GROTHENDIECK'S PAIRING FOR JACOBIANS AND BASE CHANGE

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1. Introduction

Let K be a complete field with a discrete valuation v, ring of integers $\mathcal{O}_K$, and maximal ideal (π_K). Let $k := \mathcal{O}_K/(\pi_K)$ be the residue field, assumed to be separably closed of characteristic $p \geq 0$. Let A/K be any abelian variety of dimension g. Let $\mathcal{A}/\mathcal{O}_K$ denote its Néron model, with special fiber $\mathcal{A}_k/k$ and group of components $\Phi_{A,K}$. Let A'/K denote the dual abelian variety, with Néron model $\mathcal{A}'/\mathcal{O}_K$ and group of components $\Phi_{A',K}$. Grothendieck's pairing

$$\langle\ ,\ \rangle_K : \Phi_{A,K} \times \Phi_{A',K} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is introduced in [9], IX, 1.2. This pairing is known to be perfect in many cases ([2], [5], [6], [13], and [9], IX, 11.3 and 11.4, completed in [3] and [18]). The pairing is conjectured to be perfect when the residue field k is perfect. It was shown not to be perfect[1] in general in [4]. A further counter-example when k is not perfect is provided in 2.2.

Let X/K be a smooth proper geometrically connected curve of genus $g \geq 1$. Let J/K denote its Jacobian. Let $\mathcal{X}/\mathcal{O}_K$ denote a regular model of X/K. Let $\mathcal{X}_k = \sum_{i=1}^{v} r_i C_i$ be its special fiber, where C_i/k is an irreducible component of $\mathcal{X}_k$ of multiplicity r_i. Let $M = M(\mathcal{X}) = ((C_i \cdot C_j))$ denote the associated intersection matrix. When $\gcd(r_1, \ldots, r_v) = 1$ and the components of $\mathcal{X}_k$ are geometrically reduced, the group $\Phi_{J,K}$ is described in terms of the matrix M. Under the hypothesis that $X(K) \neq \emptyset$, the authors of [6] described how Grothendieck's pairing can be identified with an explicit pairing induced by M. Our goal in this paper is to prove that this identification remains possible in certain cases where $X(K) = \emptyset$. More precisely, we will show that it is possible to describe Grothendieck's pairing only in terms of M when X/K has a regular model $\mathcal{X}/\mathcal{O}_K$ whose special fiber has smooth components with normal crossings, and there exist two such components (C_i, r_i) and (C_j, r_j) with $\gcd(r_i, r_j) = 1$ intersecting in at least one point. As a consequence, we obtain that the pairing is perfect for the Jacobians of such curves. To prove our result, we apply the results of [6] to the curve X_F/F obtained after an appropriate base change F/K such that $X(F) \neq \emptyset$. The effect of the base change F/K on the model $\mathcal{X}$ and on Grothendieck's pairing are studied in sections 2 and 4.

We are very much indebted to A. Bertapelle for providing us with Example 2.4, and to Qing Liu for discussions leading to Lemma 3.5. We also thank the referee for valuable suggestions.

2. Grothendieck's pairing under base change

Let A/K denote any abelian variety. The pairing $\langle\ ,\ \rangle_K$ behaves very nicely under extensions of the ground field. Let F/K be any finite extension with residue field k_F. Denote by $\Phi_{A,F}$ the group of components of the Néron model of A_F/F. Let $e_{F/K}$ denote

Date: December 10, 2007.

[1]Note that it follows from the perfectness of the pairing that $\Phi_{A,K}$ and $\Phi_{A',K}$ are isomorphic. It is not known in general whether $\Phi_{A,K}$ and $\Phi_{A',K}$ are always isomorphic.

the ramification index of F/K, with $e_{F/K}[k_F : k] = [F : K]$. Let $\gamma : \Phi_{A,K} \to \Phi_{A,F}$ and $\gamma' : \Phi_{A',K} \to \Phi_{A',F}$ denote the natural maps induced by the base change map from $\mathcal{A} \times_{\mathcal{O}_K} \mathcal{O}_F$ to the Néron model of A_F/F, and from $\mathcal{A}' \times_{\mathcal{O}_K} \mathcal{O}_F$ to the Néron model of A'_F/F, respectively. The following key formula can be inferred from [9], VIII, (7.3.5.2) and (7.3.1.2). Let $x \in \Phi_{A,K}$ and $y \in \Phi_{A',K}$. Then

$$\langle \gamma(x), \gamma'(y) \rangle_F = e_{F/K} \langle x, y \rangle_K. \tag{2.0.1}$$

Our next proposition is an immediate consequence of this formula.

