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# Models of Curves and Wild Ramification

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*Dedicated to John Tate*

**Abstract:** Let $K$ be a complete discrete valuation field with ring of integers $\mathcal{O}_K$ and residue field $k$ of characteristic $p \geq 0$, assumed to be algebraically closed. Let $X/K$ denote a smooth proper geometrically connected curve of genus $g \geq 1$, and let $\mathcal{X}/\mathcal{O}_K$ denote its minimal regular model. When $g \geq 2$, or $g = 1$ and $X(K) \neq \emptyset$, there exists a finite Galois extension $L/K$ minimal with the property that $X_L/L$ has semi-stable reduction. Let $\mathcal{X}'/\mathcal{O}_L$ denote the minimal regular model of $X_L/L$. We discuss in this article properties of the special fiber of $\mathcal{X}'$ and of the extension $L/K$ that can be inferred from the knowledge of the combinatorial properties of the special fiber of $\mathcal{X}$.
**Keywords:** Curve, discrete valuation field, regular model, semi-stable model, potentially good reduction, potentially multiplicative reduction, wild ramification.

## 1. Introduction

Let $K$ be a complete discrete valuation field with ring of integers $\mathcal{O}_K$ and residue field $k$ of characteristic $p \geq 0$, assumed to be algebraically closed. Let $X/K$ denote a smooth proper geometrically connected curve of genus $g \geq 1$, and let $\mathcal{X}/\mathcal{O}_K$ denote its minimal regular model. When $g \geq 2$, or $g = 1$ and $X(K) \neq \emptyset$, there exists a finite Galois extension $L/K$ minimal with the property

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that $X_L/L$ has semi-stable reduction. Let $\mathcal{X}'/\mathcal{O}_L$ denote the minimal regular model of $X_L/L$.

Which properties of the special fiber of $\mathcal{X}'$ or of the extension $L/K$ can be inferred from the knowledge of the combinatorial properties of the special fiber of $\mathcal{X}$? Let us consider first the case of an elliptic curve $E/K$. Tate noted the following in the summary of his famous algorithm[2] [26]:

*Let $p \neq 2, 3$. Let $E/K$ be an elliptic curve with additive reduction over $\mathcal{O}_K$. Then $E/K$ has potentially multiplicative reduction if and only if it has reduction of type $\mathrm{I}_n^*$ for some $n > 0$. In other words, $E/K$ has potentially good reduction if and only if its reduction type over $\mathcal{O}_K$ is either* II, II$^*$, III, III$^*$, IV, IV$^*$ *or* $\mathrm{I}_0^*$.

A slight refinement when $p = 3$ is noted in the summary of the algorithm given in [24], page 365, and it is well-known to the experts that the above statement is true if $p \neq 2$. When $p = 2$, the 'if and only if' statement does not hold anymore, as it is easy to find examples of elliptic curves with potentially good reduction and reduction type $\mathrm{I}_n^*$ $(n > 0)$. On the other hand, we will show in 2.8 that if $E/K$ has potentially multiplicative reduction, then $[L : K] = 2$ and $E/K$ has reduction of type $\mathrm{I}^*_{\nu+4s_{L/K}}$, where $\nu > 0$ is such that $E_L/L$ has reduction of type $\mathrm{I}_{2\nu}$, and $s_{L/K} + 1$ is the valuation of the different of the extension $L/K$.

As the example of elliptic curves suggests, the cases where $p \mid [L : K]$ are the most difficult to analyze. It is well-known that if $p \mid [L : K]$, then $p \leq 2g + 1$. We note in 2.1 that if $g + 1 < p \leq 2g + 1$, $[L : K] = p^s$, and the reduction of the Jacobian $A/K$ of $X/K$ is purely additive, then $A/K$ has potentially good reduction. When $p = g+1$, the reduction is either potentially good or potentially purely multiplicative. In the latter case we provide in Theorem 2.2 a precise description, similar to the case $g = 1$, for the possible types of reduction of a curve $X/K$ of genus $g$ with $p = g + 1$ such that $p \mid [L : K]$ and $\mathrm{Jac}(X_L)$ has purely multiplicative reduction. This theorem provides some evidence that the general questions raised below may have a positive answer.

[2]Given a Weierstrass equation for $E/K$, Tate's algorithm produces a minimal such equation, and an explicit description of the type of reduction of $E/K$ when $k$ is perfect. The possible types of reduction are denoted by the symbols $\mathrm{I}_0$ (good reduction), $\mathrm{I}_n$, $n > 0$ (multiplicative reduction), and II, II$^*$, III, III$^*$, IV, IV$^*$ and $\mathrm{I}_n^*$, $n \geq 0$ (additive reduction). In the case of imperfect residue fields, see [25] for an analogue algorithm, and [11], Appendix, for the list of reduction types of curves of genus 1.

To state these questions, we need to recall the following notation. Let $\mathcal{X}_k = \sum_{i=1}^{v} r_i C_i$ denote the special fiber of a regular model $\mathcal{X}/\mathcal{O}_K$, where $C_i$ is an irreducible component of multiplicity $r_i$ and has geometric genus $g(C_i)$. *All curves $X/K$ discussed in this article will be assumed, unless stated otherwise, to have a regular model with* $\gcd(r_1, \ldots, r_v) = 1$. This condition holds for instance if $X(K) \neq \emptyset$. Let ${}^tR := (r_1, \ldots, r_v)$, and let $M(\mathcal{X}) := ((C_i \cdot C_j))$ denote the intersection matrix of $\mathcal{X}_k$; then $MR = 0$. We call the model *good* if each component $C_i/k$ is smooth, and the reduced special fiber $(\mathcal{X}_k)^{red}$ has normal crossings. Such a model can be obtained from the minimal regular model by a sequence of blow-ups of closed points. There exists a minimal good regular model. To a model $\mathcal{X}$ we associate a graph $G := G(\mathcal{X})$ as follows: The vertices of $G$ are the curves $C_i$'s, and $C_h$ is linked to $C_j$ by $(C_h \cdot C_j)$ edges. The main invariant of this graph is its first Betti number $\beta(G)$. When the model $\mathcal{X}$ is good, $\beta(G)$ is independent of the choice of a good model, as we now recall.

Let $A/K$ be an abelian variety of dimension $g$. Let $\mathcal{A}/\mathcal{O}_K$ denote its Néron model. The connected component of zero $\mathcal{A}_k^0$ of the special fiber $\mathcal{A}_k$ of $\mathcal{A}/\mathcal{O}_K$ is an extension of an abelian variety $B/k$ by the product of a torus $T/k$ and a unipotent group $U/k$. We shall call the dimensions of $B/k$, $T/k$, and $U/k$ respectively, the abelian, toric, and unipotent ranks of $A/K$, denoted by $a_K$, $t_K$, and $u_K$. When $A_L/L$ has semi-stable reduction, $u_L = 0$. We shall say that $A/K$ has potentially good reduction if $a_L = g$, and potentially purely multiplicative reduction if $t_L = g$. Let now $A = \mathrm{Jac}(X/K)$. Given a regular model $\mathcal{X}/\mathcal{O}_K$ of $X/K$, Raynaud [20] has shown that $a_K = \sum_{i=1}^{v} g(C_i)$, and that when $\mathcal{X}$ is good, $t_K = \beta(G(\mathcal{X}))$.

The *degree* of a vertex $C_i$ on $G$ is the integer $d_i := \sum_{j \neq i} (C_j \cdot C_i)$. A vertex of degree 1 is called a terminal vertex, and a vertex of degree $d > 2$ is called a *node*. The topological space obtained by removing all nodes from a graph is a union of connected components. A *chain* in $G$ is the closure of such a connected component. If a chain contains a terminal vertex, we call it a *terminal chain*. It also contains exactly one node, unless the graph is reduced to a single chain. The other chains are called *connecting chains*, and contain two nodes. The *weight* of a chain $\mathcal{D}$ is the integer $w(\mathcal{D}) := \gcd(r_j, C_j \text{ a vertex on } \mathcal{D})$. When $g = 1$, all graphs associated with good models have only a single node[3], except for the graphs $\mathrm{I}_n^*$, $n > 0$, which have two nodes linked by a chain of weight 2.

**Question 1.1** Let $X/K$ be a curve with a good model $\mathcal{X}/\mathcal{O}_K$ with $a_K = \sum_{i=1}^{v} g(C_i) = 0$ and such that its graph $G(\mathcal{X})$ is a tree. If all chains of $G(\mathcal{X})$ have weight equal to 1, is it true that $\mathrm{Jac}(X)/K$ has potentially good reduction?

When the multiplicities of all nodes of $G(\mathcal{X})$ are coprime to $p$, it is known that the extension $L/K$ is tame[4]; when one or more nodes has multiplicity divisible by $p$, the extension $L/K$ is wild (see 1.3 below). While trying to generalize further the above question when $L/K$ is wild, we were lead to ask the following optimistic question.

**Question 1.2** Let $X/K$ be a curve with a good regular model $\mathcal{X}$ and associated intersection matrix $M(\mathcal{X})$. Winters' Theorem [27] proves the existence of a discrete valuation field $F$ of equicharacteristic zero and of a curve $Y/F$ with a regular model $\mathcal{Y}/\mathcal{O}_F$ such that $M(\mathcal{Y}) = M(\mathcal{X})$.

Consider all discrete valuation fields $F$ of equicharacteristic zero and all curves $Y/F$ with a regular model $\mathcal{Y}/\mathcal{O}_F$ having intersection matrix $M(\mathcal{Y}) = M(\mathcal{X})$. Let $t_{M(\mathcal{X})}$ denote the maximum of the semi-stable toric ranks achieved by the abelian varieties of the form $\mathrm{Jac}(Y)/F$, where $Y/F$ is any curve with a regular model such that $M(\mathcal{Y}) = M(\mathcal{X})$ (by semi-stable toric rank of $\mathrm{Jac}(Y)/F$, we mean the toric rank of the Néron model of $\mathrm{Jac}(Y_{F'})/F'$ over any extension $F'/F$ such that $\mathrm{Jac}(Y_{F'})/F'$ has semistable reduction). Is it true that $t_L(\mathrm{Jac}(X)) \leq t_{M(\mathcal{X})}$?

When $\mathcal{X}$ is as in Question 1.1, then one can show that $t_{M(\mathcal{X})} = 0$. Thus, a positive answer to Question 1.2 implies a positive answer to Question 1.1. It should be noted that even when $L/K$ is tame, it is possible that the graph $G(\mathcal{X})$ has a connecting chain with weight bigger than 1 while the Jacobian has potentially good reduction (see [14]). We note also (3.11) that the special fiber

[3]The statement 'if a good model for $X/K$ has a graph with exactly one node, then the curve $X/K$ has potentially good reduction' holds for instance if all components have multiplicities coprime to $p$, but does not hold in general. When $p = 3$, there are curves $X/K$ of genus 2 with good models having one node only (types III, IV, or V in the classification of [19]), but whose semi-stable models have a special fiber containing two elliptic curves (see [9], (6.4.1) and (6.4.2)). However, in these examples, $\mathrm{Jac}(X)/K$ has potentially good reduction.

