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Two-Variable Zeta-Functions on Graphs and Riemann–Roch Theorems

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We investigate, in this article, a generalization of the Riemann–Roch theorem for graphs of Baker and Norine, with a view toward identifying new objects for which a two-variable zeta-function can be defined. To a lattice Λ of rank $n-1$ in $\mathbb{Z}^n$ and perpendicular to a positive integer vector R, we define the notions of g-number and of canonical vector, in analogy with the notions of genus and canonical class in the theory of algebraic curves. When Λ is the full sublattice of $\mathbb{Z}^n$ perpendicular to R, its g-number turns out to be the classical Frobenius number of the coefficients of R. We investigate the existence of canonical vectors—in particular, in the context of arithmetical graphs—where we obtain an existence theorem using methods from arithmetic geometry. We show that a two-variable zeta-function can be defined when a canonical vector exists.

1 Introduction

We investigate, in this article, a generalization of the Riemann–Roch theorem for graphs of Baker and Norine [3], with a view toward identifying new objects for which a zeta-function can be defined.

The theorem of Baker and Norine was motivated by the classical Riemann–Roch theorem for smooth projective curves. The main topological invariant of such a curve is its *genus* $g \geq 0$. The theory of functions on such curves motivates the introduction of the

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notion of a *divisor* D on the curve, having degree $\deg(D)$, and each curve is endowed with a *canonical divisor* K, of degree $2g-2$. To each divisor, D, is associated a vector space of functions having dimension $h(D)$, with $h(D) = \deg(D) + 1 - g$ when $\deg(D) > 2g-2$.

We propose a generalization of the Riemann–Roch theorem in [3] for integer lattices Λ of rank $n-1$ in $\mathbb{Z}^n$ and perpendicular to a positive vector R. When G is a connected graph on n vertices with $(n \times n)$-Laplacian matrix M, our results apply to the lattice Λ_G consisting of the image of M in $\mathbb{Z}^n$. This lattice is perpendicular to (the transpose of) the vector $(1, \ldots, 1)$ (see Example 2.2).

In analogy with the theory of algebraic curves, we introduce in the next section the *g-number* $g(\Lambda)$ of Λ (Definition 2.1). When Λ_R is the full sublattice of $\mathbb{Z}^n$ perpendicular to R, the integer $g(\Lambda_R)$ is equal to the classical *Frobenius number* of the coordinates of R (Lemma 2.4). We define a notion of *canonical divisor* K *of a lattice* Λ in Definition 2.6.

To state our proposed generalization of the Riemann–Roch theorem, we introduce in Section 3 the notion of *Riemann–Roch structure* on a lattice in Definition 3.1, with its associated Riemann–Roch function h. A Riemann–Roch function is the weakest type of function for which a nontrivial, meaningful Riemann–Roch theorem could possibly be expected to hold. With this definition, a lattice has a Riemann–Roch structure if and only if a Riemann–Roch theorem holds for this lattice and for the associated function h. If the lattice has a canonical divisor, then it carries a Riemann–Roch structure (Proposition 3.4), but a lattice could carry several distinct Riemann–Roch structures.

To a lattice endowed with a Riemann–Roch structure, we associate a *two-variable zeta-function* in Definition 3.6, and the main properties of this function, such as its functional equation, are proved in Proposition 3.10. This definition is motivated by the theory of zeta-functions for curves over finite fields. A two-variable zeta-function for curves along these ideas was introduced by Pellikaan [26].

In the case of a finite connected graph, such a zeta-function can always be defined, owing to the theorem of Baker and Norine. A partial evaluation of this associated two-variable Riemann–Roch zeta-function is shown, using results of Biggs and Merino, to equal a partial evaluation of the Tutte polynomial of the graph (Proposition 3.12). It would be interesting to understand better what are the graph-theoretical properties shared by two graphs having the same zeta-function (Proposition 3.14).

In Section 4, we discuss the case of arithmetical graphs. An arithmetical graph (G, M, R) consists of a usual connected graph G on n vertices, plus some extra structure, including a natural lattice Λ_M of rank $n-1$ in $\mathbb{Z}^n$. Arithmetical graphs arise naturally in arithmetic geometry in the context of degenerations of curves [18].

The main integer associated with an arithmetical graph is denoted by $g_0(G, M, R)$ in [17], and is given by an explicit formula in terms of the data (G, M, R) (see Definition 4.1). Arithmetical graphs, with $g_0(G, M, R) = 1$, 2, and 3, have been completely enumerated [25, 31]. Given an arithmetical graph (G, M, R), we show in Theorem 4.2 that the g-number of the lattice Λ_M is at most equal to $g_0(G, M, R)$, and that when we have equality, $g(\Lambda_M) = g_0(G, M, R)$, then Λ_M has an explicit canonical divisor, and its zeta-function can be defined. Our proof uses techniques from arithmetic geometry. It would be of interest to understand better when the equality $g(\Lambda_M) = g_0(G, M, R)$ occurs. The problem of classifying the arithmetical graphs with small g-numbers seems to be quite difficult (see [13] for some partial results).

In Section 5, we produce some classes of lattices Λ beyond those associated with the Laplacian of a graph and for which we can compute explicitly the g-number $g(\Lambda)$ and prove that Λ has a canonical divisor (Proposition 5.3). In view of the fact that $g(\Lambda)$ is a generalization of the classical Frobenius number, it is not surprising that closed formulas for $g(\Lambda)$ are not expected to exist in general. This explains our interest in Theorem 4.2, where an explicit formula for $g(\Lambda)$ is shown to exist in some cases. In general though, given a lattice of moderate size explicitly, experiments seem to indicate that it is computationally quite expensive to determine its g-number.

We also show in Section 5 that a lattice $\Lambda \subset \mathbb{Z}^n$ perpendicular to a positive vector R has a canonical vector if and only if an associated lattice perpendicular to the vector J_n, the transpose of $(1, \ldots, 1)$, has a canonical vector. Lattices perpendicular to J_n are studied in great detail in a paper by Amini and Manjunath [1], where such lattices are called sublattices of the root lattice A_{n-1}. We thank an anonymous referee for providing us with the reference [1]. In [1], the authors discuss the existence of a Riemann–Roch theorem for lattices perpendicular to J_n and for a specific Riemann–Roch function h. Two integers, $g_{\min}(\Lambda) \leq g_{\max}(\Lambda)$, are introduced in [1] for such lattices, and it turns out that $g_{\max}(\Lambda) = g(\Lambda)$ (Proposition 5.7). It is shown in [1, 5.5], that if $g_{\min}(\Lambda) = g_{\max}(\Lambda)$ and the lattice is reflection invariant, then it satisfies a Riemann–Roch theorem.

2 The g-Number of a Lattice

We introduce in this section the basic terminology used in this article, and define the *g-number* and *canonical vector* of certain lattices Λ of rank $n-1$ in $\mathbb{Z}^n$. The g-number can be seen as a generalization to lattices of the Frobenius number of an n-tuple of positive integers $(r_1, \ldots, r_n)$.

Let $R \in \mathbb{Z}^n$ be a vector with strictly positive integers entries. We denote its transpose by ${}^{t}R = (r_1, \ldots, r_n)$. In this article, unless specified otherwise, any integer vector denoted R is assumed to have $\gcd(r_1, \ldots, r_n) = 1$.

Let $D \in \mathbb{Z}^n$. We define the *degree of* D as $\deg_R(D) := D \cdot R$. When the context makes the reference to R unnecessary, we will denote $\deg_R$ simply by deg. The kernel of the degree homomorphism $\mathbb{Z}^n \to \mathbb{Z}$ is the lattice in $\mathbb{Z}^n$ perpendicular to R:

$$\Lambda_R := \{D \in \mathbb{Z}^n, D \cdot R = 0\}.$$

For any sublattice $\Lambda \subseteq \Lambda_R$ of rank $n-1$, we define $\mathrm{Pic}(\Lambda) := \mathbb{Z}^n/\Lambda$. If $D \in \mathbb{Z}^n$, then we denote by $[D]$ the class of D in $\mathrm{Pic}(\Lambda)$. By construction, $\deg(\Lambda) = \{0\}$, so that we have a group homomorphism

$$\deg : \mathrm{Pic}(\Lambda) \to \mathbb{Z}, \deg([D]) := D \cdot R.$$

This homomorphism is *surjective* since we assume that $\gcd(r_1, \ldots, r_n) = 1$. Its kernel is the finite abelian group

$$\mathrm{Pic}^0(\Lambda) := \Lambda_R/\Lambda.$$

If $D \in \mathbb{Z}^n$, then we will write $D \geq 0$ if all coefficients of D are nonnegative, and we write $D > 0$ if all coefficients of D are strictly positive. Note that if $D \geq 0$, then $\deg(D) \geq 0$. We will say that $D \geq 0$ is *effective*. An element $D \in \mathbb{Z}^n$ may be called a *divisor*, and $[D] \in \mathrm{Pic}(\Lambda)$ a *divisor class*, in keeping with the notation used in the Riemann–Roch theorem for curves.

Definition 2.1. Let Λ be as above. The *g-number of* Λ, denoted $g(\Lambda)$, or simply g, is the smallest nonnegative integer γ such that, for any vector $D \in \mathbb{Z}^n$ such that $\deg(D) \geq \gamma$, there exists $E \geq 0$ with $D - E \in \Lambda$. □

We show in Proposition 2.5 that the integer $g(\Lambda)$ exists. The above definitions are motivated by the following key example.

Example 2.2. Let G be a finite unweighted connected multigraph on n vertices and m edges, without loop edges. Choose an ordering $v_1, \ldots, v_n$ for the vertices of G. Let d_i denote the valency of v_i. Let A denote the associated adjacency matrix. Set $\mathcal{D} := \mathrm{diag}(d_1, \ldots, d_n)$, the diagonal matrix of the valencies. Let $M := \mathcal{D} - A$, the Laplacian matrix of G. By definition, $(1, \ldots, 1)M = 0$. It follows that $\Lambda_G := \mathrm{Im}(\mathrm{M}) \subset \mathbb{Z}^n$ is a lattice of rank $n-1$.

The order of the group $\mathrm{Pic}^0(\Lambda_G)$ is well known to be the number $\kappa(G)$ of spanning trees of G [6, 6.3]. The group $\mathrm{Pic}^0(\Lambda_G)$ occurs in the literature under different names, depending on the context in which it is used: *group of components* [19] (1989), *sandpile group* [11] (1990), *jacobian group* [2] (1997), or *critical group* [7] (1999). See [22] for the relationships between this group and the eigenvalues of the Laplacian.

Recall that the integer $\beta(G) := m - n + 1$ is the first Betti number of the graph. The work of Baker and Norine [3] completely determines the integer $g(\Lambda_G)$. □

Proposition 2.3. Let G be a graph as above. Then $g(\Lambda_G) = m - n + 1$. □

Proof. For any $D \in \mathbb{Z}^n$, Baker and Norine introduce an integer $r(D)$ with the following property [3, 2.1]: $r(D) \geq -1$ and $r(D) > -1$, if and only if D is equivalent to an effective. Theorem 1.12 in [3] states that for all $D \in \mathbb{Z}^n$, $r(D) - r(K - D) = \deg(D) + 1 - \beta(G)$. Then $\deg(D) + 1 - \beta(G) = r(D) - r(K - D) \leq r(D) + 1$. Assume that $\deg(D) \geq \beta(G)$. Then $\deg(D) - \beta(G) \geq 0$, so that $r(D) \geq 0$. It follows that $g(\Lambda_G) \leq \beta(G)$. A complete description of the set of $D \in \mathbb{Z}^n$ of degree $\beta(G) - 1$ that are not equivalent to an effective divisor is given in [3, 3.4], showing in particular that this set is not empty. Hence, $g(\Lambda_G) = \beta(G)$. ■

Fix positive integers $r_1, \ldots, r_n$. We define $g(r_1, \ldots, r_n)$ to be one more than the largest integer that does not belong to the additive semigroup of $\mathbb{Z}$ generated by $r_1, \ldots, r_n$. In other words, every integer $N \geq g(r_1, \ldots, r_n)$ can be written in the form $\sum_{i=1}^n x_i r_i$ with $x_i \geq 0$ for all $i = 1, \ldots, n$, and $g(r_1, \ldots, r_n) - 1$ cannot be written in this form.

