QUESTIONS ON WILD $\mathbb{Z}/p\mathbb{Z}$-QUOTIENT SINGULARITIES IN DIMENSION 2

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1. Some questions

Let A denote a regular local ring of dimension 2 with maximal ideal $\mathcal{M}_A$. Let p be prime, and let $H = \mathbb{Z}/p\mathbb{Z}$ act on A. Let $\mathcal{Z} := \operatorname{Spec}(A^H)$. Assume that the action of H on $\operatorname{Spec}(A)$ is free off the closed point, and that $\mathcal{M}_A^H$ is the only singular point of $\mathcal{Z}$. When the residue characteristic of $A/\mathcal{M}_A$ is equal to p, $\mathcal{M}_A^H$ is called a *wild* cyclic quotient singularity.

Let $f : \mathcal{X} \to \mathcal{Z}$ be a resolution of the singularity, minimal with the property that the irreducible components of $f^{-1}(\mathcal{M}_A^H)$ are smooth with normal crossings. Attached to this resolution are two natural objects that we now describe, the *intersection matrix* N, and the *resolution graph* G. The exceptional divisor $f^{-1}(\mathcal{M}_A^H)$ consists in n irreducible components C_i, $i = 1, \dots, n$. Denote by $N := ((C_i \cdot C_j)_{\mathcal{X}})$ the associated symmetric matrix. The matrix N is negative-definite and, in particular, $\det(N) \neq 0$. Let G denote the graph whose vertices are the n irreducible components of $f^{-1}(\mathcal{M}_A^H)$, and where two vertices C and D are linked by $(C \cdot D)_{\mathcal{X}}$ edges.

For future reference, recall that the *degree* of a vertex C in a graph G is the number of edges connected to C, and a vertex of degree at least 3 on a graph is called a *node*. A vertex of degree 1 is a *terminal vertex*. The closure in G of a connected component of $G \setminus \{\text{all nodes of G}\}$ is called a *chain* of G. If the chain contains a terminal vertex, it is called a *terminal chain*.

Wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities of surfaces are expected to have resolution graphs which are trees, with associated intersection matrices N satisfying $\det(N) = p^s$ for some $s \geq 0$. The Smith group $\Phi_N := \mathbb{Z}^n/\operatorname{Im}(N)$ is expected to be killed by p ([1], 2.6). Finally, we should expect that the fundamental cycle Z of N has self-intersection $|Z^2| \leq p$. This latter combinatorial result follows from the algebraic result that the multiplicity of the singularity is expected to be at most p ([1], 2.3).

Question 1.1 Classify the matrices N which can occur as intersection matrices associated with minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension 2. This question is probably too broad to be useful. Here are some more focused sub-questions.

Minimal resolutions with graphs having only one node are exhibited in [2], 6.6.

a) Can you produce, for each p, examples of intersection matrices N associated with resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension 2 whose graphs $G(N)$ do not have a node?
b) Can you produce, for each p, examples of intersection matrices N associated with minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension 2 whose graphs $G(N)$ have more than one node? (One such example is exhibited in mixed characteristic 2 in [1], 4.10. See also question 1.2 (a)). Is there a bound on the possible number of nodes that such a graph can have?

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c) Can you produce, for each p and each $s \geq 0$, examples of intersection matrices N associated with minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension 2 with $|\det(N)| = p^s$ (examples with $s-1>0$ and coprime to p are given in [3], 3.12)?

Question 1.2 Is there a difference between the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities Spec A^H in dimension 2 when A is of equicharacteristic p, and the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities Spec A^H in dimension 2 when A is of mixed characteristic $(0,p)$? One related such difference is found in [1], 4.1. A second instance, related also to 1.1 (b), is in [1], 4.10, and prompts the following question:

(a) Produce discrete valuations fields of residue characteristic p, and curves X/K with potentially good reduction after an extension L/K of degree p, such that 1) the special fiber of the smooth model of X_L/L has p-rank 0, and 2) the graph of the special fiber of the regular model of X/K has more than one node. The case of elliptic curves E/K in equicharacteristic 2 with reduction of type I_n^* with $n>0$ is open.
(b) Is it possible to exhibit the Dynkin diagrams D_n (having n vertices) with $n \equiv 2 \mod 4$ as the resolution graphs of $\mathbb{Z}/2\mathbb{Z}$-quotients singularities in dimension 2 when A is of equicharacteristic 2?

Question 1.3 Is there a difference between the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension 2, and the set of intersection matrices associated with the minimal resolutions of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities occurring on the normal quotient $\mathcal{Z}$, model of a curve X/K with potentially good reduction after a degree p extension, as in [2], 5.3.

We suggest in 6.2 of [2] some extra structure that one may be able to attach to the intersection matrix in the case of a model of a curve. In particular, can the intersection matrix exhibited in [2], 6.14, occur as the intersection matrix associated with the minimal resolution of a wild $\mathbb{Z}/5\mathbb{Z}$-quotient singularity in dimension 2.

Question 1.4 Consider an intersection matrix N, and assume that for some prime p, it satisfies all known conditions that would have to be satisfied if this intersection matrix was associated with the resolution of a $\mathbb{Z}/p\mathbb{Z}$-singularity: its graph $G(N)$ is a tree, $\det(N)$ is a power of p and the Smith group is killed by p, and the fundamental cycle Z has $|Z^2| \leq p$.

If $\det(N) = 1$ and $G(N)$ is a tree, then the above conditions are satisfied for every prime at least equal to $|Z^2|$. In particular, when $\det(N) = 1$, the matrix N could potentially be associated with the resolution of a $\mathbb{Z}/p\mathbb{Z}$-singularity for infinitely many primes p. Can this actually happen? A related question: consider the existence of examples of such N occurring as $\mathbb{Z}/p\mathbb{Z}$-quotient singularities for more than one prime. See [2], 6.13, for a related discussion.

References

[1] D. Lorenzini, *Wild quotient singularities of surfaces*, Math. Zeit. **275** Issue 1 (2013), 211–232.
[2] D. Lorenzini, *Wild models of curves*, Algebra Number Theory **8** (2014), no 2, 331–367.
[3] D. Lorenzini, *Wild quotients of products of curves*, Preprint.

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