[4]When $L/K$ is tame, an extension $F/K$ where $X_F/F$ has semi-stable reduction can be explicitly given (1.3), and Question 1.1 has a positive answer in this case, obtained by explicitly desingularizing the normalization of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_F$.

of a good model $\mathcal{X}$ does not determine in general the semi-stable toric rank $t_L(\mathrm{Jac}(X))$.

Since we would like to infer properties of the matrix $M(\mathcal{X}')$ associated with a good semi-stable model of $X_L/L$ from the properties of the matrix $M(\mathcal{X})$ associated with a good model of $X/K$, we should find ways to relate these two matrices. When $L/K$ is tame, one is often successful in constructing a regular model for $X_L/L$ by proceeding as follows. Given a good model $\mathcal{X}/\mathcal{O}_K$, consider the normalization $\mathcal{W}$ of the scheme $\mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_L)$. When $L/K$ is tame, the singularities of $\mathcal{W}$ can be explicitly resolved; let $\mathcal{V} \to \mathcal{W}$ denote this resolution of singularities, so that $\mathcal{V}/\mathcal{O}_L$ is a regular model for $X_L/L$. For instance, when $p \neq 2$ and an elliptic curve $E/K$ has reduction $\mathrm{I}_n^*$, $n > 0$, this process shows that $[L : K] = 2$ and that $E_L/L$ has reduction $\mathrm{I}_{2n}$.

Another way of relating the matrices $M(\mathcal{X})$ and $M(\mathcal{X}')$ is through the components groups of the associated Néron models. Indeed, let $A := \mathrm{Jac}(X/K)$, and denote by $\mathcal{A}/\mathcal{O}_K$ and $\mathcal{A}'/\mathcal{O}_L$ the Néron models of $A/K$ and $A_L/L$, respectively. Then the component group $\Phi_{A,K}$ is isomorphic to the torsion subgroup of the group $\mathbb{Z}^v/M(\mathcal{X})(\mathbb{Z}^v)$, and $\Phi_{A_L,L}$ is isomorphic to the torsion subgroup of $\mathbb{Z}^{v'}/M(\mathcal{X}')(\mathbb{Z}^{v'})$, where $v$ and $v'$ denote the sizes of the matrices $M(\mathcal{X})$ and $M(\mathcal{X}')$, respectively. Consider the canonical map

$$\mathrm{can}_{K,L} : \mathcal{A} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{A}'$$

induced by the universal property of the Néron model. This map induces a canonical group homomorphism $\Phi_{A,K} \to \Phi_{A_L,L}$. When the group $\Phi_{A,K}$ is not trivial, one may hope that the homomorphism $\Phi_{A,K} \to \Phi_{A_L,L}$ will provide some information on the group $\Phi_{A_L,L}$ and, hence, on $M(\mathcal{X}')$ itself.

Let $\Psi_{K,L}$ denote the kernel of $\Phi_{A,K} \to \Phi_{A_L,L}$. If $A/K$ has potentially good reduction, then $\Psi_{K,L} = \Phi_{A,K}$, since in this case $\Phi_{A_L,L} = (0)$. Some evidence that Question 1.1 may have a positive answer is provided by the following statements:

**Theorem 3.1/3.4** *Let $X/K$ be a curve with a good regular model $\mathcal{X}/\mathcal{O}_K$ whose graph $G(\mathcal{X})$ is a tree with $m \geq 1$ nodes, all with multiplicity $p$ and degree* 3, *and such that all chains of $G$ have weight* 1. *Let $A = \mathrm{Jac}(X)$ and assume that $a_K = 0$. Then*

(1) $\Psi_{K,L} = \Phi_{A,K}$ *when* $\mathrm{char}(K) = 0$.

(2) *If $m$ is odd, then $a_L > 0$.*
(3) *If $m = 1$ or $2$, then $A/K$ has potentially good reduction.*

Possible relationships between the statements '$\Psi_{K,L} = \Phi_{A,K}$' and '$A/K$ has potentially good reduction' are discussed in 3.6.

**1.3** Recall that some information on the extension $L/K$ can be read on a good regular model $\mathcal{X}/\mathcal{O}_K$ for $X/K$. Let us call *principal* a component $C_i$ of $\mathcal{X}_k$ such that either $g(C_i) > 0$, or $C_i$ is a node of $G(\mathcal{X})$. When the model $\mathcal{X}$ does not contain any principal components with multiplicity divisible by $p$, one may show, using the base change/normalization/desingularization method described above, that $[L : K]$ divides the least common multiple of the multiplicities of the principal components of $G(\mathcal{X})$ (see, e.g., [8], 10.4.6, for a related statement). In this case, $L/K$ is then tame and cyclic.

A theorem of T. Saito ([21], 3.11) shows that when $g > 1$, or $g = 1$ and $X(K) \neq \emptyset$, the extension $L/K$ is wild[5] if and only if at least one principal component of the special fiber of the minimal good regular model $\mathcal{X}$ of $X/K$ has multiplicity divisible by $p$.

**Question 1.4** In view of Saito's theorem, it is natural to wonder whether, when $p^r$ divides the multiplicity of a principal component of the minimal good model, then $p^r$ also divides $[L : K]$.

Some evidence that this question may have a positive answer is provided in 4.1 and 4.2.

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[5]Note that when $L/K$ is wild, it is not true in general that $[L : K]$ divides the least common multiple of the multiplicities of the principal components of $G(\mathcal{X})$. For instance, it may happen when $p = 2$ that an elliptic curve has reduction of type $\mathrm{I}_0^*$, but $6 \mid [L : K]$. Note also that when $L/K$ is wild, the least common multiple of the multiplicities of the principal components of $G(\mathcal{X})$ does not divide $[L : K]$ in general (see, e.g., 3.12 with $m = 3$).

## 2. The case where $p = g + 1$.

**Proposition 2.1.** *Let $A/K$ be an abelian variety of dimension $g$ achieving semi-stable reduction after a minimal extension $L/K$. Let $\ell$ be an odd prime dividing $[L : K]$. Then*

(1) *If $\ell = 2g + 1$, then $A/K$ has potentially good reduction.*
(2) *If $t_K = 0$, $[L : K] = \ell^s$, and $g + 1 < \ell \leq 2g + 1$, then $A/K$ has potentially good reduction.*
(3) *If $u_K = g$, $[L : K] = \ell^s$, and $2g/3 + 1 < \ell \leq g + 1$, then the reduction of $A/K$ is either potentially good or potentially purely multiplicative.*
(4) *If $\ell = 2g + 1$, then the $\ell$-part of* $\mathrm{Gal}(L/K)$ *is killed by $\ell$ and $u_M = g$ for any extension $M/K$ of degree prime to $\ell$. If $q$ is a prime dividing $|\Phi_{A,K}|$, then either $q = \ell$ or $q = p$. If $\ell = p$ and in addition $A/K$ is a Jacobian, then $|\Phi_{A,K}| = 1$ or $p$.*
(5) *If $\ell = g + 1$ and $t_L = g$, then the $\ell$-part of* $\mathrm{Gal}(L/K)$ *is killed by $\ell$, and $u_M = g$ for any extension $M/K$ of degree prime to $\ell$. If $q$ is a prime dividing $|\Phi_{A,K}|$, then either $q = \ell$ or $q = p$. If $\ell = p$ and in addition $A/K$ is a Jacobian, then $|\Phi_{A,K}| = 1$ or $p^2$.*

*Proof.* (1), (2), and (3): Assume that $\mathrm{Gal}(L/K)$ contains an element of order $\ell^r$, $r \geq 1$. Then, looking at the action of this element on the monodromy filtration of the Tate module of $A$ (see, e.g., [13], proof of 3.1), we find that either $\ell - 1 \leq t_L - t_K$, or $\ell - 1 \leq 2(a_L - a_K)$. If $\ell = 2g + 1$, we find that we must have $\ell - 1 = 2(a_L - a_K)$, so $a_L = g$. Suppose now that $[L : K]$ is a power of $\ell$. Then, if $t_L - t_K > 0$, then $\ell - 1 \leq t_L - t_K$, and if $a_L - a_K > 0$, then $\ell - 1 \leq 2(a_L - a_K)$. Clearly, if $\ell - 1 > g$, then $t_L = t_K = 0$, and $A/K$ has potentially good reduction. It also follows that if both $t_L - t_K > 0$ and $a_L - a_K > 0$, then $\ell - 1 \leq 2(g - a_K - t_K)/3 \leq 2g/3$. Hence, if $2g/3 + 1 < \ell$, either $t_L = t_K = 0$ or $a_L = a_K = 0$.

(4) Assume that $\ell = 2g + 1$. Then $t_L = 0$, and the bound $\ell^{r-1}(\ell - 1) \leq 2(a_L - a_K)$ ([13], 3.1) shows that $r = 1$. For any extension $M/K$ of degree prime to $\ell$, we find that $A_M$ has potentially good reduction after an extension $L'/M$ of degree divisible by $\ell = 2g + 1$, and $u_M = g$ follows from the bound $\ell^{r-1}(\ell - 1) \leq 2(a_{L'} - a_M)$.

Assume that $q \mid |\Phi_{A,K}|$ with $q \neq p, \ell$. Let $N/K$ denote the maximal tamely ramified extension of $K$ in $L$ of order coprime to $\ell$. Since $[L:N]$ kills $\Psi_{N,L}$ [7], we find that $q \mid |\Psi_{K,N}|$. Recall the bound ([15], 3.1 (5) and (10))

$$\mathrm{ord}_q(|\Psi_{K,N}|)(q-1) \leq 2(a_N - a_K) + (t_N - t_K). \tag{2.1.1}$$

Then (2.1.1) implies that $a_N > 0$ (since $t_N = 0$ here), which is a contradiction since $N \neq L$.

Assume now that $\ell = p$ and that $A/K$ is a Jacobian. It is shown in [13], 2.4, that when a Jacobian $A/K$ has $t_K = 0$, then

$$\sum_{q \ \text{prime}} \mathrm{ord}_q(|\Phi_{A,K}|)(q-1) \leq 2u_K. \tag{2.1.2}$$

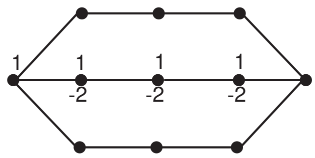
If $\ell \mid |\Phi_{A,K}|$, then the above bound shows that $\Phi_{A,K}$ is cyclic of order $\ell$. Otherwise, $|\Phi_{A,K}| = 1$.

(5) Assume that $\ell = g+1$ and that $a_L = 0$. Then the bound $\ell^{r-1}(\ell-1) \leq t_L - t_K$ ([13], 3.1) shows that $r = 1$. For any extension $M/K$ of degree prime to $\ell$, we find that $A_M$ has potentially purely multiplicative reduction after an extension $L'/M$ of degree divisible by $\ell = g+1$, and $u_M = g$ follows from the bound $\ell^{r-1}(\ell-1) \leq t_{L'} - t_M$.

Assume that $q \mid |\Phi_{A,K}|$ with $q \neq p, \ell$. Since $[L:N]^2$ kills $\Phi_{A_N,N}$ when $t_N = 0$ ([10], 1.8), we find that $q \mid |\Psi_{K,N}|$. Then (2.1.1) implies that $t_N > 0$, a contradiction since $N \neq L$.