In this article, we call $g(r_1, \ldots, r_n)$ *the Frobenius number* of $r_1, \ldots, r_n$, even though it is the integer $g(r_1, \ldots, r_n) - 1$ which is classically called the Frobenius number of $r_1, \ldots, r_n$ in the literature. We chose this rescaling so that we have the property $g(r_1, \ldots, r_n) \geq 0$. In particular, if $r_i = 1$ for some i, $g(r_1, \ldots, r_n) = 0$.

Our next lemma shows that $g(\Lambda_R) = g(r_1, \ldots, r_n)$. We can thus interpret the integer $g(\Lambda)$ as a generalization of the Frobenius number to lattices.

Lemma 2.4. Let $R > 0$ be as above, and $\Lambda' \subseteq \Lambda \subseteq \Lambda_R$ be lattices of rank $n-1$.

(a) $g(\Lambda_R) = g(r_1, \ldots, r_n) \geq 0$, and $g(\Lambda_R) \neq 1$. In particular, if $|\mathrm{Pic}^0(\Lambda)| = 1$, then $\Lambda = \Lambda_R$, and $g(\Lambda) = g(r_1, \ldots, r_n)$.

(b) If $\Lambda' \subseteq \Lambda \subseteq \Lambda_R$, then we have a natural surjective homomorphism $\mathrm{Pic}(\Lambda') \to \mathrm{Pic}(\Lambda)$, and $g(\Lambda') \geq g(\Lambda) \geq g(\Lambda_R)$.

(c) If $g(\Lambda) = 0$, then $\Lambda = \Lambda_R$. □

Proof. (a) By construction, $\mathrm{Pic}^0(\Lambda_R) = (1)$, and $\deg : \mathrm{Pic}(\Lambda_R) \to \mathbb{Z}$ is an isomorphism. Thus, if there exists $E \geq 0$ of degree d, every $D \in \mathbb{Z}^n$ of degree d is such that $[D] = [E]$ in $\mathrm{Pic}(\Lambda_R)$. And if there is no $E \geq 0$ of degree d, then no element $D \in \mathbb{Z}^n$ of degree d is equivalent to an effective. This shows that $g(\Lambda_R) = g(r_1, \ldots, r_n)$. By definition, $g(r_1, \ldots, r_n) \neq 1$.

(b) Follows from the definitions.

(c) If $g(\Lambda) = 0$, we find that every D of degree 0 is equivalent to an effective. But there is only one effective $E \geq 0$ with $\deg(E) = 0$, the zero vector. Hence, $\mathrm{Pic}^0(\Lambda) = (1)$, and $\Lambda = \Lambda_R$. ■

Our next proposition implies that the integer $g(\Lambda)$ exists. Given any positive integer x, we let $x\Lambda := \{xD,\, D \in \Lambda\}$. Denote by $e = e(\Lambda)$, the exponent of the group $\mathrm{Pic}^0(\Lambda)$. In particular, when $\Lambda \subseteq \Lambda_R$, $e\Lambda_R \subseteq \Lambda$.

Proposition 2.5. Let $R > 0$ be as above, and $\Lambda \subseteq \Lambda_R$ be a lattice of rank $n-1$ and exponent e. Then

$$g(\Lambda) \leq eg(\Lambda_R) + (e-1)\left(-1 + \sum_{i=1}^{n} r_i\right),$$

and $e(\Lambda) \geq \frac{g(\Lambda)-1+\sum_{i=1}^{n} r_i}{g(\Lambda_R)-1+\sum_{i=1}^{n} r_i}$. □

Proof. The second inequality is immediate from the first. To prove the first, we note that, since $e\Lambda_R \subseteq \Lambda$, $g(\Lambda) \leq g(e\Lambda_R)$. We claim that $g(e\Lambda_R) \leq eg(\Lambda_R) + (e-1)(-1 + \sum_{i=1}^{n} r_i)$. Indeed, let $D \in \mathbb{Z}^n$. Write $D = eD' + {}^{t}(y_1, \ldots, y_n)$ with $0 \leq y_i \leq e-1$. Suppose that $\deg(D) > e(g(\Lambda) - 1) + (e-1)(\sum_{i=1}^{n} r_i)$. Then $\deg(eD') > e(g(\Lambda) - 1)$. It follows that $\deg(D') \geq g(\Lambda)$. Hence, there exists $E' \geq 0$ and $V' \in \Lambda$ such that $D' = E' + V'$. Thus, $D = eE' + {}^{t}(y_1, \ldots, y_n) + eV'$, and D is $e\Lambda$-equivalent to an effective. Hence, $g(e\Lambda) \leq 1 + e(g(\Lambda) - 1) + (e-1)(\sum_{i=1}^{n} r_i)$. ■

The following rescaling makes for a prettier formula. Let $f(\Lambda) := g(\Lambda) - 1 + \sum_{i=1}^{n} r_i$. Then $f(\Lambda_R)$ is the largest integer not representable as a linear combination of $r_1, \ldots, r_n$ in *positive* integers. Proposition 2.5 implies that $f(\Lambda) \leq ef(\Lambda_R)$.

Let G be a graph as in Example 2.2. Let e denote the exponent of the group $\mathrm{Pic}^0(\Lambda_G)$. Then Example 2.5 shows that $e \geq \frac{m}{n-1}$. This bound is achieved when G is a graph on two vertices linked by m edges. Note that when G has vertex connectivity at least 2, $\kappa(G) \geq m$ [22, 4.3].

Definition 2.6. Let $\Lambda \subseteq \Lambda_R$ be a lattice of rank $n-1$. A *canonical vector* $K \in \mathbb{Z}^n$ for Λ is a vector of degree $\deg(K) = 2g(\Lambda) - 2$ such that, for all $D \in \mathbb{Z}^n$ of degree $g(\Lambda) - 1$, either both D and $K - D$ are equivalent to an effective, or neither D nor $K - D$ is equivalent to an effective.

Let $D \in \mathbb{Z}^n$. Set $\epsilon_\Lambda(D)$ to be 1, if there exists $E \geq 0$ such that $[D] = [E]$ in $\mathrm{Pic}(\Lambda)$, and set $\epsilon_\Lambda(D)$ to be 0 if there does not exist any $E \geq 0$ such that $[D] = [E]$ in $\mathrm{Pic}(\Lambda)$. Then K is a canonical divisor if, for all $D \in \mathbb{Z}^n$ of degree $g(\Lambda) - 1$, $\epsilon_\Lambda(D) = \epsilon_\Lambda(K - D)$. □

The Riemann–Roch theorem of Baker and Norine [3] shows that a canonical vector exists for the lattice Λ_G associated to any graph G.

Proposition 2.7. Let G be a graph as in Example 2.2, and let $\Lambda_G \subseteq \mathbb{Z}^n$ be its associated Laplacian lattice. Let $K := (d_1 - 2, \ldots, d_n - 2)$. Then $\deg(K) = 2g(\Lambda_G) - 2$ and K is a canonical vector for Λ_G. □

Proof. The condition $\epsilon_\Lambda(D) = \epsilon_\Lambda(K - D)$ for all D of degree $g(\Lambda) - 1$ is equivalent to condition RR2 in [3, 2.2], and is proved to hold in the proof of 1.12 of [3]. ■

The motivation for introducing the notions of g-number and canonical vector is found in Definition 3.1 and Proposition 3.4. The existence of a canonical vector for Λ allows for the existence of a Riemann–Roch structure on Λ, to which one associates a two-variable zeta-function. These topics are discussed in the next section.

Example 2.8. Let $R = {}^{t}(r_1, \ldots, r_n) > 0$ with $\gcd(r_1, \ldots, r_n) = 1$. The example below exhibits many lattices $\Lambda \subset \Lambda_R$, with $\Lambda \neq x\Lambda_R$ for all $x > 0$, and which have a canonical vector (lattices of the form $x\Lambda$ are considered in Corollary 5.5). Consider the following vectors in Λ_R:

$$
\begin{aligned}
w_1 &:= {}^{t}(r_n, 0, 0, \ldots, 0, -r_1), \\
w_2 &:= {}^{t}(0, r_n, 0, \ldots, 0, -r_2), \\
&\ldots \\
w_{n-1} &:= {}^{t}(0, \ldots, 0, r_n, -r_{n-1}).
\end{aligned}
$$

Note that when $r_n = 1$, this set of vectors generates Λ_R. Choose positive integers $y_1, \ldots, y_{n-1}$. Let $\Lambda \subset \Lambda_R$ denote the sublattice of Λ_R generated by $y_1 w_1, \ldots, y_{n-1} w_{n-1}$. We claim that

$$g(\Lambda) - 1 = \sum_{i=1}^{n-1} (r_n y_i - 1) r_i - r_n,$$

and $\mathrm{Pic}(\Lambda)$ contains a single class of degree $g(\Lambda) - 1$ which is not equivalent to an effective, the class of

$$D_0 := (r_n y_1 - 1, \ldots, r_n y_{n-1} - 1, -1).$$

Thus, $2D_0$ is a canonical vector for Λ. Moreover,

$$|\mathrm{Pic}^0(\Lambda)| = r_n^{n-2} y_1 \cdot \ldots \cdot y_{n-1}. \qquad \square$$

Proof. Given any vector $D \in \mathbb{Z}^n$, we can write it as $D = \sum_{i=1}^{n-1} c_i (y_i w_i) + {}^{\mathrm{t}}(z_1, \ldots, z_n)$ with $0 \le z_i \le r_n y_i - 1$ for all $i = 1, \ldots, n-1$. In other words, the class $[D]$ in $\mathrm{Pic}(\Lambda)$ contains a vector of the form ${}^{\mathrm{t}}(z_1, \ldots, z_n)$ with $0 \le z_i \le r_n y_i - 1$ for all $i = 1, \ldots, n-1$. Suppose now that $\deg(D) = \sum_{i=1}^{n} z_i r_i > \sum_{i=1}^{n-1} (r_n y_i - 1) r_i - r_n$. Then, z_n being an integer, we find that $z_n \ge 0$, and D is equivalent to an effective. When $\deg(D) = \sum_{i=1}^{n} z_i r_i = \sum_{i=1}^{n-1} (r_n y_i - 1) r_i - r_n$, we find that $z_n \ge 0$ unless $z_i = r_n y_i - 1$ for all $i = 1, \ldots, n-1$, and $z_n = -1$, that is, unless $[D] = [D_0]$. It is easy to show that D_0 is not equivalent to an effective.

The order of $\mathrm{Pic}^0(\Lambda)$ can be computed as the greatest common divisor of the determinants of the $(n-1) \times (n-1)$-minors of the matrix whose columns are the vectors $y_1 w_1, \ldots, y_{n-1} w_{n-1}$. ■

Let $J_n := {}^{\mathrm{t}}(1, \ldots, 1) \in \mathbb{Z}^n$. Let $\Lambda \subset \Lambda_{J_n}$ be a sublattice. If Λ is of the form Λ_G for some graph G with Laplacian M, then $\Lambda = \mathrm{Im}(M)$ for some symmetric positive-semidefinite matrix.

