The Index of an Algebraic Variety

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Plan of the talk

- Definition of the index and examples.
- A completely different perspective on the index.
- Index in a local family.
- Index in a global family.
- Some moving lemmas.

The results in the talk are mostly joint work:

found in papers of Gabber-Liu-L.,
and papers of Liu-L.-Raynaud.

Definition of the index

Let K be any field.
Let X/K be any algebraic variety
(or any scheme of finite type over K).
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$\mu(X/K) :=$ smallest element in the set $\mathcal{D}(X/K)$.

$\delta(X/K) :=$ gcd of the elements of $\mathcal{D}(X/K)$.

The integer $\delta(X/K)$ is called the index of X/K.

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The index $\delta(X/K)$ of X/K is the gcd of the set $\mathcal{D}(X/K)$.

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This follows from the Weil bounds, which imply the existence of an integer $n_0 > 0$ such that X has a $\mathbb{F}_{p^n}$-point for all $n \geq n_0$, i.e.,

$$\mathcal{D}(X/K) \supset \{n, n \geq n_0\}.$$

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Conjecture (Cassels and Swinnerton-Dyer for $n = 4$, Coray (1975)). Assume that $d = 3$. Then $\delta(X_F/K) = 1$ implies $1 \in \mathcal{D}(X_F/K)$.

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Intersect F_p with the line $x + y + 1 = 0$:

$$(x+1)^p - x^p - 1 = x(x+1)(x^2+x+1)^b \cdot E_p(x),$$

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Putting these two conjectures together, we would find that

$$\mathcal{D}(F_p/\mathbb{Q}) = \{1, 2, \deg(E_p(x)), p-1, p, ?, \dots\}.$$

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Can the smallest such n_0 be bounded in terms of some geometrical invariants of X/K?

Modular curves

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Theorem (Merel). Let $d \neq 1, (p-1)/2$, be any integer. Then there exists $p_0 = p_0(d)$ such that for all primes $p \geq p_0(d)$, then $d \notin \mathcal{D}(X_1(p)/\mathbb{Q})$.

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$e(I,A)$ is an *integer*, the *Hilbert-Samuel multiplicity of I*.

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Is there an algorithm to compute $\gamma(A)$?

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of degree $d > 1$.

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There is a canonical map

$$Z \setminus \{P\} \longrightarrow X,$$

sending $(a, b, c) \mapsto (a : b : c)$.

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In particular, the sets $\mathcal{D}(X/K)$ and $\mathcal{E}(A)$ are distinct in general.

Theorem (GLL). Let X/K be a smooth projective curve of degree d. Let A be the local ring at the vertex of the cone Z/K. Then

$$\delta(X/K) = \gamma(A).$$

In other words, $\gcd(\mathcal{D}) = \gcd(\mathcal{E})$.

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Example Consider the curve $\Gamma/\mathbb{R}$ given by $f(x,y) = x^2 + y^2$. It has a unique singular point $(0,0)$, which is also the unique $\mathbb{R}$-rational point on Γ. Thus, $\delta(\Gamma/\mathbb{R}) = 1$, but $\delta(\Gamma^{reg}/\mathbb{R}) = 2$.

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Using the intersection of Cartier divisors with 1-cycles on the regular scheme $\mathcal{X}$, we easily find that $\gcd_i\{r_i\delta(\Gamma_i/k)\}$ divides $\delta(X/K)$. Our next theorem strengthens this divisibility.

Let K be a discrete valuation field, and $f : \mathcal{X} \to S$ be a one-dimensional local family as above.

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(a) Then $\gcd(r_i\delta(\Gamma_i^{reg}/k), i = 1, \ldots, n)$ divides $\delta(X/K)$.

(b) When $\mathcal{O}_K$ is Henselian, then

$$\delta(X/K) = \gcd\{r_i\delta(\Gamma_i^{reg}/k), i = 1, \ldots, n\}.$$

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So $f : \mathcal{X} \to V$ can be thought of as a 1-parameter family of curves, parameterized by $v \in V$.

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Question. How do the integers $\delta(X/K)$ and $\delta(X_{K_v}/K_v)$, $v \in V$, fit together?

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Conjecture of Birch-Swinnerton-Dyer (as in Tate's 1965 Bourbaki Seminar). The Shafarevich-Tate group $\text{Ш}(A)$ of A/K is finite.

Our next theorem generalizes the following:

Theorem (Milne, 1982). If $|\text{Ш}(A)|$ is finite and $\delta_v = 1$ for all v, then $|\text{Ш}(A)| = \delta(X/K)^2 |\mathrm{Br}(\mathcal{X})|$.

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Theorem (Liu-L.-Raynaud). Assume that $|Ш(A)|$ is finite. Then $|Ш(A)| \prod_v \delta_v \delta'_v = \delta(X/K)^2 |\mathrm{Br}(\mathcal{X})|$.

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Theorem (LLR). Let $\mathcal{X}/k$ be a smooth geometrically connected surface. Assume that for some prime ℓ, the ℓ-part of $\mathrm{Br}(\mathcal{X})$ is finite. Then $|\mathrm{Br}(\mathcal{X})|$ is a square.

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• The index of the generic fiber in a local family can be computed with data pertaining only to the special fiber.

• The indices of the fibers in a global family are expected to satisfy $|Ш(A)| \prod_v \delta_v \delta'_v = \delta(X/K)^2 |\mathrm{Br}(\mathcal{X})|$.

THANKS!

Mahalo!