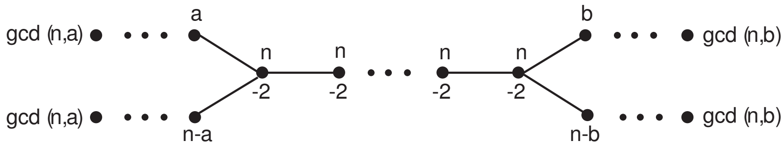
Assume now that $\ell = p$ and that $A/K$ is a Jacobian. If $\ell \mid |\Phi_{A,K}|$, then $\ell^2 \mid |\Phi_{A,K}|$ (3.13, (2)), and then the above bound (2.1.2) shows that $|\Phi_{A,K}| = \ell^2$. Otherwise, $|\Phi_{A,K}| = 1$. □

To discuss the case where $\ell = g+1$, we introduce the following notation. The graph $\mathrm{I}(\nu, n)$ with $\nu, n \geq 1$, has two nodes of degree $n$ connected by $n$ chains with $\nu$ edges each. All vertices have multiplicity 1. The graph $\mathrm{I}(4,3)$ is represented below. For any arithmetical graph $(G, M, R)$, a positive number near a vertex $v_i$ of $G$ denotes the corresponding coefficient $r_i$ in $R$, and a negative number near $v_i$ denotes the coefficient $(C_i \cdot C_i)$ in $M$.



The properties of the graph $\mathrm{I}(\nu, n)$ are discussed in [5], p. 283, or [16], 2.5. The component group of $\mathrm{I}(\nu, n)$ has order $n\nu^{n-1}$, and when $\gcd(n, \nu) = 1$, is isomorphic to $\mathbb{Z}/n\mathbb{Z} \times (\mathbb{Z}/\nu\mathbb{Z})^{n-1}$.

The graph $\mathrm{I}^*(\nu, n, a, b)$ for some $\nu, n \geq 1$ and $1 \leq a, b < n$ with $\gcd(n, a, b) = 1$, is described below. The integer $\nu$ is the number of edges on the chain linking the two nodes of degree 3.



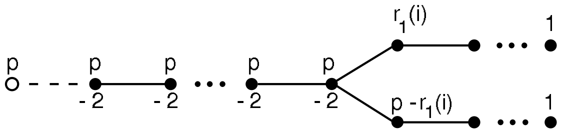

(The multiplicities of the vertices on a terminal chain are uniquely determined by the multiplicity of the initial component and the multiplicity of the node.) The component group of this graph has order $(\frac{n}{\gcd(a,n)\gcd(b,n)})^2$ ([12], 1.5). Its structure depends on $\gcd(n, \nu)$. When $\gcd(n, a) = \gcd(n, b) = 1$, the group is cyclic of order $n^2$ if and only if $\gcd(n, \nu) = 1$ ([7], Lemma 3). An example of a curve $X/K$ having reduction of type $\mathrm{I}^*(\nu, p, 1, 1)$ and whose Jacobian has potentially good reduction is given in [7], Lemmata 4 and 5.

**Theorem 2.2.** *Let $g \geq 2$ be an integer such that $\ell = g + 1$ is an odd prime. Let $X/K$ be a curve of genus $g$ whose Jacobian $A/K$ achieves purely multiplicative reduction after a minimal extension $L/K$ of degree divisible by $\ell$. Then*

(1) *The graph of the minimal regular semi-stable model of $X_L/L$ is of type* $\mathrm{I}(\nu, \ell)$ *for some $\nu \geq 1$, and* $\mathrm{ord}_\ell([L : K]) = 1$.

(2) *Assume that $[L : K] = \ell$. Then the graph of the minimal regular model of $X/K$ is $\mathrm{I}^*(\nu + \ell s, \ell, a, b)$ for some $s \geq 0$, and $\nu$ as in* (1)*. Moreover, $s > 0$ if and only if $\ell = p$.*

*More precisely, consider the quotient $\mathcal{Z}/\mathcal{O}_K$ of the action of* $\mathrm{Gal}(L/K)$ *on the minimal regular semi-stable model $\mathcal{Y}/\mathcal{O}_L$. The special fiber $\mathcal{Z}_k$ consists of a chain of $\nu + 1$ projective lines of multiplicity $\ell$. When $\ell = p$, $\mathcal{Z}$ is singular at only two points $Q_1$ and $Q_2$, one on each terminal component of the chain $\mathcal{Z}_k$. The resolution of singularities of each singular point $Q_i$ is of type $N(p, \alpha_i, r_1(i))$, whose graph is given below,*




*The integer $\alpha_i$ is the number of (bold) vertices of multiplicity $p$ in this resolution. We find that the type of reduction of $X/K$ is $\mathrm{I}^*(\nu+\alpha_1+\alpha_2, p, r_1(1), r_1(2))$. Moreover, $p \mid \alpha_1$ and $p \mid \alpha_2$.*

(3) *Assume that $\ell = p$. Then the graph of the minimal regular model of $X/K$ is either $\mathrm{I}^*(\mu, p, a, b)$, or $\mathrm{I}^*(\mu, 2p, a', b')$ for some $1 \leq a', b' < 2p$ with $\gcd(2p, a') = 2$ and $\gcd(2p, b') = p$. In the first case $|\Phi_{A,K}| = p^2$, and in the latter case, $\Phi_{A,K}$ is trivial.*

*Proof of* (1). To show that the semi-stable reduction is of type $\mathrm{I}(\nu, \ell)$, we may assume, without loss of generality, that $[L : K] = \ell$. The action of $H := \mathrm{Gal}(L/K)$ on $X_L/L$ extends to an action on the minimal regular semi-stable model $\mathcal{Y}/\mathcal{O}_L$ of $X_L/L$. The graph $G := G(\mathcal{Y}_k)$ has at most $\ell - 1 = g$ independent cycles, and $\beta(G) = \ell - 1$ if and only if all the components of $\mathcal{Y}_k$ are smooth. We shall show below that this is indeed the case.

We claim that this graph can only be of the form $\mathrm{I}(\nu, \ell)$. Indeed, note first that since we assume that $A_L/L$ has $a_L = 0$, the graph $G$ has no terminal vertices, as such a vertex would have self-intersection $-1$, and this could only happen if the genus of the corresponding component was bigger than 0, implying then that $a_L > 0$.

The graph $G$ has by hypothesis an action of the group $H$ of order $\ell$. We claim that the action of $H$ on $G$ is not trivial. Indeed, consider the normal quotient $\mathcal{Z} := \mathcal{Y}/H$, and a desingularization $\mathcal{X} \to \mathcal{Z}$ with $\mathcal{X}_k$ with normal crossings and smooth components. If the action of $H$ is trivial, then the first Betti number of the graph of $\mathcal{X}$ will not be 0, contradicting the fact that $t_K = 0$ (2.1 (5)). This fact is obvious if $\beta(G) > 0$. If $\beta(G) = 0$, then any of the irreducible components of $\mathcal{Y}_k$ with an ordinary double point as a singularity is mapped under the quotient map $\mathcal{Y} \to \mathcal{Z}$ to an irreducible component whose generalized Jacobian contains a torus. Again, this contradicts the fact that $t_K = 0$. In fact, even when $H$ acts non-trivially, a component of $\mathcal{Y}_k$ with an ordinary double point is mapped onto $\mathcal{Z}$ to a component whose generalized Jacobian contains a torus. Thus, we find that all components of $\mathcal{Y}_k$ are smooth.

Since $H$ acts non-trivially, we can find a vertex of $G$ that is not fixed by this action. Since $0 < 2\ell - 4 = 2\beta(G) - 2 = \sum_i (d_i - 2)$ (where $d_i$ denotes the degree of the vertex $v_i$), we find that there exists a node that is fixed by the action (only nodes contribute to this sum, and a node that is not fixed along with its conjugates contribute $\ell(d_i - 2) \geq \ell$ to this sum). Now that we have a node that is fixed, we can find a node $C$ that is fixed to which is attached an edge that is not fixed. Thus, at least $\ell$ edges are attached to $C$. Each such edge starts a chain of $G$, which must end with another node $C'$ of $G$ since $G$ has no terminal vertices. But then $C'$ is also the end-node for the chains attached to all $\ell$ edges conjugated under $G$. We now have two nodes $C$ and $C'$ with $d(C), d(C') \geq \ell$. The formula $2\ell - 4 = 2\beta(G) - 2 \geq (d(C) - 2) + (d(C') - 2)$ shows that $G$ has exactly two nodes, linked by $\ell$ connecting chains each with $\nu - 1$ vertices.

The special fiber $\mathcal{W}_k$ of the stable model of $X_L/L$ is thus the union of two rational curves meeting in $\ell$ points, and $\mathrm{Gal}(L/K)$ injects into $\mathrm{Aut}(\mathcal{W}_k)$. The automorphism group of $\mathcal{W}_k$ injects into $S_\ell \times \mathbb{Z}/2\mathbb{Z}$, where an automorphism $\sigma$ is sent to $(\sigma_1, \sigma_2)$, with $\sigma_1$ being the permutation induced by $\sigma$ on the $\ell$ singular points of the stable fiber, and $\sigma_2 = -1$ if and only if $\sigma$ permutes the two irreducible components. In particular, when $\ell$ is odd, $\mathrm{ord}_\ell([L : K]) = 1$.

*Proof of* (2). Consider the quotient $\mathcal{Z}/\mathcal{O}_K$ of $\mathcal{Y}/\mathcal{O}_L$ by the action of an element $\sigma$ of order $\ell$ in $\mathrm{Gal}(L/K)$. The graph of $\mathcal{Y}_k$ is of the form $\mathrm{I}(\nu, \ell)$, and the special fiber $\mathcal{Z}_k$ thus consists of a chain $\nu + 1$ projective lines, each of multiplicity $\ell$. The element $\sigma$ acts on the two irreducible components $E_1$ and $E_2$ of the stable model. When $\ell \neq p$, this action has two fixed points on each (rational) component, $P_1$ and $P_1'$ on $E_1$, and $P_2$ and $P_2'$ on $E_2$. When $\ell = p$, this action has exactly one fixed point on each component, $P_i$ on $E_i$. The quotient $\mathcal{Z}$ is singular exactly at the images $Q_i$ and $Q_i'$ of the fixed points (indeed, the morphism $\mathcal{Y} \to \mathcal{Z}$ is unramified in codimension 1. Thus, the points $Q_i$ are singular on the normal model $\mathcal{Z}$, using the purity of the branch locus.).