Lemma 2.9. Let $R \in \mathbb{Z}^n$. Any lattice Λ perpendicular to R is of the form $\Lambda = \mathrm{Im}(N)$ for some *symmetric positive-semidefinite* matrix $N \in M_n(\mathbb{Z})$. $\square$

Proof. Note first that any lattice Λ is of the form $\Lambda = \mathrm{Im}(L)$ for some matrix $L \in M_n(\mathbb{Z})$. Simply take n vectors in $\mathbb{Z}^n$ which generate Λ, and let L denote the matrix whose columns are these vectors. To prove the lemma, it suffices to show that given any matrix, $L \in M_n(\mathbb{Z})$, there exists $W \in \mathrm{GL}_n(\mathbb{Z})$ such that LW is symmetric positive semidefinite. Then $\Lambda = \mathrm{Im}(L) = \mathrm{Im}(LW)$. Given L, there exist $P, Q \in \mathrm{GL}_n(\mathbb{Z})$ such that $PLQ = D$ is diagonal with nonnegative entries. Thus, in particular, D is symmetric positive-semidefinite. Then $L \cdot Q(P^{-1})^{\mathrm{t}} = P^{-1} D (P^{-1})^{\mathrm{t}}$ is symmetric positive-semidefinite. ■

Example 2.10. Examples of lattices $\Lambda \subset \Lambda_{J_n}$ which are not the lattices Λ_G associated with any graph G, but which are nevertheless endowed with a canonical vector, are given in [1, Section 6.4].

Hower communicated to us an example of two distinct lattices $\Lambda' \subsetneq \Lambda$ in $\mathbb{Z}^6$ with $g(\Lambda') = g(\Lambda)$. He also found examples of lattices coming from arithmetical graphs which do not have a canonical vector, and examples of other lattices Λ with two canonical vectors K and K' such that $[K] \neq [K']$ in $\mathrm{Pic}(\Lambda)$ [13, 11.1, 11.2]. □

3 Riemann–Roch Structures

Can the Riemann–Roch theorem in [3] be extended to apply to structures other than graphs? We propose in this section the following extension. Fix $R > 0$. Let $\Lambda \subseteq \Lambda_R$ be a sublattice of rank $n-1$ as in Section 2, with g-number $g = g(\Lambda)$ and degree function $\deg_R$.

Definition 3.1. A *Riemann–Roch structure* on Λ is a function $h : \mathbb{Z}^n \to \mathrm{Pic}(\Lambda) \to \mathbb{Z}_{\geq 0}$ satisfying the following properties (a)–(c):

(a) There exists a vector $K \in \mathbb{Z}^n$ such that for all $D \in \mathbb{Z}^n$,

$$h(D) - h(K - D) = \deg(D) + 1 - g.$$

(b) If $\deg(D) \leq 0$, then $h([D]) = 0$, unless $[D] = [0]$, in which case $h([0]) = 1$.

(c) $h(D) \geq 1$ if and only if there exists $E \geq 0$ with $[D] = [E]$ in $\mathrm{Pic}(\Lambda)$. In other words, using the notation introduced in Definition 2.6, $h(D) \geq 1$ if and only if $\epsilon_\Lambda(D) = 1$, and $h(D) = 0$ if and only if $\epsilon_\Lambda(D) = 0$. □

Lemma 3.2. Let h be a Riemann–Roch structure on Λ. Then

(i) We have $\deg(K) = 2g - 2$, $h(K) = g$, and the class of K is uniquely determined by h.

(ii) If $\deg(D) > 2g - 2$, then $h(D) = \deg(D) + 1 - g$. Moreover, if $\deg(D) = 2g - 2$ and $[D] \neq [K]$, then $h(D) = g - 1$. □

Proof. (i) and (ii): Plugging $D = 0$ and $D = K$ in the formula in (a) gives $h(0) - h(K) = 1 - g$ and $h(K) - h(0) = \deg(K) + 1 - g$. It follows that $\deg(K) = 2g - 2$.

Using (b), we find that $h(K) = g$, and if $\deg(D) > 2g - 2$, then $h(D) = \deg(D) + 1 - g$. Moreover, if $\deg(D) = 2g - 2$ and $[D] \neq [K]$, then $h(D) = g - 1$. It also follows that the class of K is uniquely determined by h. Indeed, suppose that K and K' both satisfy the condition in the Riemann–Roch formula. Then $h(K') - h(K - K') = \deg(K') + 1 - g = g - 1$. Since $h(K') = g$, we find that $h(K - K') = 1$, so that $[K - K'] = [0]$, as desired. ■

Lemma 3.3. Let h be a Riemann–Roch structure on Λ. Consider the properties (1) and (2) below, which are satisfied by the classical Riemann–Roch function in the context of curves:

(1) Let $E \geq 0$. For all D, $h(D) \leq h(D + E)$.
(2) For all $D, D' \geq 0$, $h(D) + h(D') \leq h(D + D') + 1$.

Then:

(i) If h satisfies (2), then h satisfies (1). Indeed, if $h(D) = 0$, then $h(D + E) \geq h(D) = 0$ is always true. Assume now that $h(D) \geq 1$. If $E \geq 0$, then $h(E) \geq 1$. Using (2), we find that $h(D + E) \geq h(D) + h(E) - 1 \geq h(D)$.

(ii) Assume that the Riemann–Roch structure h satisfies (1). Then

$$\text{If } 0 \leq \deg(D) \leq 2g - 2, \text{ then } h(D) \leq \min(g, \deg(D) + 1).$$

Indeed, this bound trivially holds if $h(D) = 0$. When $h(D) \geq 1$, we find that D is equivalent to an effective. Hence, applying (1), $h(K - D) \leq h(K) = g$. It follows from (a) that $h(D) = \deg(D) + 1 + (h(K - D) - g)$, so that $h(D) \leq \deg(D) + 1$.

If $h(K - D) = 0$, then we find from (a) that $h(D) = \deg(D) + 1 - g \leq g - 1$. If $h(K - D) > 0$, then $K - D$ is equivalent to an effective and (1) implies $h(D) \leq h(K) = g$.

(iii) Assume that the Riemann–Roch structure h satisfies (2). Then it satisfies the analogue of *Clifford's Theorem*:

$$\text{If } 0 \leq \deg(D) \leq 2g - 2, \text{ then } h(D) \leq \tfrac{1}{2}\deg(D) + 1.$$

Assume that $D \geq 0$ and $K - D \geq 0$. By (2), $h(D) + h(K - D) \leq h(K) + 1 = g + 1$, and by (a) $h(D) - h(K - D) = \deg(D) + 1 - g$. Adding these two relations gives $2h(D) \leq 2 + \deg(D)$, as desired.

If D is not equivalent to an effective, $h(D) = 0$ and the inequality is obvious. If $K - D$ is not equivalent to an effective, $h(K - D) = 0$ and the

Riemann–Roch formula implies that $h(D) = \deg(D) + 1 - g = \deg(D)/2 - (g - 1 - \deg(D)/2) \leq \deg(D)/2$. $\square$

Proposition 3.4. Let $\Lambda \subseteq \Lambda_R$ be a lattice of rank $n-1$ with g-number g and $|\mathrm{Pic}^0(\Lambda)| > 1$. Then Λ has a Riemann–Roch structure if and only if there exists a canonical divisor K for Λ. $\square$

Proof. Suppose that Λ has a Riemann–Roch structure h with associated divisor K. We claim that K is a canonical divisor for Λ. Indeed, for all D of degree $g-1$, the Riemann–Roch formula gives $h(D) = h(K - D)$. The equality $\epsilon_\Lambda(D) = \epsilon_\Lambda(K - D)$ follows from the fact that $h(D) = 0$ if and only if $\epsilon_\Lambda(D) = 0$

Suppose the existence of a canonical divisor K for Λ. Let us define a Riemann–Roch function h_ϵ as follows. If $\deg(D) \leq 0$, then set $h_\epsilon([D]) := \epsilon([D]) = 0$, unless $[D] = [0]$, in which case set $h_\epsilon([D]) := \epsilon(D) = 1$ (property (b) is satisfied). If $0 < \deg(D) \leq g - 1$, then set $h_\epsilon([D]) := \epsilon([D])$ and $h_\epsilon(K - D) := g - \deg(D) - 1 + \epsilon(D)$. When $\deg(D) = g - 1$, this is well defined, since $h_\epsilon(K - D) = \epsilon(D) = \epsilon(K - D)$ by hypothesis. The Riemann–Roch formula trivially holds for $\deg(D) \leq g - 1$.

When $g - 1 \leq \deg(D) < 2g - 2$, the above rule determines $h_\epsilon(D)$ since it determines $h_\epsilon(K - D)$ already, because $0 < \deg(K - D) \leq g - 1$. In fact, $h_\epsilon(K - D) = \epsilon_\Lambda(K - D)$, and $h_\epsilon(D) = h_\epsilon(K - (K - D)) := g - \deg(K - D) - 1 + \epsilon(K - D)$. It follows that $h_\epsilon(D) = \deg(D) + 1 - g + h_\epsilon(K - D)$, and the Riemann–Roch formula holds.

When $\deg(D) = 2g - 2$, set $h_\epsilon(D) := g - 1$ if $D \neq K$, and $h_\epsilon(K) := g$. If $\deg(D) > 2g - 2$, set $h_\epsilon(D) := \deg(D) + 1 - g$. We leave it to the reader to check that h_ϵ and the divisor K define a Riemann–Roch structure on Λ. $\blacksquare$

Lemma 3.5. Let $\Lambda \subseteq \Lambda_R$ be a lattice of rank $n-1$ with g-number g and canonical vector K. The associated Riemann–Roch function h_ϵ satisfies condition (2) in Lemma 3.3 and, thus, satisfies the analog of Clifford's Theorem (Lemma 3.3(iii)). $\square$

Proof. Let $D \geq 0$ and $D' \geq 0$. Then $D + D' \geq 0$. If $\deg(D) \leq g - 1$ and $\deg(D') \leq g - 1$, then $h_\epsilon(D) = h_\epsilon(D') = 1$ and $h_\epsilon(D + D') \geq 1$. Hence, $h_\epsilon(D) + h_\epsilon(D') \leq h_\epsilon(D + D') + 1$, as desired.

If $\deg(D) \geq g$ and $\deg(D') \leq g - 1$, then $h_\epsilon(D') = 1$, and $h_\epsilon(D) = h_\epsilon(K - D) + \deg(D) + 1 - g$ with $h(K - D) \leq 1$. The inequality holds if $h(K - D) \leq h(K - D - D') + \deg(D')$. If $\deg(D') \geq 1$, then this latter inequality clearly holds. If $\deg(D') = 0$, $D' = 0$, and the inequality (3.3) is obvious.

If $\deg(D), \deg(D') \geq g$, (2) is satisfied if $h_\epsilon(K-D) + h_\epsilon(K-D') \leq h_\epsilon(K-D-D') + g$. Recall that our assumption on the degrees imply that $h(K-D), h(K-D') \leq 1$. This inequality is thus trivially satisfied if $g \geq 2$. When $g \geq 1$, $K = 0$. We leave the details of the cases $g = 1$ and $g = 0$ to the reader. ■

Definition 3.6. Let $\Lambda \subseteq \Lambda_R$ be a lattice of rank $n-1$ with g-number g and a Riemann–Roch structure h. The *zeta-function* of h is defined as follows:

$$Z_h(\Lambda, t, u) := \sum_{[D] \in \mathrm{Pic}(\Lambda)} \frac{u^{h(D)} - 1}{u - 1} t^{\deg(D)},$$

where we set $u^0 := 1$.

