$ as we did in 5.3. Let C be the matrix "direct sum of $C_{12}, \ldots, C_{2\tilde{\beta}-1, 2\tilde{\beta}}, I_{n-2\tilde{\beta}}$". By construction, $N = M + C$. It is not hard to check that if $k \geqq \tilde{\beta} + 1$, the determinant of any k-minor of N is a linear combination of determinants of square $(k - \tilde{\beta})$-matrices that are simultaneously minors of N and M. This statement is true because $\det(C_{ij}) = 0$ for all $(i, j) \in S$. Hence $\Delta_{k-\tilde{\beta}}(M)$ divide $\Delta_k(N)$. In particular

$$|\tilde{\Upsilon}| = \Delta_{n-1-\tilde{\beta}}(M) \quad \text{divides} \quad \Delta_{n-1}(N) = \phi(T).$$

We computed that $\Phi(T) = \tilde{v}(T) \cdot \left(\prod_{(i,j) \in T, i<j} c_{ij} \right)$ in 2.3, 4.12; By construction, $\tilde{v}(G) = \tilde{v}(T) \cdot (r_1 \cdots r_{\tilde{\beta}})^2$. By definition, $\tilde{v}(G)$ divides $v(G)$ and this last number is an integer. Hence $l(|\tilde{\Upsilon}|) \leqq \prod_{i<j, c_{ij} \neq 0,} (c_{ij} - 1) + l(v(G)) \leqq \beta - \tilde{\beta} + 2g_0 - 2\beta$.

Remark 6.3. We saw in the above proof that for simple graphs in $\mathscr{G}$, $|\Upsilon|$ divides v. We do not know if this is the case for simple graphs not belonging to $\mathscr{G}$. We note that Υ and $2g_0 - 2\beta$ are not invariant under blow-up.

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