Assume that $\ell = p$. Then the singularity at $Q_i$ is called a wild quotient singularity, and we claim that its resolution graph is of the type $N(p, \alpha_i, r_1(i))$, with $p \mid \alpha_i$. Let $\mathcal{X}$ denote the regular model of $X/K$ obtained by desingularizing $\mathcal{Z}$ and minimal with the property that $\mathcal{X}_k$ has smooth components and normal crossings. Let $D_1$ (resp. $D_2$) denote the components of the desingularization of $Q_1$ (resp. of $Q_2$) that meet the strict transform of $\mathcal{Z}_k$ in $\mathcal{X}$. The key is that the shape of the special fiber $\mathcal{Z}_k$ forces the graphs of the desingularization of $Q_1$ and of $Q_2$ to have a node. Indeed, since two (in fact, all) consecutive components of $\mathcal{Z}_k$ have multiplicity $p$, the vertices $D_1$ and $D_2$ must have multiplicity divisible by $p$. If the graph of the desingularization of $Q_i$ does have a node, all of its vertices have then multiplicities divisible by $p$; since the desingularization is assumed to be minimal, no component of the desingularization graph can be a $-1$ curve. The terminal multiplicity on a chain divides the multiplicity of the vertex $E$ immediately adjacent to it. Hence, it is smaller than the multiplicity of $E$. Since no component of the graph has self-intersection $-1$ and the components of $\mathcal{Z}_k$ have multiplicity exactly $p$, we find that all components of the desingularization graph have multiplicity $p$. But this is a contradiction, since then the terminal vertex has self-intersection $-1$. Therefore, the desingularization graph has a node. Let $C_i$ denote the node of the desingularization of $Q_i$ closest to $D_i$. Let $n_i p$ denote its multiplicity.

Recall that the genus of $X/K$ is given by the formula

$$2g = \sum_D (r(D)-1)(d(D)-2),$$

where the sum is over all irreducible components $D$, with $r(D)$ being the multiplicity of $D$ and $d(D)$ its degree in the graph. We can rewrite this formula by grouping together the contributions of each node $C$, as in [18], 7.2. For each node $C$, let

$$\mu(C) := (r(C)-1)(d(C)-2) - \sum_D (r(D)-1),$$

where the sum is over all terminal vertices $D$ of the graph which lie on a terminal chain attached to $C$. We have

$$2g = \sum_{\text{nodes } C} \mu(C).$$

Since we assume that all vertices of degree 1 or 2 have self-intersection less than $-1$, we find that $\mu(C) \geq 0$, with $\mu(C) = 0$ only when $r(C) = 1$.

In our case, $0 < \mu(C_i) < 2g = 2(p-1)$ and the multiplicity $r(C_i)$ is divisible by $p$. The list of possibilities for such a node $C$ are written below in the form $(r(C), r_1, r_2, \dots \mid s_1, s_2, \dots)$, where the terminal multiplicities are $r_1, r_2, \dots$, and the weights of the connecting chains are $s_1, s_2, \dots$. For later use, we also compute

$$\phi(C) := r(C)^{d(C)-2} \prod_D r(D)^{-1},$$

where the product is over all terminal vertices $D$ on terminal chains attached to $C$. The possibilities are:

(a) $(p, 1, 1 \mid 1)$, or $(p, 1 \mid 1, 1)$, or $(p \mid 1, 1, 1)$, all with $\mu = p - 1$ and $\phi = p$,
(b) $(p, 1, 1 \mid p)$, or $(p, 1 \mid 1, p)$, or $(p \mid 1, 1, p)$, all with $\mu = p - 1$ and $\phi = p$,
(c) $(2p, p, p, p \mid p)$ with $\mu = p + 1$ and $\phi = 4/p$,
(d) $(2p, p, 2 \mid 1)$ with $\mu = p - 1$ and $\phi = 1$,
(e) $(2p, p \mid 2, 1)$, or $(2p, p, 1 \mid 2)$ with $\mu = p$ and $\phi = 2$,
(f) $(2p, 2, 2 \mid 2)$ or $(2p, 2, 2 \mid 2p)$, with $\mu = 2p - 3$ and $\phi = p/2$,
(g) $(3p, p, p \mid p)$ or $(3p, p, p \mid 3p)$, with $\mu = p + 1$ and $\phi = 3/p$,
(h) $(2np, np, np \mid 2np)$, with $\mu = 1$ and $\phi = 2/np$, and
(i) $(2p, p \mid p, 2p)$, with $\mu = p$ and $\phi = 2$.

In our case, the nodes $C_1$ and $C_2$ are linked by a connecting chain of weight $p$. We conclude our argument now using this fact and the fact that $\mu(C_1) + \mu(C_2) \leq 2p - 2$. Since $p \neq 2$, the only possibility for the graph is to have two nodes of type $(p, 1, 1 \mid p)$. In particular, each desingularization graph is of type $N(p, \alpha_i, r_1(i))$, as desired.

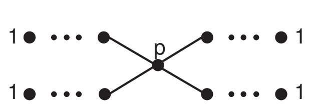
The fact that $p \mid \alpha_i$ follows from a general fact about $\mathbb{Z}/p\mathbb{Z}$-quotient singularities: such a singularity has a component group killed by $p$ ([18], 2.4 (c)), and in the case of a singularity of type $N(p, \alpha_i, r_1(i))$, this can occur only when $p \mid \alpha_i$ ([18], 3.12 and 3.10).

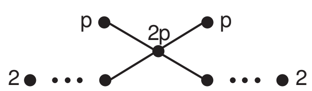
Assume now that $\ell \neq p$. Then the singularities at $Q_i$ and $Q'_i$ are tame quotient singularities, resolved by a chain of rational curves, all of multiplicities less than $\ell$. Thus, in this case, the minimal model is of type $\mathrm{I}^*(\nu, \ell, a, b)$ for some $1 \leq a, b < \ell$ and $\nu > 0$. Since the statement about the singularities being resolved by a chain of rational curves of multiplicities less than $\ell$ does not seem to be proved in the literature in our context, we will provide here a proof in our particular case. Let $\mathcal{X} \to \mathcal{Z}$ denote a good desingularization of $\mathcal{Z}$. Let $B_1$ and $B_2$ denote the strict transforms in $\mathcal{X}$ of the images in $\mathcal{Z}$ of the components $E_1$ and $E_2$. The vertices $B_1$ and $B_2$ are nodes of degree 3 on the graph $G(\mathcal{X})$. Indeed, $B_1$ and $B_2$ have multiplicity $\ell$ and are linked by a chain of rational curves of multiplicity $\ell$, and the resolutions of $Q_i$ and $Q'_i$ each intersect $B_i$. Thus, if $B_i$ does not have two terminal chains, then $\mu(B_i) = (\ell - 1)$. If $B_i$ has a terminal chain, then its terminal multiplicity is 1 and, again, $\mu(B_i) = (\ell - 1)$. Thus $\mu(B_1) + \mu(B_2) = 2g$, and we find that all additional nodes $C$ on $G(\mathcal{X})$ must have $\mu(C) = 0$. Such nodes must have multiplicity 1. Since any such node can have only connecting chains, we find that the graph $G$ can contain no such nodes.

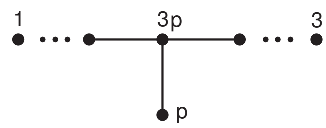
*Proof of* (3). We now turn to describing the possible reduction graphs for the reduction of a curve $X/K$ whose Jacobian $A/K$ is such that $t_L = g = p - 1$ after a wildly ramified extension $L/K$ (note that we do not assume as in (2) that $[L : K] = p$). We will use repeatedly the following facts:

(i) $A/K$ has purely additive reduction (2.1 (5)), so a good model $\mathcal{X}/\mathcal{O}_K$ has a graph which is a tree, and all its components are smooth rational curves.
(ii) The greatest common divisor of the multiplicities of the components of $\mathcal{X}_k$ is 1. Indeed, if it were not 1, it would be a power of $p$ ([11], 7.4). But then the genus formula $2g - 2 = \sum_D r(D)(d(D) - 2)$ would give $p \mid 2g - 2 = 2p - 4$, a contradiction.
(iii) The component group $\Phi_{A,K}$ has order 1 or $p^2$ (2.1 (5)).
(iv) The tree $G$ has a node of multiplicity divisible by $p$ (using Saito's Theorem recalled in 1.3).

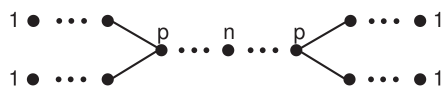
**2.3** Clearly, the graphs $\mathrm{I}^*(\mu, p, a, b)$, and $\mathrm{I}^*(\mu, 2p, a', b')$ with $\gcd(a', 2p) = p$ and $\gcd(b', 2p) = 2$, satisfy these conditions. It is easy to write down several additional arithmetical trees with the above properties. For instance:

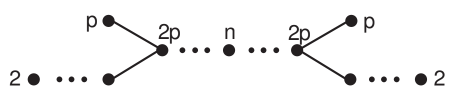







where the latter graph is defined only for $p \neq 3$. In the above drawings, on any terminal chain, only the multiplicities $r$ and $t$ of the node and of the terminal vertex are given. Thus the vertex of the chain linked to the node has implicitly multiplicity $s < r$ such that $\gcd(r, s) = t$. The sum of the multiplicities of the vertices attached to a node is divisible by the multiplicity of the node. In the next two graphs, the multiplicity $n$ is coprime to $p$.






To prove Part (3), we prove the following two claims:

(I) If the reduction graph of the minimal regular model of $X/K$ is not of type $\mathrm{I}^*(\mu, p, a, b)$, or of type $\mathrm{I}^*(\mu, 2p, a', b')$ with $\gcd(a', 2p) = p$ and $\gcd(b', 2p) = 2$, then after possibly a quadratic extension $F/K$, $X_F/F$ has a model with reduction type one of the five graphs listed above.

(II) If the reduction type of a good model of $X/K$ is one of the five graphs listed above, then there exists a tame extension $K'/K$ such that after any tame extension $H/K'$, the reduction type of the minimal good model of $X_H/H$ is also one of these five graphs.

Assume that (I) and (II) hold. Suppose that the reduction graph of $X/K$ is not of type $\mathrm{I}^*(\mu, p, a, b)$ or of type $\mathrm{I}^*(\mu, 2p, a', b')$ with $\gcd(a', 2p) = p$ and $\gcd(b', 2p) = 2$. Then there exists a tame extension $N/K$ such that over any tame extension $H/N$, $X_H/H$ does not have reduction of type $\mathrm{I}^*(\mu, p, a, b)$ or $\mathrm{I}^*(\mu, 2p, a', b')$. Consider now the maximal tame extension $L_0/K$ in $L/K$. Part (1) shows that $[NL : NL_0] = p$. Part (2) then implies that $X_{NL_0}/NL_0$ has reduction of type $\mathrm{I}^*(\mu, p, a, b)$, a contradiction.

*Proof of* (I). We assume that the good model $\mathcal{X}$ is minimal with this property, so that every vertex of the graph $G$ of degree 1 or 2 has self-intersection less than $-1$. Let $C_0$ denote a node of the graph whose multiplicity is $mp$ for some $m \geq 1$.

When $m = 1$, we find that either $d(C_0) = 4$, and the graph has only one node $(p, 1, 1, 1, 1)$ (first graph drawn in 2.3) with no connecting chains (in which case $|\Phi_{A,K}| = p^2$), or the node has $d(C_0) = 3$ and $\mu(C_0) = p - 1$. (To show that in the case $d(C_0) = 4$, $C_0$ cannot have a connecting chain, we note that in this case, $\mu(C_0) = 2p - 2$; thus, if $C_0$ has a connecting chain to a node $C$, we must have $\mu(C) = 0$. Nodes with $\mu(C) = 0$ exist (for instance any node with multiplicity 1), but then there is a path from such a node to a node $C' \neq C_0$ with a terminal chain, and $\mu(C') > 0$.) We discussed in the proof of (2) the nodes with at least one connecting chain, with $p \mid r(C)$, and with $\mu < 2p - 2$.

