PRIMES IN THREE CONSECUTIVE INTEGERS

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Consider a sequence of three consecutive integers $n-1, n, n+1$. We want to minimize the number of distinct primes appearing in the factorization of these three integers as follows.

- If n is even, we would like n to be a power of 2, and $n-1$ and $n+1$ each to be a power of a single prime. Examples are $(1, 2, 3)$, $(3, 4, 5)$, and $(7, 8, 9)$. These are in fact the only cases. Indeed, note that one of $n-1$ and $n+1$ is divisible by 3. So we are asking for $2^s - 3^r = \pm 1$, and Catalan's conjecture implies that $2^s = 8$, $3^r = 9$ is the only solution with $s, r > 1$.
- If n is odd, we would like it to be a power of a single prime, and both $n-1$ and $n+1$ to be powers of 2, or twice the power of a single prime, or four times the power of a single prime. More precisely, we want that $n = p^r$ for some odd prime p, and that $(n^2-1)/8$ has at most two distinct prime factors. We have the following possibilities:

$$n = 3, 5, 7, 9, 11, 13, 17, 19, 3^3, 37, 53, 107, 163, 3^5, 3^7, 2917, 4373, 8747, 3^{13}, 86093443.$$

  For instance, when $n = 8747$, $n-1 = 2 \cdot 4373$, and $n+1 = 4 \cdot 3^7$. When $n = 86093443$, $n-1 = 2 \cdot 3^{16}$, and $(n+1)/4$ is prime. As expected in all cases, $n-1$ or $n+1$ is divisible by 3, and thus is divisible by a large power of 3 when n is large.
- Any chance that this list of integers n is the complete list of all possible such n? Can you show that the number of such integers n must be finite?

The motivation for considering this question came from Remark 2.15 in my paper *Torsion and Tamagawa numbers*, Ann. Inst. Fourier **61** no. 5 (2011), 1995-2037.