Proposition 2.1. *Let A/K be an abelian variety. Assume that Grothendieck's pairing $\langle\ ,\ \rangle_K$ is perfect. Then, for any finite extension F/K, the kernel $\Psi_{K,F}$ of the map $\gamma : \Phi_{A,K} \to \Phi_{A_F,F}$ is killed by $e_{F/K}$.*

Proof: Let $x \in \Psi_{K,F}$ and $y \in \Phi_{A',K}$. Then

$$\langle e_{F/K} x, y \rangle_K = e_{F/K} \langle x, y \rangle_K = \langle \gamma(x), \gamma'(y) \rangle_F = 0.$$

Since $\langle\ ,\ \rangle_K$ is perfect, $e_{F/K} x = 0$.

The group $\Psi_{K,F}$ is killed by $[F : K]$ even when $\langle\ ,\ \rangle_K$ is not perfect (see [8]). We show in 2.2 that the conclusion of Proposition 2.1 does not hold when $\langle\ ,\ \rangle_K$ is not assumed to be perfect. Example 2.4 shows that the converse of 2.1 does not hold in general when k is imperfect.

Example 2.2 The example below will show that when $\langle\ ,\ \rangle_K$ is not perfect, the conclusion of Proposition 2.1 does not hold in general. This example is also a new example of an abelian variety with $\langle\ ,\ \rangle_K$ not perfect.

Let $p = 2$. Consider an abelian variety A/K of dimension g which has good ordinary reduction over $\mathcal{O}_K$ (i.e., $\mathcal{A}_k$ is an ordinary abelian variety). Assume in addition that every point of order 2 in $\mathcal{A}_k(k)$ lifts to a point of order 2 in $A(K)$. Pick a Galois extension L/K of degree 2 with associated residue field extension k_L/k. Consider the twist[2] B/K of A/K obtained from the natural map $\mathrm{Gal}(L/K) \to \{\pm \mathrm{id}\} \subset \mathrm{Aut}(A/K)$. The abelian variety B/K has purely additive reduction. Indeed, $A \times B$ is isogenous over K to the Weil restriction $R_{L/K}(A_L)$ ([14], Prop.7), and this Weil restriction has abelian rank over K equal to the abelian rank of A/K since the kernel of the norm map $R_{L/K}(A_L) \to A$ has unipotent reduction (see, e.g., [8], proof of Thm. 1).

Since the points of order 2 of $A(K)$ are invariant under the action of $\mathrm{Gal}(L/K)$, we find that $B[2](K)$ contains a subgroup C of order 2^g. Indeed, consider an isomorphism $\rho : B_L \to A_L$ defined over L. Then the map $c : G \to \mathrm{Aut}(A/K)$ given by $c_\sigma := \rho^\sigma \circ \rho^{-1}$ is the cocycle giving the twist B/K. If P is a point of order 2 in $A(K)$, then $\rho^{-1}(P)$ has order 2 in $B(\overline{K})$. To show that $\rho^{-1}(P)$ belongs to $B(K)$, we note that since P is a fixed point of the inverse map, we must have $c_\sigma(P) = P$ for all σ. Thus, $\rho^{-1}(P) = (\rho^{-1}(P))^\sigma$ for all σ, and $\rho^{-1}(P) \in B(K)$. We claim that the natural reduction map $\mathrm{red} : B(K) \to \Phi_{B,K}$ is not trivial when restricted to C, which implies that $\Phi_{B,K} \neq (0)$. Indeed, let $\mathcal{B}_k^0$ denote the connected component of the special fiber of the Néron model of B/K over $\mathcal{O}_K$. The group scheme $\mathcal{B}_k^0$ is unipotent. If $\mathrm{red}(C) \subseteq \mathcal{B}_k^0$, we find that the image of C under the reduction map $B(L) \to \mathcal{B}_{L,k_L}$, where $\mathcal{B}_L/\mathcal{O}_L$ is the Néron model of B_L/L, is trivial, contradicting the hypothesis that the points of C reduce to the points of order 2 in $\mathcal{B}_{L,k_L} = \mathcal{A}_k \times_k k_L$. Thus, $\Phi_{B,K} \neq (0)$.

[2] In the equicharacteristic case with $g = 1$, we can choose A/K to be given by a Weierstrass equation $y^2 + xy = x^3 + 1$, with 2-torsion point $(0, 1)$. When $L = K(z)$ with $z^2 + z + D = 0$, $D \in K$, the twist is $y^2 + xy = x^3 + Dx^2 + 1$.

Now choose L/K such that $e_{L/K} = 1$. By construction, the map $\Phi_{B,K} \to \Phi_{B,L} = (0)$ is not injective. Hence, $\mathrm{Ker}(\Phi_{B,K} \to \Phi_{B,L})$ is not killed by $e_{L/K}$. Thus, $\langle\ ,\ \rangle_K$ is not perfect.