When $m > 1$ and the node has no connecting chains, it is easy to show that $d(C_0) \leq 4$ (using in one case that the gcd of the multiplicities is 1). Consider now the case where $d(C_0) = 4$. Note that a prime $q$ can divide at most two of the terminal multiplicities, otherwise it divides all of them. We have $\mu(C_0) \geq 2(mp-1) - 2(mp/2-1) - 2(mp/3-1)$. This latter expression is bigger than $2p-2$ when $m \geq 6$. Checking the remaining cases $2 \leq m \leq 5$ by hand and using the fact that the gcd of the multiplicities is 1, we find that only the case $(2p, p, p, 2, 2)$ (second graph drawn in 2.3) has $\mu \leq 2p - 2$ (in fact, in this case $\mu = 2p - 2$). This graph has trivial component group. Consider now the case where $m > 1$ and $d(C_0) = 3$. Let $a$, $b$, and $c$, denote the terminal multiplicities. We have $a, b, c \mid 3m$, with $3m/abc = 1$ or $p^2$, and $(3m - 1) - (a - 1) - (b - 1) - (c - 1) = 2p - 2$. It is not hard to check that we also must have $\gcd(a, b) = \gcd(a, c) = \gcd(b, c)$. It is a tedious computation to check that there is only one graph with these properties, $(3p, p, 3, 1)$, the third graph described in 2.3.

Consider now the case where $C_0$ has a connecting chain. Recall the formula $|\Phi_{A,K}| = \prod_D r(D)^{d(D)-2}$, where the product is taken over all vertices $D$ of the graph. Using the definition of $\phi(C)$ given in Part (2), we find that $|\Phi_{A,K}| = \prod_{\text{nodes } C} \phi(C)$.

We conclude from the fact that $|\Phi_{A,K}| = 1$ or $p^2$ that the graph $G$ must contain, in addition to $C_0$, a second node $C$ having multiplicity divisible by $p$ in all cases except possibly in cases (d), (e), (g), or (i). Noticing that the weight of the connecting chain in cases (g) or (i) is divisible by $p$ allows us to conclude that in these cases too, the graph contains a second node with multiplicity divisible by $p$. Keeping in mind that $\mu(C_0) + \mu(C) \leq 2p - 2$, we find that such a graph with a node of type (a), (b), (c), (f), (g), (h), or (i) has in fact exactly two nodes, and can only be as follows: both nodes of type (a) (fourth graph in 2.3), or both of type (b) (type $\mathrm{I}^*(\mu, p, a, b)$), or one of type (f) and one of type (h) with $n = 1$ (type $\mathrm{I}^*(\mu, 2p, a', b')$).

We are left to consider the graphs that contain a node of type (d) or (e). One obvious such possibility has two nodes of type (d), leading to the fifth graph in 2.3. Suppose that the arithmetical graph associated with the reduction of $X/K$ contains a node of type (d) or (e). We claim that after a quadratic extension $K'/K$, the arithmetical graph associated with the reduction of $X_{K'}/K'$ contains a node of type (a).

Before we can prove this claim, we need to review the base change/normalization process. Let $\mathcal{X}/\mathcal{O}_K$ be a good model of a curve $X/K$. Let $q \neq p$ be prime, and let $F/K$ denote the unique Galois extension of degree $q$. Let $\mathcal{W}$ denote the normalization of $\mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_F)$. Denote by $\rho : \mathcal{W} \to \mathcal{X}$ the natural map. Let $\mathcal{V} \to \mathcal{W}$ denote the minimal resolution of singularities of $\mathcal{W}$. The scheme $\mathcal{V}/\mathcal{O}_F$ is a regular model of $X_F/F$. The following facts are well-known.

**2.4** Write $\mathcal{X}_k = \sum r_i C_i$. Let $B := \sum_{q \nmid r_i} C_i$. The divisor $B$ has normal crossings.

- If a point $P$ is singular on $\mathcal{W}$, then $\rho(P)$ is a singular point of $B$.
- Let $C_i$ be an irreducible component of $\mathcal{X}_k$. If $q \nmid r_i$, then $\rho^{-1}(C_i) =: D_i$ is irreducible and the restricted map $\rho_{|D_i} : D_i \to C_i$ is an isomorphism. The curve $D_i$ has multiplicity $r_i$ in $\mathcal{W}_k$.
- If $q \mid r_i$ and $C_i \cap B \neq \emptyset$, then $\rho^{-1}(C_i) =: D_i$ is irreducible and the restricted map $\rho_{|D_i} : D_i \to C_i$ is a morphism of degree $q$ ramified over $|C_i \cap B|$ points of $C_i$. The curve $D_i$ has multiplicity $r_i/q$ in $\mathcal{W}_k$. Its genus is computed using the Riemann-Hurwitz formula.
- If $q \mid r_i$ and $C_i \cap B = \emptyset$, then $\rho : \rho^{-1}(C_i) \to C_i$ is an etale morphism and each irreducible component of $\rho^{-1}(C_i)$ has multiplicity $r_i/q$ in $\mathcal{W}_k$. If $\rho^{-1}(C_i)$ is not irreducible, then it is equal to the disjoint union $D_1 \sqcup \ldots \sqcup D_q$ of $q$ irreducible components, and each restricted map $D_j \to C_i$ is an isomorphism.

The proofs of the above facts are well-known to the experts, although we have not found an appropriate reference for them in the literature. We briefly review here the main ideas in these proofs. First, since $\mathcal{X}_k$ has normal crossings, the local ring $\mathcal{O}_{\mathcal{X},Q}$ at a point $Q$ can be given in a normal form ([8], 9.2.34 and 9.2.35). In particular, the completion of $\mathcal{O}_{\mathcal{X},Q}$ is isomorphic to $\mathcal{O}_K[[u,v]]/(F(u,v))$, with $F(u,v) = u^{r_1} - \pi_K a$ or $F(u,v) = u^{r_1}v^{r_2} - \pi_K a$ according to whether $\mathcal{X}_k$ is irreducible at $Q$ (with $Q$ on a component of multiplicity $r_1$), or whether $Q$ belongs to the intersection of two components, of multiplicities $r_1$ and $r_2$, respectively. The element $a$ belongs to $\mathcal{O}_K^* + (u,v)$. The above description is valid independently of the residue characteristic $p$.

When making a base change $F/K$ of degree $q$, one is lead to study the ring $\mathcal{O}_F[[u,v]]/(F(u,v))$ (see, e.g., the proof of 10.4.6 in [8]). We can write $\pi_K = \pi_F^q \lambda$, and $F(u,v) = u^{r_1} - \pi_F^q \lambda a$ or $F(u,v) = u^{r_1}v^{r_2} - \pi_F^q \lambda a$. The key to the hypothesis $q \neq p$ is that we can take a $q$-th root of $\lambda a$.

**2.5** Suppose that $P \in \mathcal{W}$ is a point whose image $Q := \rho(P)$ lies on two components $C$ and $D$ of $\mathcal{X}_k$, of multiplicities $r$ and $s$ respectively, with $q \nmid rs$. As we did not find an appropriate reference in the literature for the fact that $P$ is then singular on $\mathcal{W}$, and that the resolution of the singularity at $P$ is an explicit Hirzebruch-Young string, we will not use these facts in this article. To substitute for the explicit knowledge of the resolution of the singularity, it often suffices to consider the following construction. Let $\alpha$ denote the smallest positive integer such that $q \mid \alpha r + s$ (such an integer exists since $q \nmid rs$). A sequence of blow-ups $\mathcal{X}' \to \mathcal{X}$, starting with the blow-up of $Q$, produces a new special fiber where the strict transform of $C$ meets a component $D'$ of multiplicity $\alpha r + s$, and the component $D'$ is contracted to $Q$ under $\mathcal{X}' \to \mathcal{X}$. Now, instead of considering the base change/normalization construction for $\mathcal{X}$, we can consider it for $\mathcal{X}'$. We find that the preimage of $C$ in the normalization $\mathcal{W}'$ of $\mathcal{X}' \times_{\mathcal{O}_K} \mathcal{O}_F$ is irreducible of multiplicity $r$, and meets at a *regular* point of $\mathcal{W}'$ a component $E$ of multiplicity $(\alpha r + s)/q$.

Let us return to the proof of (I). Suppose that the arithmetical graph associated with the reduction of the minimal good model of $X/K$ contains a node of type (d) or (e). Thus $\mathcal{X}_k$ contains a component $C$ of multiplicity $2p$ which intersects a component of multiplicity $p$, a component of multiplicity $r$ with $\gcd(r, 2p) = 2$, and a component of multiplicity $r'$ with $\gcd(r', 2p) = 1$.

Let $K'/K$ be a quadratic extension, and consider the base change/normalization map $\mathcal{W} \to \mathcal{X}$. Using the facts reviewed above, the strict transform $D$ of $C$ has multiplicity $p$, all its points are regular on $\mathcal{W}$, and it meets only a component of multiplicity $p$, two components $D_1$ and $D_2$ of multiplicity $r/2$, and one component $D_3$ of multiplicity $r'$. The component of multiplicity $p$ has self-intersection $-1$ and can be blown down. Let $\mathcal{W} \to \mathcal{W}'$ be this contraction. Let $\mathcal{V} \to \mathcal{W}'$ denote the minimal desingularization of $\mathcal{W}'$. The strict transform of $D$ in $\mathcal{V}$ is thus a node of type (a). The model $\mathcal{V}$ is good.

Let $\mathcal{X}'/\mathcal{O}_{K'}$ denote the minimal good model of $X_{K'}/K'$ and consider the natural map $\mathcal{V} \to \mathcal{X}'$. For $i = 1, 2, 3$, let $G_i$ denote the connected component of $G(\mathcal{V}) \setminus \{D_i\}$ that does not contain $D$. We claim that $D_i$ and the components in $G_i$ cannot all be contracted under the map $\mathcal{V} \to \mathcal{X}'$. Indeed, we can find a rational point of degree $r/2$ or $r'$ reducing on $D_i$. If all components were contracted, this point would then have to reduce to the component $D$ of multiplicity $p$ coprime to $r/2$ or $r'$, which is impossible. It follows that either $D$ maps to a node of type (a) under $\mathcal{V} \to \mathcal{X}'$, or $D$, $D_1$, $D_2$, and $D_3$ all contract to a point under $\mathcal{V} \to \mathcal{X}'$. The following lemma shows that the latter possibility cannot happen.