Given a Riemann–Roch structure h, we may obtain a possibly different Riemann–Roch structure by considering h_ϵ defined in Proposition 3.4, keeping the same canonical divisor for both h and h_ϵ. We can thus also consider the associated zeta-function

$$Z_{h_\epsilon}(\Lambda, t, u) := \sum_{[D] \in \mathrm{Pic}(\Lambda)} \frac{u^{h_\epsilon(D)} - 1}{u - 1} t^{\deg(D)}.$$ □

Lemma 3.7. $Z_{h_\epsilon}(\Lambda, t, 0) = Z_h(\Lambda, t, 0)$. □

Proof. First $h_\epsilon(D) = h(D)$ if $\deg(D) \leq 0$ or $\deg(D) \geq 2g-2$, since both h and h_ϵ are Riemann–Roch functions with the same canonical divisor. Assume that $1 \leq \deg(D) \leq g-1$. Then $h_\epsilon(D) = \epsilon(D)$ by definition. It follows that $h(D) = 0$ if and only if $h_\epsilon(D) = 0$, since the condition $h(D) = 0$ is equivalent to $\epsilon(D) = 0$. Assume now that $\deg(D) \geq g$. Then $h(D) \geq 1$ and $h_\epsilon(D) \geq 1$. Our claim follows, since it is easy to check that $\left(\frac{u^{h(D)}-1}{u-1}\right)_{|u=0}$ equals 0, if $h(D) = 0$, and equals 1, if $h(D) > 0$. ■

Example 3.8. When $g(\Lambda) = 0$, any Riemann–Roch structure satisfies $h(D) = \deg(D) + 1$ if $\deg(D) \geq -1$. The zeta-function of Λ is

$$Z(\Lambda, t, u) = \frac{1}{(1-t)(1-ut)}.$$

When $g(\Lambda) = 1$, any Riemann–Roch structure satisfies $h(D) = \deg(D)$ if $\deg(D) \geq 1$. The zeta-function of Λ is completely determined by $\kappa := |\mathrm{Pic}^0(\Lambda)|$, with

$$Z(\Lambda, t, u) = \frac{1 + (\kappa - (u+1))t + ut^2}{(1-t)(1-ut)}.$$ □

Example 3.9. Let G be a graph as in Example 2.2, with Laplacian M and lattice $\Lambda := \mathrm{Im}(M)$. Baker and Norine attach an integer $r(D)$ to any vector $D \in \mathbb{Z}^n$ in [3, 1.6], and we set $h(D) := r(D) + 1 \geq 0$ to obtain a Riemann–Roch structure h on Λ. The Riemann–Roch theorem in [3] is motivated by the analogy between graphs and algebraic curves, and by the existence of the Riemann–Roch theorem for algebraic curves. When we refer in the remainder of this article to the zeta-function of a graph G, we will mean, unless specified otherwise, the zeta-function Z_h associated with the h-function of Baker and Norine as above. The Riemann–Roch structure h satisfies the analog of Clifford's Theorem (Lemma 3.3(iii)), since it is shown in [3, 3.5], that it satisfies (2).

It is only for $g(\Lambda_G) \geq 3$ that h_ϵ may produce a zeta-function Z_{h_ϵ} different from Z_h. This happens, for instance, for the graph G on two vertices linked by four edges. The effective divisor $D = (1, 1)$ on this graph has $h(D) = 2$, while $h_\epsilon(D) = 1$. □

Proposition 3.10. Fix $R > 0$. Let $\Lambda \subseteq \Lambda_R$ be a sublattice of rank $n-1$ with g-number g and Riemann–Roch structure h. Then

$$Z_h(\Lambda, t, u) := \frac{f(t, u)}{(1-t)(1-tu)},$$

where

$$f(t, u) = 1 + c_1(u)t + \cdots + c_g(u)t^g + uc_{g-1}(u)t^{g+1} + u^2 c_{g-2}(u)t^{g+2} + \cdots + u^g t^{2g},$$

and for all $i = 1, \ldots, g$, $c_i(u)$ is an integer polynomial; when h satisfies the analog of Clifford's Theorem (Lemma 3.3(iii)), the degree of $c_i(u)$ is at most $(i+1)/2$. Moreover,

(a) Functional equation:

$$Z\left(\Lambda, \frac{1}{ut}, u\right) = (ut^2)^{1-g} Z(\Lambda, t, u).$$

(b) $f(1, u) = |\mathrm{Pic}^0(\Lambda)|$.

(c) The leading term of $f(t, u)$ as a polynomial in u is $t^{2g} - t^{2g-1}$, and the polynomial $f(t, u)$ is irreducible in $\mathbb{C}[t, u]$. □

Proof. We let $\kappa := |\mathrm{Pic}^0(\Lambda)|$. This proposition follows formally from the properties of the function h. Our proof below follows closely the classical proof of the rationality of the zeta-function for curves over finite fields (see, e.g., [20, VIII.6 and VIII.7]).

Write $(u-1)Z(\Lambda, t, u)$ as the sum of two terms $\alpha(t, u)$ and $\beta(t, u)$, where

$$\alpha(t, u) := \sum_{[D], 0 \leq \deg(D) \leq 2g-2} u^{h(D)} t^{\deg(D)}, \quad \text{and}$$

$$\beta(t, u) := \sum_{[D], \deg(D) \geq 2g-1} u^{h(D)} t^{\deg(D)} - \sum_{[D], \deg(D) \geq 0} t^{\deg(D)}.$$

The second expression can be rewritten using Lemma 3.2(b) as

$$\begin{aligned}\beta(t, u) &= \kappa \left(\sum_{d \geq 2g-1} u^{d+1-g} t^d - \sum_{d \geq 0} t^d \right) \\ &= \kappa u^{1-g} (ut)^{2g-1} \left(\sum_{f \geq 0} (ut)^f \right) - \kappa \sum_{d \geq 0} t^d \\ &= \kappa \left(\frac{u^g t^{2g-1}}{1 - ut} \right) - \kappa \left(\frac{1}{1-t} \right).\end{aligned}$$

We find that $\alpha(t, u) + \beta(t, u) = \frac{F(t,u)}{(1-t)(1-ut)}$ for some polynomial $F(t, u)$. We can compute $\alpha(t, 1) + \beta(t, 1) = 0$ explicitly. It follows that $u-1$ divides $F(t, u)$. Hence, we find that $Z(\Lambda, t, u)$ is a rational function of the form $f(t, u)/(1-t)(1-ut)$ for some polynomial $f(t, u)$ with integer coefficients, and degree in t at most $2g$.

An easy calculation shows that

$$\lim_{t \to 1} (1-t) Z(\Lambda, t, u) = -\frac{\kappa}{u-1},$$

so that $f(1, u) = \kappa$, proving (b). It is also easy to check that

$$\beta(1/ut, u) = (ut^2)^{1-g} \beta(t, u).$$

We now turn our attention to the term $\alpha(t, u)$ and prove that the zeta-function satisfies the expected functional equation in (a). We use first the fact that the map $[D] \mapsto [K-D]$ is a bijection from the set of all divisor classes of nonnegative degree at most $2g-2$ to itself. Clearly, the map is well defined, since if $0 \leq \deg(D) \leq 2g-2$, then $0 \leq \deg(K-D) \leq$

$2g-2$. Using this remark, we can write

$$\alpha(t,u)=\sum_{[D],0\le\deg(D)\le 2g-2} u^{h(K-D)}t^{\deg(K-D)}.$$

The Riemann–Roch formula 3.1(a) is now used to show that

$$\begin{aligned}\alpha(t,u)&=\sum_{[D],0\le\deg(D)\le 2g-2} u^{h(D)-\deg(D)-1+g}t^{\deg(K)-\deg(D)}\\&=u^{g-1}t^{\deg(K)}\sum_{[D],0\le\deg(D)\le 2g-2} u^{h(D)}\frac{1}{(ut)^{\deg(D)}}\\&=u^{g-1}t^{2g-2}\alpha(1/ut,u).\end{aligned}$$

This completes the proof of (a).

We now use the fact that if $\deg(D)=0$, then $h(D)=0$, unless $D=0$, in which case $h(D)=1$. Then

$$Z(\Lambda,t,u):=1+\sum_{[D]\in\mathrm{Pic}(\Lambda),\deg(D)>0}\frac{u^{h(D)}-1}{u-1}t^{\deg(D)},$$

showing that $f(t,u)=1+t\varphi(t,u)$. The functional equation shows that

$$f(t,u)=u^g t^{2g} f(1/ut,u).$$

This equality implies that $f(t,u)$ has degree $2g$ in t, and the expected form

$$f(t,u)=1+c_1(u)t+\cdots+c_g(u)t^g+uc_{g-1}(u)t^{g+1}+u^2c_{g-2}(u)t^{g+2}+\cdots+u^g t^{2g}.$$

Let us write

$$Z(\Lambda,t,u)=1+\sum_{i\ge 1}b_i(u)t^i$$

for some integer polynomials $b_i(u)$. The analog of Clifford's Theorem (Lemma 3.3(iii)) states that $h(D)\le\deg(D)/2+1$ for all D with $0\le\deg(D)\le 2g-2$. When this inequality holds, we have $\deg(b_i(u)) \leq i/2$ when $1 \leq i \leq 2g-2$. Clearly,

$$\left(1 + \sum_{i \geq 1} b_i(u) t^i\right)(1 - (u+1)t + ut^2) = f(t,u).$$

In particular, both $c_1(u) := b_1(u) - (u+1)$ and $c_2(u) := u - (u+1)b_1(u) + b_2(u)$ have degree 1. For $i \geq 3$, $c_i(u) := ub_{i-2}(u) - (u+1)b_{i-1}(u) + b_i(u)$, and we find that $\deg(c_i(u)) \leq (i+1)/2$.

It is then clear that when writing $f(t,u)$ as a polynomial in u, only the terms

$$u^{g-2} c_2(u) t^{2g-2} + u^{g-1} c_1(u) t^{2g-1} + u^g t^{2g}$$

can contain a nontrivial multiple of the monomial u^g. Since $c_2(u)$ has degree 1 and $c_1(u)$ has leading term -1, we find that the leading term of $f(t,u)$ as a polynomial in u is $-t^{2g-1} + t^{2g}$. To end the proof of part (c), we apply [24, 2.1], to the polynomial $F(t,u) := t^{2g} f(1/t, u)$, which is monic in t, has an irreducible leading coefficient in u, and satisfies $F(1,u) \neq 0$. These conditions suffice to imply that $F(t,u)$ and, hence, $f(t,u)$ is absolutely irreducible. ■

Previous work on two-variable zeta-functions of curves and number fields can be found in [10, 15, 26, 32]. Inspired by these studies, we make the following definition.

Definition 3.11. Given a lattice $\Lambda \subseteq \Lambda_R$ of rank $n-1$ with Riemann–Roch structure h and g-number g, consider

$$W_h(\Lambda, x, y) := \sum_{[D] \in \mathrm{Pic}(\Lambda)} x^{h(D)} y^{h(K-D)}.$$

When $g = 0$, $W_h(\Lambda, x, y) = \frac{1-xy}{(1-x)(1-y)}$. A formal computation shows that

$$W_h(\Lambda, ut, t^{-1}) = (u-1)t^{1-g} Z_h(\Lambda, t, u).$$

To a connected graph G is associated its Tutte polynomial, defined as a sum taken over the set $\Sigma(G)$ of spanning trees of G:

$$\mathcal{T}(G, x, y) := \sum_{T \in \Sigma(G)} x^{i(T)} y^{j(T)},$$

where $i(T)$ and $j(T)$ are nonnegative integers associated with the spanning tree T. In [8, p. 127], Biggs associates to $\mathrm{Pic}^0(G)$ a polynomial $\mathcal{L}(t)$ of degree g, and proves using a result of Merino [23] that

$$\mathcal{T}(G, 1, t^{-1}) = t^{-g}\mathcal{L}(t).$$

A polynomial of degree g which naturally occurs in the context of our zeta-function is the numerator of $Z_h(\Lambda, t, 0)$ (Lemma 3.7). This polynomial does not depend on the existence of a canonical divisor. Using the notation of Proposition 3.10, we write

$$Z_h(\Lambda, t, 0) = \frac{f(t, 0)}{(1-t)} = 1 + a_1 t + \cdots + a_{g-1} t^{g-1} + \kappa t^g + \kappa t^{g+1} + \cdots,$$

where a_i denotes the number of elements $[D]$ in $\mathrm{Pic}(G)$ of degree i such that $h(D) > 0$. By definition, $f(t, 0) = 1 + (a_1 - 1)t + (a_2 - a_1)t^2 + \cdots + (a_g - a_{g-1})t^g$. The following proposition follows directly from the definitions. □