Remark 2.3 In the above example, the group of components of the abelian variety $R_{L/K}(A_L)$ is isomorphic to the group $\Phi_{A_L,L}$ and, thus, is trivial. In particular, Grothendieck's pairing for $R_{L/K}(A_L)$ is perfect. On the other hand, Grothendieck's pairing for $A \times B$ is not perfect, as the example above shows, even though $A \times B$ and $R_{L/K}(A_L)$ are isogenous.

Example 2.4 Let K'/K be an extension of degree $n > 1$ and ramification index $e = 1$ (in particular, k is imperfect). Let B/K' be an abelian variety with semi-stable reduction. Let A/K denote the Weil restriction of B/K' to K. From [4], Corollary 2.2, it is possible to choose B/K' such that Grothendieck's pairing is not perfect for A/K and its dual.

We claim that, nevertheless, the morphism $\Phi_{A,K} \to \Phi_{A_F,F}$ is injective (in particular, killed by $e_{F/K}$) for all finite extensions F/K. First, the morphism $\Phi_{A,K} \to \Phi_{A_{K'},K'}$ can be identified with the diagonal embedding $\Phi_{B,K'} \to (\Phi_{B,K'})^n$, and is thus injective ([4], Lemma 2.2). Since $A_{K'}/K'$ has semi-stable reduction, we find that the morphism $\Phi_{A_{K'},K'} \to \Phi_{A_{K'F},K'F}$ is always injective. It follows that the composition $\Phi_{A,K} \to \Phi_{A_{K'},K'} \to \Phi_{A_{K'F},K'F}$ is injective, and so is $\Phi_{A,K} \to \Phi_{A_F,F} \to \Phi_{A_{K'F},K'F}$. Hence, we find that the first morphism in this latter composition is injective, as desired.

3. The main result

Let X/K be a smooth geometrically connected proper curve of genus g. Let J/K denote the Jacobian of X/K, let $\mathcal{J}$ denote its Néron model over $\mathcal{O}_K$, and let $\Phi_{J,K}$ be the associated component group. The latter is a finite étale k-group scheme and, thus, constant, as k is separably closed. We will write the special fiber $\mathcal{X}_k/k$ as a Weil divisor $\mathcal{X}_k = \sum_C r(C)C$, where C runs through the irreducible components of $\mathcal{X}_k$, and where $r(C)$ is the multiplicity of C in $\mathcal{X}_k$.

3.1 We assume from now on that X/K has a regular model $\mathcal{X}/\mathcal{O}_K$ such that all irreducible components of $\mathcal{X}_k$ are smooth and such that $(\mathcal{X}_k)^{red}$ has normal crossings. In particular, all components are geometrically reduced and all intersection points of components are k-rational. Such a model always exists when k is algebraically closed, but if k is only separably closed and imperfect, not all curves possess such a model. We also assume that $\gcd(r(C), C \subset \mathcal{X}_k) = 1$.

3.2 Our hypotheses that $\gcd(r(C), C \subset \mathcal{X}_k) = 1$ and all components are geometrically reduced imply that $\delta_{\mathcal{X}/\mathcal{O}_K} = 1$, so that the hypothesis (i) in [17], (8.2.1), is satisfied. Thus, $\mathcal{X}/\operatorname{Spec}(\mathcal{O}_K)$ is cohomologically flat ([17], (8.2.1) (iv)). Theorem (8.1.4) (b) in [17] implies that $Q^\tau/\mathcal{O}_K$ is the Néron model of its generic fiber, J/K. We can then use [17], (8.1.2) (iii), to obtain a description of the group of components $\Phi_{J,K}$ in terms of M, once we note that when $\mathcal{X}$ is regular, the group schemes $Q^\tau/\mathcal{O}_K$ in (8.1.4) (b) and $Q'/\mathcal{O}_K$ in (8.1.2) (iii) are equal (the group scheme $Q'/\mathcal{O}_K$ is denoted by $Q/\mathcal{O}_K$ in the reference [7]). Let us now recall the description of $\Phi_{J,K}$ in terms of M.

Writing $\mathbb{Z}^I$ for the free $\mathbb{Z}$-module generated by the irreducible components C of $\mathcal{X}_k$, we consider the complex of $\mathbb{Z}$-modules $\mathbb{Z}^I \xrightarrow{\alpha} \mathbb{Z}^I \xrightarrow{\beta} \mathbb{Z}$, where the $\mathbb{Z}$-linear maps α, β are given by $\alpha(D) := \sum_C (D \cdot C)C$, and $\beta(C) := r(C)$. The quotient $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ is canonically identified with the component group $\Phi_{J,K}$. To be more precise, let us introduce the degree map $\rho\colon \operatorname{Pic}(\mathcal{X}) \longrightarrow \mathbb{Z}^I$, with $\mathcal{L} \longmapsto \sum_C \deg_C(\mathcal{L})C$, where $\deg_C(\mathcal{L})$ denotes the degree of a line bundle $\mathcal{L}$ on the component C. Let $P(\mathcal{X})$ be the subgroup in $\operatorname{Pic}(\mathcal{X})$ consisting of all line bundles of total degree 0 on $\mathcal{X}$. The following diagram is commutative:

$$\begin{array}{ccccc} P(\mathcal{X}) & \xrightarrow{\text{res}} & \text{Pic}^0(X) & = & J(K) = \mathcal{J}(\mathcal{O}_K) \\ \rho\downarrow & & & & \downarrow \\ \text{Ker}\,(\beta) & \longrightarrow & \text{Ker}\,(\beta)/\text{Im}\,(\alpha) & = & \Phi_{J,K}, \end{array}$$

where the vertical map on the right is the natural composition $\mathcal{J}(\mathcal{O}_K) \to \mathcal{J}_k(k) \to \Phi_{J,K}$. That $\text{Pic}^0(X) = J(K)$ follows from the fact that $\delta_{\mathcal{X}/\mathcal{O}_K} = 1$ ([17], 7.1.4).

Thus, given any point $a_K \in J(K)$, its image in $\text{Ker}\,(\beta)/\text{Im}\,(\alpha)$ is constructed as follows. Choose a divisor D_K of degree 0 on X representing a_K. Consider the schematic closure D of D_K in X, and let $[D]$ be the line bundle on $\mathcal{X}$ associated to the Weil divisor D. Then the image of a_K in $\Phi_{J,K}$ is given by the class of $\rho([D])$ in $\text{Ker}\,(\beta)/\text{Im}\,(\alpha)$.

In order to describe the above maps in terms of matrices, choose a numbering $C_1, \ldots, C_v$ of the irreducible components of the special fibre $\mathcal{X}_k$, and consider the intersection matrix $M := (C_i \cdot C_j)_{1 \le i,j \le v}$, and the vector of multiplicities $R = {}^t(r_1, \ldots, r_v)$ with $r_i := r(C_i)$. Then $\alpha\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}^v$ and $\beta\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}$ are given by the matrices M and ${}^tR$. We thus obtain

$$\Phi_M := \text{Ker}\,({}^tR)/\text{Im}\,(M) = (\mathbb{Z}^v/\text{Im}\,(M))_{\text{tors}} = \text{Ker}\,\beta/\text{Im}\,\alpha$$

as the component group of the Jacobian of X/K.

3.3 We may now consider the pairing $\langle\ ,\ \rangle_M\colon \Phi_M \times \Phi_M \to \mathbb{Q}/\mathbb{Z}$ attached to M in section 1 of [6] (see 4.4). Since the component group $\Phi_{J,K}$ of the Jacobian J of X is canonically identified with Φ_M, the pairing $\langle\ ,\ \rangle_M$ on Φ_M, gives rise to a well-defined symmetric pairing, again denoted by $\langle\ ,\ \rangle_M$:

$$\langle\ ,\ \rangle_M\colon \Phi_{J,K} \times \Phi_{J,K} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

which is independent of the chosen numbering of the components of $\mathcal{X}_k$. The pairing $\langle\ ,\ \rangle_M$ is perfect ([6], 1.3).

Assume that $X(F) \neq \emptyset$, and let $P \in X(F)$. Let $h\colon X_F \longrightarrow J_F$, $Q \longmapsto [Q] - [P]$, be the associated map from X_F into its Jacobian. We write $\mathcal{M}$ for the universal line bundle on $X_F \times J_F$ (satisfying $\mathcal{M}|_{\{P\}\times J_F} = 0$ and $\deg \mathcal{M}|_{X_F \times \{y\}} = 0$ for all points y of J_F) and $\mathcal{P}$ for the Poincaré bundle on $J_F \times J'_F$, where J'_F is the dual of J_F. There is a unique morphism $h'\colon J'_F \longrightarrow J_F$ satisfying $(\text{id} \times h')^*\mathcal{M} = (h \times \text{id})^*\mathcal{P}$ on $X_F \times J'_F$. It is given by the pull-back of line bundles with respect to $h\colon X_F \longrightarrow J_F$ and is an isomorphism (see for instance [15], Thm. 6.9).