**Lemma 2.6.** *Let $\mathcal{X}/\mathcal{O}_K$ denote either a good model of $X/K$, or a semi-stable regular model of $X/K$. Let $\mathcal{V}$ denote any regular model of $X/K$ with a contraction* *map $f : \mathcal{V} \to \mathcal{X}$. Suppose that $C$ is a node on $G(\mathcal{V})$, of multiplicity $r$ and degree $d$, that is either contracted to a point under $f$, or whose image under $f$ is not a node of $G(\mathcal{X})$. Then at least $d-2$ irreducible components of $\mathcal{V}_k$ that meet $C$ have multiplicity divisible by $r$. Moreover, consider the connected components of the topological space $G(\mathcal{V}) \setminus \{C\}$. Then at least $d-2$ such connected components have all their vertices of multiplicity divisible by $r$.*

*Proof*: We treat first the case where $\mathcal{X}$ is a good model. Suppose that $P$ is a point on a smooth component $A$ of multiplicity $a$ of a regular model. Recall that if $P$ belongs to exactly one irreducible component, then the exceptional divisor of the blow-up at $P$ has multiplicity $a$. If $P$ belongs to the intersection of two components $(A, a)$ and $(B, b)$ of a good model, then the exceptional divisor of the blow-up at $P$ has multiplicity $a + b$.

Suppose that $C$ maps surjectively in $\mathcal{X}$ to a component that is not a node of the graph of $\mathcal{X}$. Then the image corresponds to a terminal vertex or a vertex of degree 2 in the graph of $\mathcal{X}$. Suppose that $C$ contracts to a point in $\mathcal{X}$. Then the morphism $f : \mathcal{V} \to \mathcal{X}$ factors as $\mathcal{V} \to \mathcal{V}_1 \to \mathcal{V}_0 \to \mathcal{X}$, such that $\mathcal{V}_0 \to \mathcal{X}$ is obtained by a sequence of blow-ups of points, so $\mathcal{V}_0$ is a good model, and the image $C_1$ of $C$ in $\mathcal{V}_1$ has dimension 1 and contracts to a point in $\mathcal{V}_0$. Since $\mathcal{V}_0$ is a good model, we find that the component $C_1$ cannot be a node of the graph of $(\mathcal{V}_1)_k$. In this graph, $C_1$ corresponds either to a terminal vertex or a vertex of degree 2. To prove the lemma, it suffices to prove it for contractions of the form $\mathcal{V} \to \mathcal{V}_1$, where either the image of $C$ is a terminal vertex in the graph of $\mathcal{X}$, or it meets the other components of the special fiber of $\mathcal{X}_k$ in two distinct points. In the first case, $d-1$ distinct points need to be blown up on $\mathcal{V}_1$ to turn $C$ into a node of $\mathcal{V}$, and in the second case, $d-2$. Each exceptional divisor so constructed has multiplicity $r$. Any further blow-up on such a divisor produces a new exceptional divisor of multiplicity divisible by $r$.

Assume now that $\mathcal{X}$ is a semi-stable model. (If $\mathcal{X}_k$ has components that are not smooth, the model $\mathcal{X}$ is not good.) Blow-up each singularities of the irreductible components of $\mathcal{X}_k$ once to obtain the minimal good model $\mathcal{X}' \to \mathcal{X}$. There exists a regular model $\mathcal{V}'$ that dominates both $\mathcal{V}$ and $\mathcal{X}'$, through maps $\mathcal{V}' \to \mathcal{V}$ and $\mathcal{V}' \to \mathcal{X}'$. Let $C'$ denote the strict transform of $C$ in $\mathcal{V}'$. The component $C'$ is also a node of $G(\mathcal{V}')$, of multiplicity $r$ and degree at least $d$. If the image of $C'$ in $\mathcal{X}'_k$ is not a node of $G(\mathcal{X}')$, we may apply the previous discussion to the node

$C'$ and the morphism $\mathcal{V}' \to \mathcal{X}'$ to conclude the proof of the lemma. Assume now that the image of $C'$ in $\mathcal{X}'_k$ is a node of $G(\mathcal{X}')$. By hypothesis, the image of $C'$ is not a node of $G(\mathcal{X})$. The image of $C'$ cannot be a point since $\mathcal{X}_k$ is semi-stable, and cannot be a component of multiplicity 2 since these components have degree 2 in $G(\mathcal{X}')$. Hence, we find that $C'$ has multiplicity 1, and the statement of the lemma is then obvious. □

*Proof of* (II). Using similar ideas as at the end of the proof of (I), we see that after a quadratic or a cubic extension, a curve having a reduction type with a node of multiplicity $2p$ or $3p$ has now a reduction type with only nodes of multiplicity $p$.

Consider now a curve $X/K$ having as reduction type one of the two graphs in 2.3 with two nodes of multiplicity $p$. Given any prime $q \neq p$, we can use the construction explained in 2.5 to show that after an extension $F/K$ of degree $q$, a regular model of $X_F/F$ has a component of multiplicity $p$ meeting three components of multiplicities coprime to $p$. We use then Lemma 2.6 to show that such a configuration cannot be contracted into any good regular model. This concludes the proof of Theorem 2.2. □

**Remark 2.7** When $[L:K] = p$, we showed in Part (2) of 2.2 that the reduction of $X/K$ is of type $\mathrm{I}^*(\nu + \alpha_1 + \alpha_2, p, a, b)$ with $p \mid \alpha_i$. Let us write $\alpha_i = ps_i$. It would be of interest to understand what determines the integers $s_1$ and $s_2$. In view of Theorem 2.8 below, it is natural to wonder whether these integers are equal to $s_{L/K}$, where $(p-1)(s_{L/K}+1)$ is the valuation of the different of $L/K$. Recall that if $H_0 \supseteq H_1 \supseteq \dots$ denote the sequence of higher ramification groups of the Galois extension $L/K$, the valuation of the different of $L/K$ is $\sum_{i=0}^{\infty}(|H_i|-1)$.

When $X/K$ is hyperelliptic and $p > 2$, $s_1 = s_2$ because in this case the local rings $\mathcal{O}_{\mathcal{Z},Q_1}$ and $\mathcal{O}_{\mathcal{Z},Q_2}$ are isomorphic (under the hyperelliptic involution).

Note that the integers $s_1$ and $s_2$ do not enter in the structure of the group $\Phi_{A,K}$, as this group is cyclic of order $p^2$ if and only if $\nu$ coprime to $p$ (indeed, $\Phi_{A,K}$ is cyclic if and only if $\nu + p(s_1+s_2)$ is coprime to $p$ ([7], Lemma 3). The sum $s_1+s_2$ does however affect the values of Grothendieck's pairing (use [4], 5.1).

**Theorem 2.8.** *Let $p = 2$. Let $v$ denote the valuation of $K$. Let $E/K$ be an elliptic curve with $j$-invariant $j(E)$ having additive reduction and potentially multiplicative reduction. Let $L/K$ denote the extension minimal with the property that $E_L/L$ has multiplicative reduction. Then $[L:K] = 2$. Denote by $s_{L/K}+1$ the* *valuation of the different of $L/K$, and let $\nu := -v(j(E)) > 0$. Then the reduction of $E_L/L$ is of type* $\mathrm{I}_{2\nu}$*, and $E/K$ has reduction of type $I^*_{\nu+4s_{L/K}}$.*

*Proof.* The reduction type of $E_L/L$ is well-known. We have included it in the statement of the theorem only for completeness. We treat first the case of equicharacteristic 2. Since $v(j(E)) < 0$ because $E/K$ has potentially multiplicative reduction, we find that $E/K$ can be given over $K$ by an equation $y^2 + xy = x^3 + a_2x^2 + a_6$, with $a_6 \neq 0$ (see, e.g., [23], App. A, 1.1). This latter equation has $\Delta = a_6$, and $j = 1/a_6$. It follows that this curve has potentially multiplicative reduction if and only if $v(a_6) > 0$. The reduction is multiplicative when $a_2 \in \mathcal{O}_K$. In general, this curve achieves multiplicative reduction over $L := K(z)$ with $z^2+z+a_2 = 0$ (to see this, make the change of variables $y = Y + zx$). Thus, over $L$, this equation is minimal and its minimal discriminant is $a_6$. Hence, $E_L/L$ has reduction type $\mathrm{I}_{2\nu}$ with $\nu = v(a_6)$.

Any quadratic extension $L/K$ given by an Artin-Schreier equation of the form $z^2 + z + D = 0$ with $D \in K$ is ramified if and only if $v(D) < 0$, and in this case it is possible to assume that $v(D)$ is odd. Let $\pi$ denote a uniformizer of $\mathcal{O}_K$. Choose then $D = u\pi^{-r} \in K^*$ with negative odd valuation $-r$ and $u$ a unit so that $L = K(w)$ with $w^2 + w + D = 0$. Our initial elliptic curve can then be given by the equation $y^2 + xy = x^3 + Dx^2 + a_6$. Set $s$ such that $r = 2s - 1$. An integral equation for $E/K$ is given by

$$y^2 + \pi^s xy = x^3 + \pi^{2s} Dx^2 + \pi^{6s} a_6.$$

It turns out that this equation for $E/K$ is already minimal. To compute the type of reduction we use Tate's Algorithm, [24], IV 9.4, page 367. Our equation satisfies the conditions of Step 7 in loc. cit. The subprocedure in Step 7 requires us, for a curve $y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6$, to consider the polynomials $(a_2/\pi)x^2 + (a_4/\pi^{(n+1)/2})x + a_6/\pi^n$ if $n$ is odd, and $y^2 + (a_3/\pi^{n/2})x - a_6/\pi^n$ if $n$ is even.

If $v(a_6\pi^{6s})$ is odd, then we can pick up the subprocedure with $n = v(a_6) + 6s$. We choose a lift $b \in \mathcal{O}_K$ of a root of $(\pi^{2s}D/\pi)x^2 + a_6\pi^{6s}/\pi^n$ modulo $\pi$, and make the change of variable $x = X + b\pi^{3s+(v(a_6)-1)/2}$. The coefficient $a_4$ after the translation is $3(b\pi^{3s+(v(a_6)-1)/2})^2$, which has valuation $6s + v(a_6) - 1$. The coefficient $a_3$ after the translation is $b\pi^{4s+(v(a_6)-1)/2}$, with valuation $4s + (v(a_6) - 1)/2$. The constant coefficient has valuation at least $v(a_6) + 6s + 1$. We continue the subprocedure and make a translation of the form $y = Y + c\pi^{3s+(v(a_6)+1)/2}$. After this translation, the new coefficient of $x$ has valuation $4s+(v(a_6)+1)/2$. The new constant coefficient has valuation at least $v(a_6)+6s+2$. The reader will check that in all further translations required in the subprocedure, the valuation of the coefficients of $x$ and $y$ are left unchanged by the translation. When we reach the even $n = 8s+v(a_6)-1$, we consider the appropriate polynomial $y^2+(A_3/\pi^{n/2})y-A_6/\pi^n$, and find that it has distinct roots modulo $\pi$, since the coefficient of $y$ is a unit. Then the reduction is of type $\mathrm{I}^*_{n-3}$, with $n-3 = 8s-4+v(a_6) = 4r+v(a_6)$. An easy computation shows that $H_0 = \cdots = H_r = \mathbb{Z}/2\mathbb{Z}$, and $H_{r+1} = \{0\}$, so that $r = s_{L/K}$, as desired.