Proposition 3.12. Let G be a connected graph. Then $\mathcal{L}(t) = f(t, 0)$. □

Proof. Fix a vertex q of G. Given any element $[D] \in \mathrm{Pic}^0(G)$, Biggs defines a nonnegative integer $L([D])$ as follows in [8, p. 127]. Consider all possible divisors equivalent to D of the form $D' - \deg(D')q$, with D' effective: then $L([D])$ is the smallest possible degree that such a divisor D' may have, and

$$\mathcal{L}(t) := \sum_{[D]\in \mathrm{Pic}^0(G)} t^{L([D])} = \sum_{i=0}^{n} b_i t^i.$$

Since $D' = 0$ is the only effective of degree 0, we find that $b_0 = 1$. Since any divisor of degree g is equivalent to an effective, $D + gq = D'$ with D' effective, and $L([D]) \le g$. Hence, $n \le g$. Since by definition of g, there exists a divisor C of degree $g - 1$ that is not equivalent to an effective, we find that $L([C - (g-1)q]) = g$ and $n = g$. Clearly, $\sum_{i=0}^{g} b_i = \kappa$. To prove the proposition, it suffices to prove that for each $i = 1, \ldots, g$, $\sum_{j=0}^{i} b_i = a_i$. For each $i > 0$, let $\mathrm{Pic}^i(G)$ denote the subset of $\mathrm{Pic}(G)$ consisting in the classes of degree i, and consider the bijection $\mathrm{Pic}^0(G) \to \mathrm{Pic}^i(G)$ given by $[D] \mapsto [D + iq]$. By definition, if $L([D]) \le i$, then $h(D + iq) > 0$ since $D + iq$ is equivalent to an effective. Hence, $\sum_{j=0}^{i} b_i \le a_i$. Suppose now that a divisor class $[C] \in \mathrm{Pic}^i(G)$ is equivalent to an effective D'. Then $[C - iq] = [D' - iq]$, and $L([C - iq]) \le i$. So $\sum_{j=0}^{i} b_i = a_i$. ■

Corollary 3.13. Let G be a connected graph, equipped with a Riemann–Roch structure h. Then the zeta-function $Z_{h_\epsilon}(\Lambda, t, u)$ of the associated Riemann–Roch structure h_ϵ is completely determined by the partial evaluation $\mathcal{T}(G, 1, t^{-1})$ of the Tutte polynomial of G. □

Proof. The number of times the function h_ϵ takes a given value is completely determined by the integers a_i, $i = 1, \ldots, g$. As the previous proposition shows, these integers are determined by $\mathcal{L}(t) = t^{-g}\mathcal{T}(G, 1, t^{-1})$. ■

Note that the function $W_h(\Lambda, x, y)$ is symmetric:

$$W_h(\Lambda, x, y) = W_h(\Lambda, y, x),$$

and this symmetry produces the functional equation satisfied by $Z_h(t, u)$. On the other hand, the Tutte polynomial is not symmetric in general. For instance, it is well-known that if G is a planar graph and G^* denotes its dual, then

$$W(G, x, y) = W(G^*, y, x).$$

It is also known that $\mathrm{Pic}^0(G)$ and $\mathrm{Pic}^0(G^*)$ are isomorphic. From the equality $W(G, x, y) = W(G^*, y, x)$, we obtain that $W(G, 1, 1) = W(G^*, 1, 1)$, which imply that the latter two groups have same order, equal to the complexity of G. We do not know if the existence of an isomorphism of groups between $\mathrm{Pic}^0(G)$ and $\mathrm{Pic}^0(G^*)$ can be deduced from the equality $W(G, x, y) = W(G^*, y, x)$.

Suppose that two graphs G and G', both equipped with the Riemann–Roch structure h of Baker and Norine (Example 3.9), have the same zeta-function $Z(G, t, u)$. It is natural to wonder what other properties these graphs must then share. When two curves $X/\mathbb{F}_q$ and $Y/\mathbb{F}_q$ over a finite field have the same zeta-function, then their jacobians are isogenous, that is, there exists a morphism $\varphi : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$ defined over $\mathbb{F}_q$ with finite kernel. Using the degree of the numerator of the zeta-function of G and G′, Using the degree of the numerator of the zeta-function, we find that $\beta(G) = \beta(G')$. Using the residue of the pole at $t = 1$ of the zeta-function, we find that $\kappa(G) = \kappa(G')$.

Given a graph G, let us introduce the following nonnegative integers: for all $i, j \geq 0$, let

$$b(i, j) := \text{number of divisor classes } [D] \text{ of degree } i \text{ in } \mathrm{Pic}(\Lambda_G) \text{ with } h(D) = j.$$

With this definition, writing $Z(G, t, u) := 1 + \sum_{i\geq 1} b_i(u)t^i$ as in the proof of Proposition 3.10, we have

$$b_i(u) = \sum_{j>0} b(i, j)\frac{u^j - 1}{u - 1}.$$

We have $b(0, 1) = 1$, $b(2g - 2, g) = 1$, and for any fixed i,

$$\sum_{j=0}^{i+1} b(i, j) = \kappa(G).$$

It follows that $b_i(0) = \kappa(G) - b(i, 0)$. By definition, two graphs G and G' have the same zeta-function if and only if $b(i, j)(G) = b(i, j)(G')$ for all $i, j \geq 0$.

Recall that a *bridge* on a graph G is an edge e of G such that $G \setminus \{e\}$ is not connected.

Lemma 3.14. Suppose that G is a graph on n vertices without bridges. Then $b(1, 1) = n$. In particular, two graphs without bridges with the same zeta-function have the same number of vertices, the same number of edges, and same complexity. □

Proof. If $\deg(D) = 1$, then $h(D) \leq 1$ (use Clifford's Theorem). If $h(D) = 1$, then $[D] = [E]$ with $E \geq 0$. There are exactly n divisors $E \geq 0$ of degree 1. Each such divisor E has $h(E) = 1$. Given two distinct vertices v and v' of G, denote again by v and v' the corresponding elements of degree 1 in $\mathbb{Z}^n$. Then $[v] = [v']$ in $\mathrm{Pic}(\Lambda_G)$ if and only if there is a path in G linking v and v' such that all edges of the path are bridges [21, 2.3]. Thus, under our hypothesis, there are exactly n divisor classes of degree 1 containing an effective divisor, implying that $b(1, 1) = n$. ■

Example 3.15. When $\beta(G) = 2$, the zeta-function of G is completely determined by the integers $b(1, 0)$ and $b(1, 1)$, with $b(1, 0) + b(1, 1) = \kappa(G)$. Thus, when G does not have bridges, the pairs of integers (n, κ) and (m, κ) each determine the zeta-function of G.

Let x, y, and z be positive integers, and consider the family of graphs $G(x, y, z)$ consisting of three vertices u, v, and w, with x edges between u and v, y edges between v and w, and z edges between w and u. The planar dual $G^*(x, y, z)$ of $G(x, y, z)$ has $m(G^*) = x + y + z$ edges, and $g(G^*) = 2$. Its complexity is $\kappa(G^*) = xy + yz + xz$, and as we have indicated above, the integers $\kappa(G^*)$ and $m(G^*)$ completely determine the zeta-function of G^*.

Consider the two graphs $G_1 := G^*(40 + 13s, 61 + 13s, 16 + 13s)$ and $G_2 := G^*(52 + 13s, 52 + 13s, 13 + 13s)$. These graphs have the same zeta-functions, since they have the same number of edges (in both cases $x+y+z=117+39s$) and same complexity ($\kappa=4056+3042s+507s^2$). The integer solutions to the system of equations $\sum_{i=1}^3 x_i=\sum_{i=1}^3 y_i$ and $\sum_{i=1}^3 x_i^2=\sum_{i=1}^3 y_i^2$ are briefly discussed with reference in [12], XXIV, page 707. But it is worth noting that two graphs can have the same zeta-function without having isomorphic Jacobian groups. Indeed, in this example with $s=0$, the groups $\mathrm{Pic}^0(\Lambda_{G_1})$ and $\mathrm{Pic}^0(\Lambda_{G_2})$ are not isomorphic, since the group $\mathrm{Pic}^0(\Lambda_{G^*(x,y,z)})$ is cyclic if and only if $\gcd(x,y,z)=1$. When $s=0$, $\mathrm{Pic}^0(\Lambda_{G_1})$ is cyclic and $\mathrm{Pic}^0(\Lambda_{G_2})=\mathbb{Z}/13\mathbb{Z}\times\mathbb{Z}/(13\cdot 24)\mathbb{Z}$. We also note that the eigenvalues of the Laplacians of these two graphs are distinct, as, for instance, t^2-4t+2 exactly divides the characteristic polynomial of G_1, while $(t^2-4t+2)^2$ divides the characteristic polynomial of G_2. □

Let G_1 and G_2 be two connected graphs with the same Tutte polynomial. Is it possible for these two graphs to have nonisomorphic groups $\mathrm{Pic}(G_1)$ and $\mathrm{Pic}(G_2)$? Is it possible for two such graphs to have distinct zeta-functions $Z_h(G_1,t,u)$ and $Z_h(G_2,t,u)$? Note that we know from our hypothesis that $|\mathrm{Pic}^0(G_1)|=|\mathrm{Pic}^0(G_2)|$ and $Z_h(G_1,t,0)=Z_h(G_2,t,0)$ (Proposition 3.12).

Example 3.16. Let T and T' be any two nonisomorphic trees on n vertices $v_1,\ldots,v_n$. Let M and M' denote their Laplacian matrices. Given any positive integer x, the matrices xM and xM' are the Laplacians of two multigraphs without bridges, which we denote by G and G'. Both graphs have $g=(x-1)(n-1)$, and Picard group isomorphic to $(\mathbb{Z}/x\mathbb{Z})^{n-1}$. Both graphs have the same Tutte polynomial.

We claim that the graphs G and G' have the same zeta-functions, even though these graphs are not isomorphic. In fact, $\mathrm{Im}(xM)=\mathrm{Im}(xM')$ in $\mathbb{Z}^n$. Indeed, since T and T' are trees, $\mathrm{Im}(M)=\mathrm{Im}(M')=\mathrm{Ker}\,({}^{\mathrm{t}}(1,\ldots,1))$. Hence, $\mathrm{Im}(xM)$ and $\mathrm{Im}(xM')$ are both generated by ${}^{\mathrm{t}}(x,-x,0,\ldots,0)$, ${}^{\mathrm{t}}(x,0,-x,0,\ldots,0),\ldots,{}^{\mathrm{t}}(x,0,\ldots,0,-x)$. □

One finds in the literature several different definitions of a zeta-function of a graph, with contributions by many authors to the subject, including Ihara, Stark and Terras [30], Bartholdi [4], and others. It would be of interest to understand how the Riemann–Roch zeta-function relates to these other objects.

4 Arithmetical Graphs

The notion of arithmetical graph was introduced in [17], and we recall below its definition. Arithmetical graphs arise when considering degeneration of curves in algebraic geometry, and encode some of the discrete data associated with the degeneration. They consist in a “usual” graph endowed with an additional structure, providing a lattice to which one may apply the considerations of the previous sections. These objects may be similar enough to “usual” graphs that they may retain some of their properties, such as the existence of a canonical vector. Our strongest result in this direction is Theorem 4.2, the proof of which uses the theory of curves, and is thus not of a combinatorial nature.

Definition 4.1. Let G be a finite unweighted connected multigraph on n vertices $v_1, \dots, v_n$, without loop edges. Let A denote its adjacency matrix. Consider a diagonal $(n \times n)$-matrix $\mathcal{D} = \mathrm{diag}(\delta_1, \dots, \delta_n)$ with strictly positive integer diagonal entries, and an integer vector ${}^{\mathrm{t}}R = (r_1, \dots, r_n)$ with $R > 0$ and $\gcd(r_1, \dots, r_n) = 1$. Let $M := \mathcal{D} - A$. The triple (G, M, R) is called an *arithmetical graph* if $MR = 0$. When $\mathcal{D} = \mathrm{diag}(d_1, \dots, d_n)$ with d_i the valency of v_i and ${}^{\mathrm{t}}R = (1, \dots, 1)$, the matrix M is nothing but the Laplacian of G. Let $\Lambda_M := \mathrm{Im}(M) \subseteq \Lambda_R$. We may denote $g(\Lambda_M)$ and $\mathrm{Pic}^0(\Lambda_M)$ simply by $g(M)$ and $\mathrm{Pic}^0(M)$, respectively.