To describe the inverse of h', we consider the maps $h^{(i)}\colon X_F^{(i)} \longrightarrow J_F$, $i \in \mathbb{N}$, induced from h, where $X_F^{(i)}$ is the i-fold symmetric product of X_F. The image of $h^{(g-1)}$ gives rise to a divisor Θ on J_F, the so-called theta divisor, and one knows that the morphism

$$\varphi_{[\Theta]}\colon J_F \longrightarrow J'_F, \qquad a_F \longmapsto [T_{a_F}^{-1}\Theta] - [\Theta],$$

is an isomorphism. In fact, $-\varphi_{[\Theta]}$ and h' are inverse to each other by [15], Thm. 6.9. Also note that $\varphi_{[\Theta]}$ and, hence, h' are independent of the choice of the rational point P on X_F, as any change of P leads to a translate of Θ. Thus, $\varphi_{[\Theta]}$ is already defined over K by descent theory ([7], page 261).

In the remainder of this paper, we will always identify J'/K with J/K using the isomorphism $-\varphi_{[\Theta]} : J \to J'$ defined over K. Induced by this identification is an identification of the corresponding Néron models and of their component groups, so that Grothendieck's pairing associated to J and J' becomes a pairing

$$\langle\ ,\ \rangle_K\colon \Phi_{J,K} \times \Phi_{J,K} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Theorem 3.4. *Let J/K be the Jacobian of a smooth proper and geometrically connected curve X/K. Identify J/K with its dual J'/K via the map $-\varphi_{[\Theta]}\colon J \to J'$ introduced above. Assume that X/K has a proper flat $\mathcal{O}_K$-model $\mathcal{X}$ as in 3.1. Assume in addition the existence of two irreducible components C and D in $\mathcal{X}_k$ with $(C \cdot D) > 0$ and $\gcd(r(C), r(D)) = 1$. Let M be the intersection matrix of $\mathcal{X}_k$. Identify the component group $\Phi_{J,K}$ with the group Φ_M as in 3.2. Then Grothendieck's pairing*

$$\langle\ ,\ \rangle_K \colon \Phi_{J,K} \times \Phi_{J,K} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

coincides with the pairing $\langle\ ,\ \rangle_M \colon \Phi_{J,K} \times \Phi_{J,K} \longrightarrow \mathbb{Q}/\mathbb{Z}$ considered in 3.3.

We start with the following lemma.

Lemma 3.5. *Let X/K be a smooth geometrically connected proper curve of genus g. Let $\mathcal{X}/\mathcal{O}_K$ be a regular model for X/K. Assume that two smooth components C and D of $\mathcal{X}_k$, of multiplicity r and s respectively, intersect with normal crossings in at least one point of intersection P. Suppose that $\gcd(r, s) = 1$. Let $n = \alpha r + \beta s$ for some integers $\alpha, \beta > 0$. Then there exists an uniformizer π for $\mathcal{O}_K$ such that $X(F) \neq \emptyset$, with $F := K(\sqrt[n]{\pi})$. In particular, for all $n \geq rs - r - s + 1$, there exists an extension F/K of degree n with $X(F) \neq \emptyset$.*

Proof: Consider the local ring $\mathcal{O}_{\mathcal{X},P}$. Its completion is isomorphic to $\mathcal{O}_K[[x, y]]/(x^r y^s - \pi)$, for some uniformizer π. Let $\pi_F := \sqrt[n]{\pi}$. Then $x \mapsto \pi_F^\alpha$ and $y \mapsto \pi_F^\beta$ define an F-rational point of X reducing to P. It is well-known that when $\gcd(r, s) = 1$, the set $\{\alpha r + \beta s \mid \alpha, \beta \geq 0\}$ contains the set $\{n \in \mathbb{N} \mid n \geq rs - r - s + 1\}$. □

Proof of Theorem 3.4. Lemma 3.5 lets us choose a prime $\ell \neq p$ and an extension F/K of degree ℓ such that ℓ is coprime to $|\Phi_{J,K}|$ and to all multiplicities of the model $\mathcal{X}/\mathcal{O}_K$, and such that $X(F) \neq \emptyset$.

We need to show that for all $x, y \in \Phi_{J,K}$, $\langle x, y\rangle_K = \langle x, y\rangle_M$. Since our chosen prime ℓ is coprime to the order of $\Phi_{J,K}$, we find that it suffices to show that for all $x, y \in \Phi_{J,K}$, $\ell\langle x, y\rangle_K = \ell\langle x, y\rangle_M$.