If $v(a_6)$ is even, then we can pick up the subprocedure with $n = v(a_6) + 6s$, and we choose $b \in \mathcal{O}_K$ such that $\pi \mid b^2 - a_6\pi^{-v(a_6)}$ to make the change of variable $y = Y + b\pi^{3s+v(a_6)/2}$. The coefficient of $x$ after the translation is $b\pi^{4s+v(a_6)/2}$ and the constant coefficient has valuation at least $6s + v(a_6) + 1$. We continue the subprocedure and make a translation of the form $x = X + c\pi^{3s+v(a_6)/2}$. After this translation, the new coefficient of $x$ still has valuation $4s + v(a_6)/2$, and the new coefficient of $y$ has valuation $4s + v(a_6)/2$. The new constant coefficient has valuation at least $v(a_6) + 6s + 2$. The reader will check that in all further translations required in the subprocedure, the valuation of the coefficients of $x$ and $y$ are left unchanged by the translation. When we reach the odd $n = 8s+v(a_6)-1$, we consider the appropriate polynomial $(A_2/\pi)x^2 + (A_4/\pi^{(n+1)/2})x + A_6/\pi^n$, and find that it has distinct roots modulo $\pi$, since the coefficient of $x$ is a unit. Then the reduction is of type $\mathrm{I}^*_{n-3}$, with $n - 3 = 8s - 4 + v(a_6) = 4r + v(a_6)$.

Consider now the case where $K$ is of mixed characteristic 2. Since $E/K$ has potentially multiplicative reduction, $j := j(E) \in K \setminus \mathcal{O}_K$. Let $C/K$ be given by the equation

$$y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}.$$

This is an elliptic curve with $j(C) = j$, and $\Delta = j^2/(j-1728)^3$. Assume now that $j = u\pi^{-\nu}$ with $u \in \mathcal{O}_K^*$ and $\nu > 0$. Then $C/K$ has split multiplicative reduction over $\mathcal{O}_K$. Note that $v(\Delta) = \nu$, so that, in particular, after any quadratic extension $M/K$, $C_M/M$ has reduction of type $\mathrm{I}_{2\nu}$. The curves $C/K$ and $E/K$, having same $j$-invariants, are twists of each other, and since $j \neq 0, 1728$, they become isomorphic after a quadratic extension $L/K$. Pick $D \in \mathcal{O}_K$ such that

$L := K(\sqrt{D})$, and consider the quadratic twist $C_D/K$ given by the equation

$$y^2 = x^3 + \frac{D}{4}x^2 - D^2Ax - D^3B, \tag{2.8.1}$$

with $A = 36/(j-1728)$ and $B = 1/(j-1728)$. Without loss of generality, we may assume that $v(D) = 0$ or $1$. Consider first the case where $v(D) = 1$. The equation (2.8.1) is not integral, and an obvious change of variables transforms it in

$$y^2 = x^3 + Dx^2 - 2^4D^2Ax - 2^6D^3B.$$

We have $v(2^4D^2A) = 6v(2) + 2 + \nu$, and $v(2^6D^3B) = 6v(2) + 3 + \nu$.

As in the equicharacteristic case, we consider the cases where $6v(2) + 3 + \nu$ is odd and even separately. Assume that it is odd. We can pick up subprocedure 7 with $n = 6v(2) + 3 + \nu$, and choose $b \in \mathcal{O}_K$ a lift of a root of $(D/\pi)x^2 - (2^6D^3B)/\pi^{6v(2)+3+\nu}$ modulo $\pi$. We make the change of variables $x = X + b\pi^{3v(2)+1+\nu/2}$. The coefficient of $x$ in the translated equation has valuation $4v(2) + 2 + \nu/2$. The constant coefficient has valuation at least $6v(2) + 4 + \nu$. We continue the subprocedure and make a translation of the form $y = Y + c\pi^{3v(2)+2+\nu/2}$. After this translation, the new coefficient of $x$ still has valuation $4v(2)+2+\nu/2$, and the new coefficient of $y$ has valuation $4v(2)+2+\nu/2$. The new constant coefficient has valuation at least $6v(2) + 5 + \nu$. The reader will check that in all further translations required in the subprocedure, the valuation of the coefficients of $x$ and $y$ are left unchanged by the translation. When we reach the odd $n = 8v(2) + \nu + 3$, we consider the appropriate polynomial $(A_2/\pi)x^2 + (A_4/\pi^{(n+1)/2})x + A_6/\pi^n$, and find that it has distinct roots modulo $\pi$, since the coefficient of $x$ is a unit. Then the reduction is of type $\mathrm{I}^*_{n-3}$, with $n - 3 = 8v(2) + \nu$.

The valuation of the different of the extension $L/K$ is computed as follows: $\sqrt{D}$ is a uniformizing parameter, and $\sigma(\sqrt{D}) - \sqrt{D} = -2\sqrt{D}$ has valuation $v_L(2)+1$. It follows that $\sum_{i=0}^{\infty} |H_i| - 1 = v_L(2) + 1$, so that the reduction is $I^*_{4s_{L/K}+\nu}$, as desired. We leave the case where $6v(2) + 3 + \nu$ is even to the reader.

When $v(D) = 0$, consider an Eisenstein equation for $L/K$, given by the equation $z^2 + az + b$, with $v(a) > 0$ and $v(b) = 1$. It follows that $a^2 - 4b = Dc^2$ for some element with $v(c) \geq 0$. More precisely, we must have $v(a) = v(c) \leq v(2)$. The valuation of the different of the extension $L/K$ is computed as follows: a root $\beta$ of $z^2 + az + b = 0$ is a uniformizing parameter, and $\sigma(\beta) - \beta = -a - 2\beta$.

Thus, $v_L(\sigma(\beta)-\beta) = v_L(a)$. It follows that $\sum_{i=0}^{\infty} |H_i| - 1 = v_L(a)$. Our goal is to show that the reduction is then $I^*_{4v_L(a)-4+\nu}$.

Note that $\frac{D}{4} - \frac{a^2}{4c^2} = -\frac{b}{c^2}$. Make the change of variables $y = Y + \frac{a}{2c}x$ in (2.8.1) to obtain an equation of the form

$$y^2 + \frac{a}{c}xy = x^3 - \frac{b}{c^2}x^2 - D^2Ax - D^3B.$$

This equation is not integral, and an obvious change of variables transforms it into

$$y^2 + axy = x^3 - bx^2 - c^4D^2Ax - c^6D^3B.$$

We have $v(c^4D^2A) = 4v(c) + 2v(2) + \nu$, and $v(c^6D^3B) = 6v(c) + \nu$. Again, we need to distinguish the cases where $6v(c)+\nu$ is odd or even. Assume that it is even. Choose an element $d \in \mathcal{O}_K$ such that $v(d^2 + (c^6D^3B)/\pi^{6v(c)+\nu}) > 0$. Make the change of variable $y = Y + d\pi^{3v(c)+\nu/2}$. The coefficient of $x$ in the translated equation has valuation $4v(a)+\nu/2$, and the coefficient of $y$ has valuation $3v(c)+v(2)+\nu/2$. The constant coefficient has valuation at least $6v(c)\nu+1$. We continue the subprocedure and make a translation of the form $x = X + e\pi^{3v(c)+\nu/2}$. After this translation, the new coefficient of $x$ still has valuation $4v(2)+\nu/2$, and the new coefficient of $y$ has valuation at least $4v(a)+\nu/2$. The new constant coefficient has valuation at least $6v(a)+\nu+2$. The reader will check that in all further translations required in the subprocedure, the valuation of the coefficients of $x$ and $y$ are left unchanged by the translation. When we reach the odd $n = 8v(a)+\nu-1$, we consider the appropriate polynomial $(A_2/\pi)x^2 + (A_4/\pi^{(n+1)/2})x + A_6/\pi^n$, and find that it has distinct roots modulo $\pi$, since the coefficient of $x$ is a unit. Then the reduction is of type $\mathrm{I}^*_{n-3}$, with $n-3 = 8v(a)-4+\nu$, as desired. The case where $6v(c)+\nu$ is odd is similar and is left to the reader. □

**Remark 2.9** When $p=2$, an elliptic curve over $K$ with potentially multiplicative reduction is a twist of an elliptic curve with multiplicative reduction over $K$. This property is not true anymore in general when $p = g+1$ is an odd prime.

Let $X/K$ be a curve with $t_L = g$ after a wild extension $L/K$. Let $A := \mathrm{Jac}(X)/K$. Assume that there exists an abelian variety $B/K$ with purely multiplicative reduction over $K$ such that $A_L$ and $B_L$ are isomorphic over $L$. We claim that $X_L/L$ has reduction of type $\mathrm{I}(\nu, p)$ with $[L:K] \mid \nu$. Indeed, since $B/K$ has semi-stable reduction over $K$, the component group $\Phi_{B_L,L}$ has order $|\Phi_{B,K}|[L:K]^g$. A proof of this fact can be obtained using the description of

$\Phi_{B,K}$ and $\Phi_{B_L,L}$ given in [3], pp. 291-292. Since the component group $\Phi_{A_L,L}$ has order $p\nu^g$ (2.2 (1)) and $g > 1$, we find that $[L : K] \mid \nu$.

Since $\Phi_{A,K}$ is killed by $p$ if and only if $p \mid \nu$, we find that such a twist $A/K$ always has a component group killed by $p$. Note also that the order of the component group $\Phi_{B,K}$ is always divisible by $p$.

## 3. Some evidence for Question 1.1

Toward a positive answer to Question 1.1, we propose the following results. Let $X/K$ be a curve with a good regular model $\mathcal{X}/\mathcal{O}_K$ whose associated arithmetical graph $(G, M, R)$ is either of type (a) or (b):

(a) $G$ is a tree with a single node $D$, of multiplicity $p$ and degree $d \geq 3$, and such that all terminal vertices have multiplicity 1.
(b) $G$ is a tree with $m \geq 1$ nodes, all with multiplicity $p > 2$ and degree 3, and such that all chains of $G$ have weight 1.

Let $A = \mathrm{Jac}(X)$, and assume that $a_K = 0$.

**Corollary 3.1.** *Let $X/K$ be a curve as above with $a_K = t_K = 0$. Then*

(1) *If $X$ is of type (a), assume that $d$ is odd, and if $X$ is of type (b), assume that $m$ is odd. Then $a_L > 0$.*
(2) *If $X$ is of type (a), assume that $d = 3$ or $4$, and if $X$ is of type (b), assume that $m = 1$ or $2$. Then $A/K$ has potentially good reduction.*

*Proof*: In both cases (a) and (b), the graph $G$ is a tree and we find that $t_K = 0$. The order of $\Phi_{A,K}$ can be computed using [12], 2.5, and is found to be $p^{d-2}$ in case (a), and $p^m$ in case ($b$). An easy computation shows that $\mathrm{ord}_p(|\Phi_{A,K}|)(p-1) = 2u_K$. Part (1) follows immediately from 3.14.