The main geometric invariant of an arithmetical graph is the integer $g_0(M)$ defined by the expression

$$2g_0(M) - 2 = \sum_{i=1}^{n} r_i(\delta_i - 2) = \sum_{i=1}^{n} r_i(d_i - 2).$$

That $g_0(M)$ is always an integer is noted in [17, 3.6]. When ${}^{\mathrm{t}}R \neq (1, \dots, 1)$, we do not have examples of two arithmetical graphs (G_1, M_1, R) and (G_1, M_2, R) on n vertices such that $\mathrm{Im}(M_1) = \mathrm{Im}(M_2)$ and $g_0(M_1) \neq g_0(M_2)$. In other words, we do not know whether the integer g_0 depends only on the lattice spanned by the columns of the matrix M.

In the Riemann–Roch theorem for the Laplacian of a graph, the canonical class is represented by ${}^{\mathrm{t}}K_G := (d_1 - 2, \dots, d_n - 2)$, with $\sum_{i=1}^{n}(d_i - 2) = 2\beta(G) - 2$. For an arithmetical graph (G, M, R), a natural analogue to consider is

$$K := {}^{t}(\delta_1 - 2, \dots, \delta_n - 2),$$

with $\deg(K) = 2g_0(M) - 2$. Note that by adding together the columns of the matrix M we obtain the vector ${}^{t}(\delta_1 - d_1, \dots, \delta_n - d_n)$, showing that ${}^{\mathrm{t}}(d_1 - 2, \dots, d_n - 2)$ is a vector equivalent to K in $\mathrm{Pic}(\Lambda_M)$. □

Theorem 4.2. Let (G, M, R) be an arithmetical graph, with $\Lambda_M := \mathrm{Im}(M) \subseteq \Lambda_R$. Let $K := {}^{\mathrm{t}}(\delta_1 - 2, \ldots, \delta_n - 2)$. Then

(a) $g(\Lambda_M) \leq g_0(M)$.

(b) Let $D \in \mathbb{Z}^n$ with $\deg(D) = g_0(M) - 1$. Then $\epsilon_{\Lambda_M}(D) = \epsilon_{\Lambda_M}(K - D)$. In particular, K is a canonical vector for Λ_M if $g(\Lambda_M) = g_0(M)$. □

Proof. We prove this theorem by first interpreting the matrix M as the intersection matrix associated with the reduction of a curve, and then by applying the Riemann–Roch Theorem for curves.

Given M and R, there exist a complete discrete valuation ring $\mathcal{O}_F$ (with field of fractions F and algebraically closed residue field k of characteristic 0), and a smooth proper geometrically connected curve X/F of (geometric) genus $p_g(X)$ with a regular model $\mathcal{X}/\mathcal{O}_F$ satisfying the following properties (see [33, 4.3]). The special fiber $\mathcal{X}_k/k$ of $\mathcal{X}/\mathcal{O}_F$ is the union of smooth irreducible curves C_i, $i = 1, \ldots, n$ (called the components of $\mathcal{X}_k$). Each curve C_i/k has genus 0 and multiplicity r_i. Denote by $(C_i \cdot C_j)$ the intersection number of the components C_i and C_j on $\mathcal{X}$. The matrix $((C_i \cdot C_j))_{1 \leq i, j \leq n}$ is called the intersection matrix of $\mathcal{X}_k/k$, and is equal to M. These conditions, on the intersection matrix and on the genus of the components of $\mathcal{X}_k$, imply that $p_g(X) = g_0(M)$.

Our reference for the facts recalled below is [16, 9.1]. Since $\mathcal{X}$ is regular, the natural inclusion $X \to \mathcal{X}$ induces a surjective restriction map $\mathrm{res} : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$. Recall that $\mathrm{Pic}(M) := \mathbb{Z}^n/\mathrm{Im}(M)$. In keeping with the geometric notation, we write $\mathbb{Z}^n$ as $\mathrm{Div}(M) = \bigoplus_{i=1}^n \mathbb{Z}C_i$, and call an element of $\mathrm{Div}(M)$ a divisor. Two divisors D and E are said to be equivalent if $[D] = [E]$. We also have a natural homomorphism $\rho : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(M)$ defined as follows: $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ is mapped to the class in $\mathrm{Pic}(M)$ of the divisor $\deg(\mathcal{L}_{|C_1})C_1 + \cdots + \deg(\mathcal{L}_{|C_n})C_n$. There exists in $\mathrm{Pic}(\mathcal{X})$ an element $K_{\mathcal{X}/\mathcal{O}_F}$, called the canonical bundle, with the following properties:

(i) $\rho(K_{\mathcal{X}/\mathcal{O}_F}) = [K]$, and

(ii) $\mathrm{res}(K_{\mathcal{X}/\mathcal{O}_F}) = K_{X/F}$ is the canonical bundle in $\mathrm{Pic}(X)$.

For each i, we can choose a closed point P_i of X a closed point P_i of X whose closure $\overline{P}_i$ in $\mathcal{X}$ intersects $\mathcal{X}_k/k$ only in C_i; more precisely, we require the following: $(\bar{P}_i \cdot C_i) = 1$, and $(\bar{P}_i \cdot C_j) = 0$ if $j \neq i$. This shows that the map ρ is surjective. Indeed, let $D := \sum_{i=1}^n a_i C_i$ be a divisor of $\mathrm{Div}(M)$. Consider the line bundle $\mathcal{L} := \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^n a_i \overline{P_i})$. Then $\rho(\mathcal{L}) = [D]$. Moreover, the degree of the divisor $\mathrm{res}(\mathcal{L}) := \mathcal{O}_X(\sum_{i=1}^n a_i P_i)$ in $\mathrm{Pic}(X)$ is equal to the degree of $[D]$ in $\mathrm{Pic}(M)$.

(a) Let $D := \sum_{i=1}^{n} a_i C_i$ be a divisor of Div(M) of degree $r \geq g_0(M)$. Consider the divisor $D' := \sum_{i=1}^{n} a_i P_i$ in Div(X). Since $r \geq g_0(M) = p_g(X)$, the Riemann–Roch theorem on curves [20, IX.4.1] shows that the vector space $H^0(X, D')$ has positive dimension, which implies that we can find a divisor $E' = \sum_{j=1}^{s} b_j Q_j \in \mathrm{Div}(X)$ with $b_j \geq 0$ for all j and such that D' and E' are linearly equivalent. Consider the line bundle $\mathcal{L}' := \mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{s} b_j \overline{Q_j})$. Then $\mathcal{L}' \otimes \mathcal{L}^{-1}$ is trivial on the generic fiber X. Hence, $\rho(\mathcal{L}) = \rho(\mathcal{L}')$ in Pic(M). It is easy to verify that $\rho(\mathcal{L}')$ is represented in Div(M) by an effective divisor. By construction, $\rho(\mathcal{L}) = [D]$ and, thus, D is equivalent to an effective divisor. It follows that $g(\Lambda_M) \leq g_0(M)$.

To prove (b), it is sufficient to show that if $D = \sum_{i=1}^{n} a_i C_i$ is an effective divisor of degree $g_0(M) - 1$ in Div(M), then $K - D$ is equivalent to an effective divisor. Consider the divisor $D' := \sum a_i P_i$ in Div(X). Let $K' \in \mathrm{Div}(X)$ denote a canonical divisor for X/F. Since $g_0(M) = p_g(X)$, the Riemann–Roch theorem for curves implies that either both D' and $K' - D'$ are linearly equivalent to an effective divisor on X, or neither is. Since D' is clearly effective in Div(X) because D is, $K' - D'$ is equivalent to an effective divisor $E' = \sum_{j=1}^{s} b_j Q_j \in \mathrm{Div}(X)$. Consider then the line bundle $K_{\mathcal{X}/\mathcal{O}_F} \otimes \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n} a_i \overline{P_i})^{-1} \otimes \mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{s} b_j \overline{Q_j})^{-1}$ in Pic($\mathcal{X}$), which is trivial on the generic fiber X by construction. It follows that $\rho(K_{\mathcal{X}/\mathcal{O}_F} \otimes \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n} a_i \overline{P_i})^{-1})$ is equivalent to $\rho(\mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{s} b_j \overline{Q_j}))$ in Pic(M). By construction, $\rho(K_{\mathcal{X}/\mathcal{O}_F} \otimes \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n} a_i \overline{P_i})^{-1}) = [K - D]$, and $\rho(\mathcal{O}_{\mathcal{X}}(\sum_{j=1}^{s} b_j \overline{Q_j}))$ is effective. ■

It would be of interest to find a completely combinatorial proof of this theorem. It is also natural to wonder whether M has a canonical vector when $g(M) < g_0(M)$. We give below some examples of arithmetical graphs, and of the inequalities $g(R) \leq g(\Lambda_M) \leq g_0(M)$.

Example 4.3. Let $R = (r_1, \ldots, r_n)$ be an integer vector with positive entries such that $\gcd(r_1, \ldots, r_n) = 1$. Using the incidence matrix of any connected graph H on n vertices, we construct an arithmetical graph of the form (G, M, R), producing in this way for most vectors R many nonisomorphic arithmetical graphs having the same vector R.

Indeed, given a graph H, Chung and Langlands introduced a Laplacian with vertex weights in [9]. When, for all i, the weight of the vertex v_i is the square of a positive integer r_i, then the Laplacian L introduced in [9, (1), p. 317] is the matrix of an arithmetical graph with vector ${}^{\mathrm{t}}R = (r_1, \ldots, r_n)$. We recall this construction here. Let B denote the incidence matrix of H, with n rows and m columns. The Laplacian of H is equal to $B({}^{\mathrm{t}}B)$. Let B_R denote the following $(n \times m)$-matrix. Say a row of the transpose ${}^{\mathrm{t}}B$ has $+1$ in its ith entry and -1 in its jth entry, then the matrix ${}^{\mathrm{t}}B_R$ has $+r_j$ in its ith entry and $-r_i$ in its jth entry. By construction, $({}^{\mathrm{t}}B_R)R = 0$. We let $M := B_R({}^{\mathrm{t}}B_R)$, and denote by (G, M, R) the associated arithmetical graph.

Let $r := \sum_{i=1}^{n} r_i^2$. When the initial graph H is the complete graph on n vertices, the associated arithmetical graph (G, M, R) has $M = rI_n - R({}^{\mathrm{t}}R)$. The group $\mathrm{Pic}^0(\Lambda_M)$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{n-2}$ [17, 1.10]. When $R = {}^{\mathrm{t}}(1, 1, 2)$, $g(M) = g_0(M) = 4$, and for $R = {}^{\mathrm{t}}(1, 1, 3)$, $g(M) = 6$ and $g_0(M) = 9$. In the latter case, ${}^{\mathrm{t}}(5, 5, 0)$ is a canonical vector.

The simplest example of this construction is with the tree H on two vertices and the vector ${}^{\mathrm{t}}R = (r, s)$, $\gcd(r, s) = 1$. Then the associated (G, M, R) has

$$M = \begin{pmatrix} s^2 & -rs \\ -rs & r^2 \end{pmatrix}.$$

In this case, $|\mathrm{Pic}^0(\Lambda_M)| = 1$, so $g(M) = g(r, s) = (r-1)(s-1)$. An easy computation shows that $g_0(M) = \frac{1}{2}sr(r+s) - (r+s) + 1$. □

When (G, M, R) is an arithmetical graph, let us denote by $\beta(G) := m - n + 1$ the first Betti number of the graph G. It is noted in [17, 4.7], that $g_0(M) \geq \beta(G)$. It is not true in general that $g(M) \geq \beta(G)$. In the above example when $n = 2$, $\beta(G) = rs - 1$, while $g(\Lambda_M) = (r-1)(s-1)$.