Consider the morphism $\gamma : \Phi_{J,K} \to \Phi_{J_F,F}$ introduced in section 2. We find from (2.0.1) that for all $x, y \in \Phi_{J,K}$,

$$\langle\gamma(x), \gamma(y)\rangle_F = \ell\langle x, y\rangle_K. \tag{3.5.1}$$

Let $\mathcal{W} \to \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathcal{O}_F$ be the normalization map. Let $\mathcal{V} \to \mathcal{W}$ denote the minimal desingularization of $\mathcal{W}$. The model $\mathcal{V}/\mathcal{O}_F$ is a regular model of X_F/F. As we recall in 4.2, all components of $\mathcal{V}_k$ are geometrically reduced. Let $M(\mathcal{V})$ denote the intersection matrix associated with the special fiber of $\mathcal{V}/\mathcal{O}_F$. Since $X(F) \neq \emptyset$, Theorem 4.6 in [6] implies that

$$\langle\gamma(x), \gamma(y)\rangle_F = \langle\gamma(x), \gamma(y)\rangle_{M(\mathcal{V})}. \tag{3.5.2}$$

To conclude the proof of our theorem, it suffices to show that

$$\langle\gamma(x), \gamma(y)\rangle_{M(\mathcal{V})} = \ell\langle x, y\rangle_M. \tag{3.5.3}$$

This formula is proved in 4.5 and 4.6. □

Corollary 3.6. *Let X/K be as in Theorem 3.4. Then the associated Grothendieck's pairing $\langle\ ,\ \rangle_K \colon \Phi_{J,K} \times \Phi_{J,K} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is perfect.*

Proof: This follows from the fact that $\langle\ ,\ \rangle_M \colon \Phi_{J,K} \times \Phi_{J,K} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is always perfect. □

Remark 3.7 We assume in 3.4 the existence of a model with $\gcd(r_1, \dots, r_v) = 1$ and smooth components. When $g = 1$, the existence of such a model already implies that $X(K) \neq \emptyset$.

When $g = 2$ and k is algebraically closed, $X(K)$ is empty only when the reduction, following the notation of [16], is of type IV (p. 155), $II^* - II^* - \alpha$ (p. 163), $III^* - II_0$ (p. 178), or III_n^* (p. 184). But in each case, the reader may check that the associated group Φ_M is trivial, so that Grothendieck's pairing is also trivial (and determined in terms of the intersection matrix as in 3.4). Note that IV and III_n^* do not satisfy the hypothesis of 3.4 on the existence of two intersecting components with $\gcd(r(C), r(D)) = 1$.

4. Base change for models of curves

Let X/K be a smooth proper geometrically connected curve with a regular model as in 3.1.

4.1 Let F/K be an extension of prime degree $\ell \neq p$ such that ℓ is coprime to the multiplicity of any component of $\mathcal{X}_k$. Consider the normalization $\mathcal{W} \to \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathcal{O}_F$. Let $\mathcal{V} \to \mathcal{W}$ denote the minimal desingularization of $\mathcal{W}$. The model $\mathcal{V}/\mathcal{O}_F$ can be explicitly described in terms of the model $\mathcal{X}/\mathcal{O}_K$. Such a description is classical in equicharacteristic zero, and a similar description holds in our context (see for instance [10], 4.3).

Let (C, r) be a component of $\mathcal{X}_k$ of multiplicity r, and denote by (C_i, r_i), $i = 1, \dots, d$, the components of $\mathcal{X}_k$ that intersect C. We have the relation

$$|C \cdot C| r = r_1 + \cdots + r_d.$$

We let C' denote the preimage of C in $\mathcal{W}$ under the natural map $\mathcal{W} \to \mathcal{X}$. Similarly, let C_i' be the preimage of C_i in $\mathcal{W}$. Since ℓ is coprime to the multiplicity of any component of $\mathcal{X}_k$, we find that C' and C_i' are irreducible of multiplicity r and r_i, respectively (for all $i = 1, \dots, d$), and the maps $C' \to C$ and $C_i' \to C_i$ are birational.

For each $i = 1, \dots, d$, the preimage P_i' in $\mathcal{W}$ of the intersection point $P_i := C \cap C_i$ consists of a single point. The point P_i' is singular on $\mathcal{W}$ because $[F : K]$ is coprime to both r and r_i. Denote by C'' and C_i'' the strict transforms in $\mathcal{V}$ of C' and C_i', respectively.

4.2 The singularity P_i' is resolved by a chain of smooth rational curves that we now describe. Let $q_1(i)$ denote the smallest positive integer such that $\ell \mid q_1(i) r + r_i$. Since $\ell \nmid r r_i$ by hypothesis, $1 \leq q_1(i) < \ell$.

If $q_1(i) = 1$, then the singularity P_i is resolved by a single rational curve $D_1(i)$ of multiplicity $(r + r_i)/\ell$ in $\mathcal{V}_k$, with $(D_1(i) \cdot D_1(i))_{\mathcal{V}} = -\ell$. Moreover, $(C'' \cdot D_1(i))_{\mathcal{V}} = (D_1(i) \cdot C_i'')_{\mathcal{V}} = 1$.

