An Avoidance Lemma and a Moving Lemma for families.

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September 2012

Plan of the talk

- An Avoidance Lemma for families.
- Application: Existence of finite quasi-sections
- A Moving Lemma for 1-cycles.
- Application: The index of an algebraic variety
- Some ideas of the method used.

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The results are joint work with O. Gabber and Q. Liu:

The index of an algebraic variety,
Invent. Math., Aug. 2012, 59 pages.

Hypersurfaces of projective schemes and a Moving Lemma,
preprint, 60 pages.

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Remark. Let X/k be an irreducible proper scheme. Let $C \subsetneq X$ be a proper closed subset, and let $\xi_1, \ldots, \xi_r$ be points of X not contained in C. It it always possible to find a closed subset H of X of codimension 1 which contains C and does not contain $\xi_1, \ldots, \xi_r$?

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Let S be any scheme and $X \to S$ any morphism. The subscheme H_f of X is called a hypersurface (relative to $X \to S$) when no irreducible component of positive dimension of X_s is contained in H_f, for all $s \in S$.

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Lemma. Let $X \to S$ be a projective morphism with S affine noetherian. Let $\mathcal{L}$ be ample on X. Let $f \in \mathcal{L}(X)$ be such that $H := H_f$ is a non-empty hypersurface on X.

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Theorem (Gabber-Liu-L.). Let S be affine. Let $X \to S$ be quasi-projective and finitely presented. Let C be a closed subscheme of X, proper and finitely presented over S. Let A be a finite subset of X such that $A \cap C = \emptyset$. Let F be a closed subscheme of X of finite presentation. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X.

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(3) For all $s \in S$, H_f does not contain any irreducible component of positive dimension of F_s.

If S is noetherian and $C \cap \mathrm{Ass}(X) = \emptyset$, then there exists such a hypersurface H_f which is locally principal.

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Proposition. Let S be a noetherian irreducible scheme. Let $\mathcal{L}$ be invertible, and consider $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$, with associated projective morphism $\pi : X \to S$. Let C_0 and C_∞ be the images of the two natural sections of π obtained from the projections $\mathcal{O}_S \oplus \mathcal{L} \to \mathcal{O}_S$ and $\mathcal{O}_S \oplus \mathcal{L} \to \mathcal{L}$.

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Answer to the above question: NO.

Let $S = \mathrm{Spec}\, R$ with R a Dedekind domain with $\mathrm{Pic}(S)$ containing an element $\mathcal{L}$ of infinite order. If H_f exists with $H_f \cap F = \emptyset$, then an irreducible component of H_f is finite and flat over S and does not meet F. This is not possible

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When $\dim S > 1$, the process of taking the closure of any closed point of the generic fiber does not always produce a closed subset *finite* over S.

Example. Let S be an irreducible scheme. Let $P \in S$ be a closed point, and consider the blowing up X of P in S, with associated projective morphism $X \to S$. When $\dim S > 1$, $X \to S$ is not finite, and $X \to S$ does not admit any finite quasi-section.

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Proposition (GLL). Assume that S is integral. Let K be the function field of S. Let $X \to S$ be a projective morphism. Assume that no fiber X_s, $s \in S$, contains a rational curve. Assume also that S is a noetherian excellent regular scheme.

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Proof of the existence of a finite quasi-section: Simply apply the Avoidance Lemma: $C \subset H_f \subset X$, with every fiber of H_f of dimension $d-1$. Repeat the process. □

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Example. S satisfies Condition (T) if S is an affine integral curve over a finite field, or if S is the spectrum of the ring of integers in a number field K.

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Here, rationally equivalent means that there exists a curve V on X containing C and C', and a function f on V, such that the cycle $C - C'$ is equal to the divisor of f on V.

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The minimal such m which works for all F is an interesting invariant attached to the local ring $\mathcal{O}_{X,x_0}$.

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Question. Is it possible to obtain the same moving lemma in a family $X \to S$, moving a finite quasi-section $C \to S$ away from a closed subset $F \to S$ such that for each $s \in S$, the codimension of F_s in the fiber X_s is at least 1?