Example 4.4. All lattices of rank 1 in $\mathbb{Z}^2$ have a canonical vector. Indeed, let ${}^{\mathrm{t}}R := (r, s)$ with $\gcd(r, s) = 1$. The lattice Λ_R is generated in $\mathbb{Z}^2$ by the vector ${}^{\mathrm{t}}(s, -r)$. We have $g(\Lambda_R) = g(r, s) = (r-1)(s-1)$. Note that the lattice Λ_R is in fact the lattice associated with an arithmetical graph (G, M, R), namely, the arithmetical graph with M as in Example 4.3. The only sublattices of Λ_R are of the form $x\Lambda_R$. Then $g(x\Lambda_R) = xrs - r - s + 1$ and $K := (xs - 2, xr - 2)$ is a canonical vector for the lattice $x\Lambda_R$ (Corollary 5.5). □

Example 4.5. (a) Given coprime integers $2 \leq a < b$, there exists an arithmetical graph (G, M, R) such that $|\mathrm{Pic}^0(M)| = 1$ and $g_0(M) = g(M) = g(R) = g(a, b)$.

Indeed, define positive integers s_1, s_2, s_3, and s_4 by the equality $2ab = (s_1 + s_2)a + (s_3 + s_4)b = a + (b-1)a + b + (a-1)b$. We consider then an arithmetical tree (G, M, R) with a single node v of multiplicity ab, and valency 4. Attached to v are four terminal chains. The vertices linked to v have multiplicities s_1a, s_2a, s_3b, and s_4b, respectively. The terminal vertices on the chains have multiplicities a, a, b, and b. Each terminal chain is obtained using Euclid's algorithm on the pairs (ab, s_ia) and (ab, s_ib), as in [17, 4.2]. Since the terminal multiplicity of a chain divides each multiplicity on the chain, we find that $g(R) = g(a, b)$. A formula for the order of $|\mathrm{Pic}^0(M)|$ when G is a tree is given in [17, 2.5], and can be used to show that $|\mathrm{Pic}^0(M)| = 1$. Thus Lemma 2.4(a) implies that $g(M) = g(R) = g(a, b)$. An easy computation shows that $g_0(M) = g(a, b)$.

(b) Given coprime integers $2 \leq a < b < c$ with $\gcd(a, c) = 1$, it is often possible to find an arithmetical graph (G, M, R) such that $|\mathrm{Pic}^0(M)| = 1$ and $g(M) = g(R) = g(a, b, c)$. In the example below, such an arithmetical graph has $g_0(M) = 1 + \frac{1}{2}(abc - a - b - c)$; in particular, it is possible to find many instances where $g(M) < g_0(M)$.

Indeed, suppose that we found positive integers s_1, s_2, and s_3, such that $abc = s_1 a + s_2 b + s_3 c$ and $\gcd(abc, s_1 a)/a = \gcd(abc, s_2 b)/b = \gcd(abc, s_3 c)/b = 1$. For instance, since $\gcd(a, c) = 1$, find $0 < x < a$ such that $a \mid c + bx$. If $\gcd(x, c) = 1$, then take $s_1 = 1$, $s_2 = x$, and $s_3 c = abc - a - bx > 0$.

Consider the arithmetical tree (G, M, R) with a single node v of multiplicity abc and valency 3 constructed as follows. Attached to v are three terminal chains. The vertices linked to v have multiplicities $s_1 a$, $s_2 b$, and $s_3 c$. The terminal vertices on the chains have multiplicities a, b, and c. Each terminal chain is obtained using Euclid's algorithm on the pairs $(abc, s_1 a)$, $(abc, s_2 b)$, and $(abc, s_3 c)$, as in [17, 4.2]. Since the terminal multiplicity of a chain divides each multiplicity on the chain, we find that $g(R) = g(a, b, c)$. The formula for the order of $|\mathrm{Pic}^0(M)|$, when G is a tree given in [17, 2.5], can be applied and shows that $|\mathrm{Pic}^0(M)| = 1$. Thus Lemma 2.4(a) implies that $g(M) = g(R) = g(a, b, c)$. Recall [5, p. 20] that

$$g(a, b, c) \leq 1 + \tfrac{1}{2}(\sqrt{abc(a + b + c)} - a - b - c).$$

In particular, in this example, the difference $g_0(M) - g(M)$ can be arbitrarily large. $\square$

The previous examples show how to construct arithmetical trees where $g(R) = g(\Lambda_M) = g_0(M)$, and also where $g(\Lambda_M) < g_0(M)$. Hower [13] has produced classes of examples of arithmetical graphs with $g(\Lambda_M) = g_0(M)$.

Some earlier works on the Frobenius number can be seen to have a connection to arithmetical graphs. For instance, in [14], Kan considers a connecting chain of length $s + 1$ in an arithmetical graph (G, M, R), and bounds the Frobenius number of the multiplicities of the chain in terms of some invariants of the corresponding $(s \times s)$-principal minor of the matrix M. In [29], an exact formula is given for the Frobenius number of the set of multiplicities of a connecting chain when all self-intersections are equal to 2 (i.e., when the multiplicities are of the form $r_0, r_0 + d, \ldots, r_i := r_0 + id$).

5 Existence of a Canonical Vector

In this section, we introduce several classes of lattices Λ of rank $n-1$ for which we can compute $g(\Lambda)$ and show the existence of a canonical vector. We show in Corollary 5.6 that the existence of a canonical vector for a lattice Λ perpendicular to a vector R is equivalent to the existence of a canonical vector for an associated lattice perpendicular to ${}^t(1, \dots, 1)$.

Let us note first the following easy facts. Let $R \in \mathbb{Z}^n$ be an integer vector with strictly positive entries, as in Section 2.

Lemma 5.1. Let $\Lambda \subseteq \Lambda_R$ be a lattice of rank $n-1$.

(a) Assume that $\mathrm{Pic}^0(\Lambda)$ has at most two distinct classes of degree $g(\Lambda)-1$ which contain an effective divisor, or has at most two distinct classes of degree $g(\Lambda)-1$ which do not contain any effective divisor. Then Λ has a canonical vector K. In particular, if $|\mathrm{Pic}^0(\Lambda)| \leq 5$, then Λ has a canonical vector K.

(b) If $g(\Lambda) \leq 1$, then Λ has a canonical vector K, and the class of K in $\mathrm{Pic}(\Lambda)$ is uniquely determined. □

Proof. (a) When $|\mathrm{Pic}^0(\Lambda)| = 1$, any vector K of degree $2g(\Lambda)-2$ is a canonical vector (and there is only one class in $\mathrm{Pic}(\Lambda)$ of any given degree). If all divisors D of degree $g(\Lambda)-1$ are not equivalent to an effective, then any divisor of degree $2g(\Lambda)-2$ is a canonical vector. If $[D]$ is the only class of degree $g(\Lambda)-1$ which is either equivalent to an effective, or not equivalent to an effective, then $2D$ is a canonical vector. If $[D]$ and $[D']$ are the only classes of degree $g(\Lambda)-1$ which are equivalent to an effective, or if they are the only classes which are not equivalent to an effective, then we can take $D+D'$ as a canonical vector. When $|\mathrm{Pic}^0(\Lambda)| \leq 5$, there are at most five classes of degree $g(\Lambda)-1$, and the result follows from the above considerations.

(b) When $g(\Lambda) = 0$, $|\mathrm{Pic}^0(\Lambda)| = 1$, and the result follows from (a). When $g(\Lambda) = 1$, the 0-vector $\mathbf{0} := (0, \dots, 0)$ is the unique vector of degree $g(\Lambda)-1$ that is effective. Then $K = (0, \dots, 0)$ is a canonical vector. If K' is another canonical vector, $\mathbf{0}$ and $K' - \mathbf{0}$ both need to be effective, so that $K' - \mathbf{0}$ is equivalent to $\mathbf{0}$, and $[K'] = [\mathbf{0}]$. ■

Let us note that if $\mathrm{Pic}^0(\Lambda)$ is killed by 2 and there are exactly two distinct classes $[D]$ and $[D']$ of degree $g(\Lambda)-1$ which are not equivalent to an effective, then Λ has two canonical vectors that are not equivalent. Indeed, $2[D-D'] = [0]$ in $\mathrm{Pic}^0(\Lambda)$. Taking

$K := 2D$, we find that $K - D = D$ and $[K - D'] = [D']$, so K is a canonical vector. Taking now $K' := D + D'$, we find first that $K - D = D'$ and $K - D' = D$, so that K' is a canonical vector. We have $[K] \neq [K']$, since otherwise, $[2D] = [D + D']$ implies that $[D] = [D']$, a contradiction.

Definition 5.2. Let $R > 0$ and consider $\Lambda \subseteq \Lambda_R$, a lattice of rank $n-1$. Let $x_1, \ldots, x_n$ be positive integers, and let $\ell := \mathrm{lcm}(x_1, \ldots, x_n)$. We now introduce a new lattice $X\Lambda$, new lattice $X\Lambda$, whose g-number and canonical vector can be computed in terms of the g-number and the canonical vector of Λ.

Let $X := \mathrm{diag}(x_1, \ldots, x_n)$, and consider the map $X : \mathbb{Z}^n \to \mathbb{Z}^n$. Let $X\Lambda$ denote the image of Λ under the map X. Clearly, $X\Lambda$ has rank $n-1$. Let $d := \gcd(\ell r_1/x_1, \ldots, \ell r_n/x_n)$. Set $R' := {}^{\mathrm{t}}(\ell r_1/dx_1, \ldots, \ell r_n/dx_n)$. Then $X\Lambda \subseteq \Lambda_{R'}$.

We claim that $d \mid \ell$. Indeed, let p be prime, and assume that p^s is the exact power of p that divides d. Since $\gcd(r_1, \ldots, r_n) = 1$, there exists r_i with $p \nmid r_i$. Hence, $p^s \mid \ell/x_i$, and so $p^s \mid \ell$.

For convenience, we will use the following notation. Let J_n denote the transpose of the vector $(1, \ldots, 1)$. As usual, I_n denotes the $(n \times n)$-identity matrix. Given a lattice $\Lambda \subseteq \Lambda_R$, let

$$\mathcal{N}(\Lambda) := \{[D] \in \mathrm{Pic}(\Lambda) \mid \deg_R([D]) = g(\Lambda) - 1, \epsilon_\Lambda(D) = 0\},$$

where ϵ_Λ is as in Definition 2.6. Note that a vector K of degree $2g(\Lambda) - 2$ is a canonical vector for Λ if and only if, for all $[D] \in \mathcal{N}(\Lambda)$, $[K - D] \in \mathcal{N}(\Lambda)$. □

Proposition 5.3. Let $\Lambda \subseteq \Lambda_R$. Let X and R' be as above, and consider $X\Lambda \subseteq \Lambda_{R'}$. Then

(a) $g(X\Lambda) - 1 = \frac{\ell}{d}(g(\Lambda) - 1) + \sum_{i=1}^{n}(x_i - 1)\ell r_i/\mathrm{d}x_i$.

(b) The map $\mathcal{N}(\Lambda) \to \mathcal{N}(X\Lambda)$, $[D] \mapsto [XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)]$, is a bijection.

(c) If K is a canonical vector for Λ, then $XK + 2(X - I_n)J_n$ is a canonical vector for $X\Lambda$.

(d) If $K(X\Lambda)$ is a canonical vector for $X\Lambda$, then $K(X\Lambda)$ is $X\Lambda$-equivalent to $XK + 2(X - I_n)J_n$ with K a canonical vector for Λ. □

Proof. (a) Write $D' \in \mathbb{Z}^n$ as $D' = XD + {}^{\mathrm{t}}(y_1, \ldots, y_n)$ with $0 \le y_i \le x_i - 1$ for all $i = 1, \ldots, n$. If $\deg_{R'}(D') > \frac{\ell}{d}(g(\Lambda) - 1) + \sum_{i=1}^{n}(x_i - 1)\ell r_i/\mathrm{d}x_i$, then $\deg_{R'}(XD') > \frac{\ell}{d}(g(\Lambda) - 1)$ and, hence, $\deg_R(D) > g(\Lambda) - 1$. It follows that D is Λ-equivalent to an effective, so that XD and D' are $X\Lambda$-equivalent to an effective. Therefore, $g(X\Lambda) - 1 \le \frac{\ell}{d}(g(\Lambda) - 1) + \sum_{i=1}^{n}(x_i - 1)$ $\ell r_i/\mathrm{d}x_i$.