When $q_1(i) > 1$, the pair $(\ell, q_1(i))$ uniquely determines the following $(n_i \times n_i)$-square matrix N_i and vector ${}^t R_i = (q_1(i), q_2(i), \dots, q_{n_i}(i))$, where the coefficients $b_j(i)$ and $q_j(i)$ are positive integers, with $\ell > q_1(i) > \cdots > q_{n_i}(i) = 1$:

$$\begin{pmatrix} -b_1(i) & 1 & 0 & \dots & 0 \\ 1 & -b_2(i) & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -b_{n_i-1}(i) & 1 \\ 0 & \dots & 0 & 1 & -b_{n_i}(i) \end{pmatrix} \begin{pmatrix} q_1(i) \\ q_2(i) \\ \vdots \\ \vdots \\ q_{n_i}(i) = 1 \end{pmatrix} = \begin{pmatrix} -\ell \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

In $\mathcal{V}$, the preimage of P_i' consists in a chain of rational curves $D_j = D_j(i)$, $j = 1, \dots, n_i$, such that

$$(C'' \cdot D_1)_{\mathcal{V}} = (D_j \cdot D_{j+1})_{\mathcal{V}} = (D_{n_i} \cdot C_i'')_{\mathcal{V}} = 1$$

for all $j = 1, \dots, n_i - 1$. Moreover, for all $j = 1, \dots, n_i$,

$$(D_j \cdot D_j)_{\mathcal{V}} = -b_j(i).$$

The resolution of P_i' can be computed explicitly because the completion of the local ring $\mathcal{O}_{\mathcal{W},P_i'}$ is isomorphic to $\mathcal{O}_F[[x,y]]/(x^r y^{r_i} - \pi_F^{\ell})$. The complex case is discussed in [1], ii) after 5.2. See also [11], pp. 206-212.

The multiplicity of $D_j(i)$ in $\mathcal{V}_k$ is described as follows. Given the matrix N_i, we can uniquely determine a vector $R_i' = (q_{n_i}(i)', \dots, q_2(i)', q_1(i)')$, with $\ell > q_1(i)' > \dots > q_{n_i}' = 1$, and

$$N_i R_i' = {}^t(0, \dots, 0, -\ell).$$

We find that $\det(N_i) = \pm\ell$, and that $q_1(i)q_1(i)' \equiv 1 \pmod{\ell}$ ([12], 2.6). When $n_i > 1$, there exists a unique vector T such that $N_i T = {}^t(-r, 0, \dots, 0, -r_i)$, and it is easy to check that

$$N_i(\frac{r}{\ell}R_i + \frac{r_i}{\ell}R_i') = {}^t(-r, 0, \dots, 0, -r_i).$$

We note that the coefficients of the vector $\frac{r}{\ell}R_i + \frac{r_i}{\ell}R_i'$ are integers; this can be checked by induction, using the fact that the first coefficient $(q_1(i)r + r_i)/\ell$ is an integer by construction. The multiplicity of D_j in $\mathcal{V}_k$ is the j-th coefficient of the vector $(\frac{r}{\ell}R_i + \frac{r_i}{\ell}R_i')$. We have

$$|C'' \cdot C''| = \frac{\sum_{i=1}^{d}(q_1(i)r + r_i)}{r\ell} = \frac{(\sum_{i=1}^{d} q_1(i)) + |C \cdot C|}{\ell}.$$

4.3 Consider the intersection matrix $M(\mathcal{X})$ as a linear map $\oplus_{C\subseteq\mathcal{X}_k}\mathbb{Z}C \to \oplus_{C\subseteq\mathcal{X}_k}\mathbb{Z}C$. Similarly, consider the matrix $M(\mathcal{V})$ as a linear map $\oplus_{B\subseteq\mathcal{V}_k}\mathbb{Z}B \to \oplus_{B\subseteq\mathcal{V}_k}\mathbb{Z}B$. Let $\Phi_{M(\mathcal{X})}$ and $\Phi_{M(\mathcal{V})}$ denote the torsion subgroup of $(\oplus_{C\subseteq\mathcal{X}_k}\mathbb{Z}C)/\mathrm{Im}(M(\mathcal{X}))$ and $(\oplus_{B\subseteq\mathcal{V}_k}\mathbb{Z}B)/\mathrm{Im}(M(\mathcal{V}))$, respectively. Consider the purely combinatorial group homomorphism

$$\varphi : (\oplus_{C\subseteq\mathcal{X}_k}\mathbb{Z}C) \longrightarrow (\oplus_{B\subseteq\mathcal{V}_k}\mathbb{Z}B), \quad C \mapsto C''.$$