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Then there exists a horizontal 1-cycle C' on X with $f_{|C'} : \mathrm{Supp}(C') \to S$ finite, rationally equivalent to C, and such that $\mathrm{Supp}(C') \cap F = \emptyset$.

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If the moving lemma holds for the morphism $X \to S$ with the 1-cycle C_0 and the closed set F, then there would exist a finite quasi-section $Y \to S$ that is disjoint from F. Hence, this can always occur only if $\mathrm{Pic}(S)$ is a torsion group. This motivates the definition:

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Recall **Condition (T)** for a Dedekind domain:

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Find interesting multiplicative subsets $T \subset \overline{\mathbb{Q}}[x]$ such that $T^{-1}\overline{\mathbb{Q}}[x]$ is pictorsion (and not semi-local).

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The moving lemma for $X \to S$ allows for F to be of codimension 1, as long as F_s has codimension 1 in X_s for all $s \in S$.

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When $X \to S$ has relative dimension 1 and $X \to S$ is projective, the closed set F to be avoided is finite over S. We exploit then the hypothesis that $\mathrm{Pic}(F)$ is torsion.

Application to the index

Let K be any field.

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Proposition (GLL). Let K be a Hilbertian field. Let V/K be an irreducible regular generically smooth algebraic variety of positive dimension. Then there exists $n_0 > 0$ such that

$$\{n\delta(V/K), n \geq n_0\} \subseteq \mathcal{D}(V/K).$$

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Example Consider the curve $\Gamma/\mathbb{R}$ given by $f(x, y) = x^2 + y^2$. It has a unique singular point $(0, 0)$, which is also the unique $\mathbb{R}$-rational point on Γ. Thus, $\delta(\Gamma/\mathbb{R}) = 1$, but $\delta(\Gamma^{reg}/\mathbb{R}) = 2$.

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Example. Consider the projective curve X_K over $K = \mathbb{R}((t))$ given by the equation $x^2 + y^2 + tz^2 = 0$. This equation defines a regular model over $\mathbb{R}[[t]]$, with integral special fiber $\Gamma/\mathbb{R}$ given by $x^2 + y^2 = 0$. So here $\delta(X_K/K) = 2 = r(\Gamma)\delta(\Gamma^{reg}/\mathbb{R})$, while $r(\Gamma)\delta(\Gamma/\mathbb{R}) = 1$.

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(2) Keep P; change the model X by blow-ups centered at points in the special fiber, so that in the new model Y, the closure of P in Y meets the new special fiber only in regular points. Show that $\gcd_i\{r_i\delta(\Gamma_i^{reg}/k)\}$ is invariant under blow-ups using the moving lemma for closed points on Γ_i.

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Fix a system of generators $f_1, \ldots, f_N$ of $H^0(X, \mathcal{I}(n))$. For $s \in S$, let $\Sigma(s) \subset \mathbb{A}^N(k(s))$ consists of all the vectors $(\alpha_1, \ldots, \alpha_N)$ such that $\sum_i \alpha_i f_{i|X_s}$ does NOT have the desired properties.

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We show then that all these subsets $\Sigma(s)$ are contained in a single constructible subset T of $\mathbb{A}^N/S$ (which depends on n).

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Theorem (GLL) *Let $S = \mathrm{Spec}\, R$ be a noetherian affine scheme. Let T be a constructible subset of $\mathbb{A}^N_S$. Suppose that:*

(1) $\dim(T) < N$.

(2) *For all $s \in S$, there exists a $k(s)$-rational point in $\mathbb{A}^N_{k(s)}$ which does not belong to T_s.*

Then there exists a section σ of $\pi : \mathbb{A}^N_S \to S$ such that $\sigma(S) \cap T = \emptyset$.

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The section σ whose existence follows from the proposition below provides the vector $(a_1, \ldots, a_N) \in R^N$ such that the section $f := \sum a_i f_i$ in $\mathcal{I} \otimes \mathcal{O}(n)$ has the desired properties.

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Moving Lemma for Families. Let $X \to S$ be a projective morphism with S a semi-local affine scheme. Given a horizontal cycle C finite over S, and a closed subset F, one can consider finding a rationally equivalent horizontal cycle C' which avoids F.

Thank you very much!

THE END