Assume now that $\deg_{R'}(D') = \frac{\ell}{d}(g(\Lambda) - 1) + \sum_{i=1}^{n}(x_i - 1)\ell r_i/\mathrm{d}x_i$. If $(y_1, \ldots, y_n) \neq (x_1 - 1, \ldots, x_n - 1)$, we find that $\deg_{R'}(XD) > \frac{\ell}{d}(g(\Lambda) - 1)$ and conclude as before that D' is $X\Lambda$-equivalent to an effective.

If $(y_1, \ldots, y_n) = (x_1 - 1, \ldots, x_n - 1)$, then $\deg_{R'}(XD) = \frac{\ell}{d}(g(\Lambda) - 1)$ and $\deg_R(D) = g(\Lambda) - 1$. We claim that D' is $X\Lambda$-equivalent to an effective if and only if D is Λ-equivalent to an effective. It is clear that if D is Λ-equivalent to an effective, then D' is $X\Lambda$-equivalent to an effective. Suppose now that D has degree $g(\Lambda) - 1$, and is such that there does not exist $V \in \Lambda$ and $E \geq 0$ such that $D = V + E$. Consider $D' := XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)$. If there exist $E' \geq 0$ and $XV' \in X\Lambda$ such that $D' + XV' = E'$, then we find that $X(D + V') + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1) \geq 0$. This can only occur if all coefficients of $D' + V$ are nonnegative, which contradicts our assumption on D. Since there exists a D of degree $\deg_R(D) = g(\Lambda) - 1$ that is not Λ-equivalent to an effective, we find that $g(X\Lambda) = 1 + \frac{\ell}{d}(g(\Lambda) - 1) + \sum_{i=1}^{n}(x_i - 1)\ell r_i/\mathrm{d}x_i$.

(b) Note first that the map $\mathcal{N}(\Lambda) \to \mathcal{N}(X\Lambda)$, $[D] \mapsto [XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)]$, is well defined . We have shown in the proof of (a) that if $[D] \in \mathcal{N}(\Lambda)$, then $[XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)] \in \mathcal{N}(X\Lambda)$. Moreover, if $[D_1] = [D]$, then $[XD_1 + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)] = [XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)]$, since $D_1 - D \in \Lambda$ implies that $XD_1 - XD \in X\Lambda$. The map is injective, since if $[XD_1 + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)] = [XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)]$ for some D_1 and D, we have $XD_1 - XD \in X\Lambda$, which implies that $D_1 - D \in \Lambda$. We proved the surjectivity of the map in (a), since we showed that every D' that is not $X\Lambda$-equivalent to an effective is of the form $XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)$ for some D not Λ-equivalent to an effective.

(c) By hypothesis, the map $\mathcal{N}(\Lambda) \to \mathcal{N}(\Lambda)$, $[D] \mapsto [K - D]$ is well-defined and bijective. We leave it to the reader to check that this implies that the map $\mathcal{N}(X\Lambda) \to \mathcal{N}(X\Lambda)$, $[D'] \mapsto [XK + 2\,{}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1) - D']$, is well-defined and bijective.

(d) Let K' be a canonical vector for $K(X\Lambda)$. Then the map $\mathcal{N}(X\Lambda) \to \mathcal{N}(X\Lambda)$, $[D'] \mapsto [K' - D']$, is well defined and bijective. Starting with any $[D_0] \in \mathcal{N}(\Lambda)$, we obtain that there exists $[D] \in \mathcal{N}(\Lambda)$ such that $[K' - XD_0 - {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)] = [XD + {}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)]$. It follows that K' is $X\Lambda$-equivalent to $X(D_0 + D) + 2\,{}^{\mathrm{t}}(x_1 - 1, \ldots, x_n - 1)$. We leave it to the reader to show that $D_0 + D$ is a canonical vector for Λ. ■

Corollary 5.4. Let G be a graph with n vertices, m edges, and adjacency matrix A. Let $\mathcal{D} = \mathrm{diag}(d_1, \ldots, d_n)$ be the diagonal matrix of the valencies of the vertices, and let $M := \mathcal{D} - A$. Let $\Lambda_G := \mathrm{Im}(M) \subseteq \Lambda_{J_n}$, with canonical vector $K = {}^{\mathrm{t}}(d_1 - 2, \ldots, d_n - 2)$ and $|\mathrm{Pic}^0(\Lambda_G)| = \kappa(G)$. Choose X as in Definition 5.2. Then

(a) $g(X\Lambda_G) = 1 + \ell m - \sum_{i=1}^{n} \ell/x_i$.

(b) The lattice $X\Lambda_G$ has canonical vector ${}^{\mathrm{t}}(x_1 d_1 - 2, \ldots, x_n d_n - 2)$.

(c) The group $\mathrm{Pic}^0(X\Lambda_G)$ has order $\kappa(G)\big(\prod_{i=1}^n x_i\big)/\ell$. □

Proof. Since $\gcd(\ell/x_1, \ldots, \ell/x_n) = 1$, we may apply the previous proposition to obtain (a) and (b) with $R' = {}^{\mathrm{t}}(\ell/x_1, \ldots, \ell/x_n)$ and $g(\Lambda_G) = m - n + 1$.

The adjoint matrix $(XM)^* = M^* X^*$ is easy to compute. The matrix M^* has all its coefficients equal to $\kappa(G)$. The matrix X^* equals

$$\mathrm{diag}\left(\left(\prod_{i=1}^n x_i\right) \bigg/ x_1, \ldots, \left(\prod_{i=1}^n x_i\right) \bigg/ x_n\right).$$

The group $\mathrm{Pic}^0(X\Lambda_G)$ has order equal to the greatest common divisors of the coefficients of $(XM)^*$. This integer is computed as

$$|\mathrm{Pic}^0(X\Lambda_G)| = \kappa(G) \gcd\left(\left(\prod_{i=1}^n x_i\right) \bigg/ x_1, \ldots, \left(\prod_{i=1}^n x_i\right) \bigg/ x_n\right) = \kappa(G) \frac{(\prod_{i=1}^n x_i)}{\ell}. \qquad \blacksquare$$

When $|\mathrm{Pic}^0(X\Lambda_G)| = 1$, we find that $\kappa(G) = 1$ and $\ell = \prod_{i=1}^n x_i$; in particular, G is a tree, with $m = n - 1$. Let $R := {}^{\mathrm{t}}(\ell/x_1, \ldots, \ell/x_n)$. Then $X\Lambda_G = \Lambda_R$, and $g(X\Lambda_G) = g(R)$.

It turns out that the Frobenius number $g(R)$ is well understood already: when $\ell = \prod_{i=1}^n x_i$, the sequence $\ell/x_1, \ldots, \ell/x_n$, is strongly flat, and $g(R) = 1 + \ell(n-1) - \sum_{i=1}^n \ell/x_i$ (see [28, 3.2.2(b)], or the original reference [27]). Corollary 5.4(a) lets us then interpret $1 + \ell(n-1) - \sum_{i=1}^n \ell/x_i$ as a g-number even when $\ell < \prod_{i=1}^n x_i$; it is the g-number of $X\Lambda_G$ when G is a tree.

Corollary 5.5. Let $\Lambda \subseteq \Lambda_R$. Let $x > 0$ be an integer and consider $x\Lambda \subseteq \Lambda$. Then

(a) $g(x\Lambda) - 1 = x(g(\Lambda) - 1) + (x - 1)(\sum_{i=1}^n r_i)$.

(b) Λ has a canonical vector if and only if $x\Lambda$ has a canonical vector. □

Our next corollary shows that a search for a canonical vector for $\Lambda \subseteq \Lambda_R$ can be reduced to a similar search for a lattice in Λ_{J_n}.

Corollary 5.6. Let $R > 0$ and set $\tilde{R} := \mathrm{diag}(r_1, \ldots, r_n)$. Let $\Lambda \subseteq \Lambda_R$, and $\tilde{R}\Lambda \subseteq \Lambda_{J_n}$. Then

(a) $g(\tilde{R}\Lambda) = g(\Lambda) + \sum_{i=1}^n (r_i - 1)$.

(b) Λ has a canonical vector if and only if $\tilde{R}\Lambda$ has a canonical vector. □

Let $\Lambda \subset \mathbb{Z}^n$ be a lattice perpendicular to J_n. In [1], such a lattice is called a *sublattice of the root lattice* A_{n-1}, and the authors study the existence of a Riemann–Roch theorem for the following function h associated with Λ (this function is closely related with the h-function that Baker and Norine attach to a graph). Let $D \in \mathbb{Z}^n$. Define

$$|D| := \{E \in \mathbb{Z}^n \mid E \geq 0, [E] = [D] \text{ in } \mathrm{Pic}(\Lambda)\}.$$

Let (see [1, 2.1])

$$h(D) := \min\{\deg(E) \mid E \geq 0, |D - E| = \emptyset\}.$$

To a lattice Λ, Amini and Manjunath [1] associate in 2.4 two integers $g_{\min}(\Lambda) \leq g_{\max}(\Lambda)$. They define a lattice to be *uniform* when $g_{\min} = g_{\max} = g$ [1, 2.11]. They further introduce the notion of *reflection invariance* [1, 5.1], and show in 5.5 that a lattice that is both uniform and reflection invariant satisfies the Riemann–Roch theorem $h(K - D) - h(D) = g - 1 - \deg(D)$ for some canonical vector K. When the lattice is not uniform, they provide in [1, 5.2], a two-sided Riemann–Roch inequality for $h(K - D) - h(D) + \deg(D)$ and for some canonical vector K. It is shown in [1, 5.7], that a lattice satisfies a Riemann–Roch formula for the function h if and only if it is uniform and reflection invariant.

Let $g(\Lambda)$ denote the g-number as defined in Definition 2.1. We thank the referee for strengthening our original Proposition 5.7.

Proposition 5.7. Let $\Lambda \subset \mathbb{Z}^n$ be a lattice perpendicular to J_n. Then $g_{\max}(\Lambda) = g(\Lambda)$. □

Proof. Consider the set $S(\Lambda)$ of all divisors $D \in \mathbb{Z}^n$ which are not equivalent to an effective. By definition of $g(\Lambda)$, the maximum degree of a divisor in $S(\Lambda)$ is $g(\Lambda) - 1$. In [1, Definition 2.1], the authors define the Sigma-Region $\Sigma(\Lambda)$, and it follows from [1, 2.2(i)], that $\Sigma(\Lambda) = -S(\Lambda)$. A subset $\mathrm{Ext}(\Lambda) \subseteq \Sigma(\Lambda)$ of extremal points is defined in [1, Definition 2.5], and $g_{\max}(\Lambda) - 1$ is defined to be the maximum of the set $\{-\deg(D) \mid D \in \mathrm{Ext}(\Lambda)\}$ [1, Definition 2.9]. This proves $g_{\max}(\Lambda) \leq g(\Lambda)$.

The equality $g_{\max}(\Lambda) = g(\Lambda)$ follows from [1, Theorem 2.6]. Indeed, this theorem states that given any point $D \in \Sigma(\Lambda)$, there exists $A \in \mathrm{Ext}(\Lambda)$ such that $D - A \geq 0$. It follows that every element $B \in S(\Lambda)$ of degree $g(\Lambda) - 1$ is such that $-B$ is an extremal point in $\Sigma(\Lambda)$. Since there exists an extremal point of degree $-(g(\Lambda) - 1)$, the equality $g_{\max}(\Lambda) = g(\Lambda)$ follows from the definition of $g_{\max}(\Lambda)$. ■

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