We claim that this homomorphism induces a homomorphism $\varphi : \Phi_{M(\mathcal{X})} \longrightarrow \Phi_{M(\mathcal{V})}$. To prove this claim, it suffices to exhibit, for each relation $M(\mathcal{X})S = T$, a vector S' such that $M(\mathcal{V})S' = \varphi(T)$. Given $M(\mathcal{X})S = T$, an appropriate vector S' can be constructed as follows. Let C be any component of $\mathcal{X}_k$, and let $s(C)$ denote the coefficient of S corresponding to C. Set the coefficient $s'(C'')$ of S' corresponding to C'' to be $\ell s(C)$. If C_i is a component of $\mathcal{X}_k$ that meets C, let $D_1(i), \dots, D_{n_i}(i)$ denote the components of $\mathcal{V}_k$ of the desingularization of $C' \cap C_i'$. Then set the coefficient $s'(D_j(i))$ of S' corresponding to $D_j(i)$ to be:

$$s'(D_j(i)) := j\text{-th coefficient of } s(C)R_i + s(C_i)R_i'.$$

(When $q_1(i) = 1$, we have $s'(D_1(i)) = s(C) + s(C_i)$.) The reader will check that the vector S' is well-defined, and that $M(\mathcal{V})S' = \varphi(T)$.

4.4 Given a $(v \times v)$-intersection matrix M, recall the pairing $\langle\ ,\ \rangle_M : \Phi_M \times \Phi_M \longrightarrow \mathbb{Q}/\mathbb{Z}$ defined in [6], 1.1. Let τ_1 and τ_2 denote two elements of Φ_M represented by vectors T_1 and T_2 in $\mathbb{Z}^v$. Pick two vectors S_1 and S_2 such that there exist non-zero integers s_1 and s_2 with $MS_1 = s_1T_1$ and $MS_2 = s_2T_2$. Then, by definition,

$$\langle\tau_1, \tau_2\rangle_M \equiv {}^t(S_1/s_1)M(S_2/s_2) \pmod{\mathbb{Z}}.$$

Lemma 4.5. *Consider the matrices $M(\mathcal{X})$ and $M(\mathcal{V})$, and the associated pairings $\langle\ ,\ \rangle_{M(\mathcal{X})}$ and $\langle\ ,\ \rangle_{M(\mathcal{V})}$. Let $\varphi : \Phi_{M(\mathcal{X})} \to \Phi_{M(\mathcal{V})}$ denote the homomorphism defined in 4.3. Then, for all τ_1 and τ_2 in $\Phi_{M(\mathcal{X})}$,*

$$\langle\varphi(\tau_1), \varphi(\tau_2)\rangle_{M(\mathcal{V})} = \ell\langle\tau_1, \tau_2\rangle_{M(\mathcal{X})}.$$

Proof: Let T_1 and T_2 be vectors in $\mathbb{Z}^v$ representing τ_1 and τ_2. Pick a vector S_1 such that there exists an integer s_1 with $MS_1 = s_1T_1$. Then

$$\langle\tau_1, \tau_2\rangle_{M(\mathcal{X})} \equiv {}^t(S_1/s_1)T_2 \pmod{\mathbb{Z}}.$$

Let S_1' be the vector associated as in 4.3 to the vector S_1, so that $M(\mathcal{V})S_1' = \varphi(s_1T)$. Then

$$\langle\varphi(\tau_1), \varphi(\tau_2)\rangle_{M(\mathcal{V})} \equiv {}^t(S_1'/s_1)\varphi(T_2) \pmod{\mathbb{Z}}.$$

To prove our lemma, it suffices to note that the coordinates of $\varphi(T_2) \in \oplus\mathbb{Z}B$ are zero, except for those which correspond to a coordinate C of $\oplus\mathbb{Z}C$. By construction, the coordinate of the vector S_1' corresponding to the basis vector C of $\oplus\mathbb{Z}C$ is $\ell s(C)$, where $s(C)$ denote the coordinate of C of the vector S_1. Our claim follows immediately. □

4.6 To conclude the proof of 3.4, it suffices to note the following fact. As recalled in 3.2, the group $\Phi_{J,K}$ can be canonically identified with $\Phi_{M(\mathcal{X})}$, and $\Phi_{J_F,F}$ can be identified with $\Phi_{M(\mathcal{V})}$. We claim that under these natural identifications, the morphism $\varphi : \Phi_{M(\mathcal{X})} \to \Phi_{M(\mathcal{V})}$ defined in 4.3 is identified with the natural morphism $\psi : \Phi_{J,K} \to \Phi_{J_F,F}$ introduced in section 2. To see this, we lift an element x of $\Phi_{J,K}$ to an element y in $\mathrm{Pic}^0(X) = \mathrm{Jac}(X)(K)$, we use the natural inclusion map $\mathrm{Jac}(X)(K) \to \mathrm{Jac}(X)(F)$, which is identified with the pull-back of line bundles under the natural morphism $X_F \to X$, to obtain the image z of y in $\mathrm{Jac}(X_F)(F)$, and then reduce z to an element of $\Phi_{J_F,F}$ to obtain $\psi(x) = \varphi(x)$. We leave the details to the reader. □

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