Both $d = 3$ and $m = 1$ produce the same graph, and in this case, $p = 2g + 1$. Since $[L : K]^2$ kills $\Phi_{A,K}$ when $t_K = 0$ ([10], 1.8), we find that $p \mid [L : K]$. Part (2) follows then in these cases from 2.1 (1). In the cases $d = 4$ and $m = 2$, we have $p = g+1$. Using the facts recalled in 2.4, we find that for any tame extension $F/K$, $A_F$ has purely additive reduction. We may thus apply 2.1 when $\mathrm{Gal}(L/F)$ is the $p$-Sylow subgroup of $\mathrm{Gal}(L/K)$ and find that $A$ has either potentially purely multiplicative or potentially good reduction. Part (2) follows from 2.2 when $p > 2$, and from 2.8 when $p = 2$. □

**3.2** To state our next theorem, let us recall the following facts. Let $A/K$ be any abelian variety. The group $\Phi_{A,K}$ is endowed with two filtrations, the $\Theta$-filtration and the $\Sigma$-filtration, where in the following diagram all maps are inclusions:

$$\begin{array}{ccccccc} \{0\} & \longrightarrow & \Theta^3 & \longrightarrow & \Theta^2 & \longrightarrow & \Theta^1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^3 & \longrightarrow & \Sigma^2 & \longrightarrow & \Sigma^1 & \longrightarrow & \Phi_{A,K} \end{array}$$

(see [6], and [15] for the prime-to-$p$ part of $\Phi_{A,K}$). The cokernel of each vertical map can be generated by $t_K$ elements. Thus, when $t_K = 0$, all vertical maps are isomorphisms.

**3.3** It is not hard to show that $\Theta^2 \subseteq \Psi_{K,L}$ ([10], proof of 1.8). When $\ell \neq p$, it is likely that the $\ell$-parts of these two groups coincide (see [15], 3.22, for some evidence). When $\ell = p$, little is known about a possible equality $\Theta^2 = \Psi_{K,L}$.

Let $A'/K$ denote the abelian variety dual of $A/K$. Grothendieck's pairing

$$< \, , \, >: \Phi_{A,K} \times \Phi_{A',K} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is known to be perfect when $A/K$ is a Jacobian and $X(K) \neq \emptyset$ ([4], 4.6). The filtrations $\Sigma$ and $\Theta$ are dual to each other under $< \, , \, >$ ([2], 6.1) when Grothendieck's pairing is perfect for all abelian varieties $B/K$. Grothendieck's pairing is known to be perfect when, for instance, $K$ is of mixed characteristic [1], and it is conjectured to be always perfect when $k$ is perfect. When $A/K$ is principally polarized, fix a principal polarization and consider the associated pairing $< \, , \, >: \Phi_{A,K} \times \Phi_{A,K} \to \mathbb{Q}/\mathbb{Z}$. Then the duality of the filtrations means that $\Sigma^i$ is the orthogonal of $\Theta^{4-i}$ for $i = 1, 2, 3$. When $t_K = 0$, $\Theta^3$ is the orthogonal of $\Theta^2$.

**Theorem 3.4.** *Assume that Grothendieck's pairing is perfect for all abelian varieties $B/K$. Let $X/K$ be a curve as above, and such that the associated graph is of type* (b)*. Then $\Psi_{K,L} = \Phi_{A,K}$.*

*Proof*: Under our hypothesis, $t_K = 0$. Assume that $\Psi_{K,L} \neq \Phi_{A,K}$. Then, since $\Theta^2 \subseteq \Psi_{K,L}$, we find that $\Phi_{A,K}/\Theta^2$ is not trivial. Our hypothesis on Grothendieck's pairing allows us to use the duality of the filtration to obtain that $\Theta^3$ is not trivial. We will show below that the group $\Phi_{A,K}$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space of dimension $m$ (this fact also follows from [12], 2.5). Let $V$ be any proper subspace of $\Phi_{A,K}$ such that $\Theta^2 \subseteq V$. Then the pairing restricted to $V$ is degenerate. Indeed, consider a basis for $V$ consisting of a basis for $\Theta^3$, completed into a basis for $\Theta^2$, completed again into a basis for $V$. The matrix $((< e_i, e_j >))$ of the pairing in this basis has its first $\dim(\Theta^3)$ columns linearly dependent, since $\dim(\Theta^3) + \dim(\Theta^2) > \dim(V)$. Our goal is to produce a contradiction to the hypothesis $\Psi_{K,L} \neq \Phi_{A,K}$ by exhibiting such a vector space $V$ on which the pairing is non-degenerate.

Consider the graph $G(\mathcal{X})$. Each connecting chain contains a vertex of degree prime to $p$ by hypothesis. Pick one such vertex for each chain, and let $F/K$ denote the unique tame extension whose order is the least common multiple of the multiplicities of the chosen vertices. We leave it to the reader to check (using the facts recalled in 2.4) that $X_F/F$ has again reduction of type (b) with the same number of nodes, but now in addition each connecting chain on the reduction graph has a vertex of multiplicity 1. Since we assume that $\Psi_{K,L} \neq \Phi_{A,K}$, and since the map $\Phi_{A_L,L} \to \Phi_{A_{FL},FL}$ is injective for any finite extension $F/K$, we find that $\Psi_{F,FL} \neq \Phi_{A_F,F}$.

Assume from now on that each connecting chain on the graph $(G(\mathcal{X}), M, R)$ contains a vertex of multiplicity 1. Fix such a vertex for each connecting chain, and label these vertices $E_1, \dots, E_t$. All terminal vertices of the graph also have multiplicity 1. For each node $D_i$ on the graph, we choose two vertices $C_{i,1}$ and $C_{i,2}$ in the set consisting of the vertices $E_i$ and of the terminal vertices, such that $C_{i,1}$ and $C_{i,2}$ are each on a chain attached to $D_i$ but not on the same chain. We can do so such that $\{C_{i,1}, C_{i,2}, i = 1, \dots, m\}$ contains the vertices $E_1, \dots, E_t$.

The component group $\Phi_{A,K}$ can be identified with the torsion subgroup of $\mathbb{Z}^v/M(\mathbb{Z}^v)$. Let $E(C_{i,1}, C_{i,2}) \in \mathbb{Z}^v$ be the vector with null components everywhere, except for a $+1$ in the $C_{i,1}$-component, and a $-1$ in the $C_{i,2}$-component. Since $C_{i,1}$ and $C_{i,2}$ have multiplicity 1, the vector $E(C_{i,1}, C_{i,2})$ defines a torsion element in $\mathbb{Z}^v/M(\mathbb{Z}^v)$ that we will denote by $\tau_i$.

We claim that $\{\tau_i, i = 1, \dots, m\}$ generates $\Phi_{A,K}$. Indeed, it follows from [17], 4.4, that each element $\tau_i$ has order $p$. The proof of [17], 4.4, also shows that there can be no linear relations among $\{\tau_i, i = 1, \dots, m\}$. Since $|\Phi_{A,K}| = p^m$ ([12], 2.5), we find that $\{\tau_i, i = 1, \dots, m\}$ is a basis over $\mathbb{Z}/p\mathbb{Z}$ for $\Phi_{A,K}$, as desired. The matrix of the pairing associated with this set of vectors can be computed using

Proposition 5.1 in [4]. We find that $< \tau_i, \tau_j >= 0$ if $i \neq j$. Since the pairing is perfect, we must then have $< \tau_i, \tau_i > \neq 0$ for all $i = 1, \ldots, m$.

If $\Psi_{K,L} \neq \Phi_{A,K}$, then we may assume that the image of some $\tau_i$, say $\tau_m$, in $\Phi_{A_L,L}$ is not trivial. Let $V$ denote the subvector space generated by $\{\tau_1, \ldots, \tau_{m-1}\}$. We claim then that $\Psi_{K,L} \subseteq V$. Once this claim is proved, we have a contradiction since the pairing on $V$ is clearly still diagonal and non-degenerate.

Consider the model $\mathcal{V}/\mathcal{O}_L$ obtained as the minimal desingularization of the normalization $\mathcal{W}$ of the model $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_L$. Let $f : \mathcal{V} \to \mathcal{X}$ denote the natural morphism. Since the multiplicity of the components $C_{i,j}$ is 1, we find that the preimage in $\mathcal{W}$ of $C_{i,j}$ consists of a single component. We denote by by $C'_{i,j}$ the proper transform in $\mathcal{V}$ of this component. We are going to break the graph $G(\mathcal{V})$ as follows. Choose an irreducible $C'_{i,j}$, and consider the $s_{ij}$ (topological) connected components of $G(\mathcal{V}) \setminus \{C'_{i,j}\}$. Since the multiplicity of $C'_{i,j}$ is 1, we can formally add a vertex $C'_{i,j}$ to each of the $s_{ij}$ connected components of $G(\mathcal{V}) \setminus \{C'_{i,j}\}$ to obtain $s_{ij}$ new arithmetical graphs $G_s$, $s = 1, \ldots, s_{ij}$. It is not hard to check that $\Phi_{G(\mathcal{V})}$ is isomorphic to $\prod_{s=1}^{s_{ij}} \Phi_{G_s}$. Repeat the process with these new graphs and the other vertices $C'_{k,\ell}$, $(k, \ell) \neq (i, j)$. We obtain in this way new arithmetical graphs $G_s$, $s = 1, \ldots, \sigma$, and a decomposition of $\Phi_{A_L,L}$ into a product $\prod_{s=1}^{\sigma} \Phi(G_s)$. The key is that no graph $G_s$ can contain more than two elements of the form $C'_{i,j}$.

Consider now the maps $\Phi_{A,K} \to \Phi_{A_L,L} \to \prod_{s=1}^{\sigma} \Phi(G_s)$. Let $v'$ denote the number of irreducible components in the special fiber $\mathcal{V}_k$. Identify $\Phi_{A_L,L}$ with the torsion subgroup of $\mathbb{Z}^{v'}/M(\mathcal{V})(\mathbb{Z}^{v'})$. Let $\tau'_i$ denote the image of the vector $E(C'_{i,1}, C'_{i,2}) \in \mathbb{Z}^{v'}$ in $\mathbb{Z}^{v'}/M(\mathcal{V})(\mathbb{Z}^{v'})$. Then $\tau'_i$ is identified with the image of $\tau_i$ under the map $\Phi_{A,K} \to \Phi_{A_L,L}$. This can be easily proved by picking two points $P_{i,1}$ and $P_{i,2}$ in $X(K)$ which reduce to $C_{i,1}$ and $C_{i,2}$ respectively, and noting that the reduction in $\mathcal{V}_k$ of the two $L$-rational points $P_{i,1}$ and $P_{i,2}$ are in $C'_{i,1}$ and $C'_{i,2}$, respectively. Under the appropriate identifications, the point in $A(K)$ defined by the divisor $P_{i,1} - P_{i,2}$ reduces in $\Phi_{A,K}$ to $\tau_i$. Similarly, the (same) point in $A(L)$ defined by the divisor $P_{i,1} - P_{i,2}$ reduces in $\Phi_{A_L,L}$ to $\tau'_i$.

Note now that by construction, $\tau'_i$ is mapped under the isomorphism $\Phi_{A_L,L} \to \prod_{s=1}^{\sigma} \Phi(G_s)$ to a vector in $\prod_{s=1}^{\sigma} \Phi(G_s)$ with at most one non-zero component. Our assumption is that the image of $\tau_m$ in $\prod_{s=1}^{\sigma} \Phi(G_s)$ is not zero. It follows easily then that the image of any element $\sum_{j=1}^{m} c_j \tau_j$ with $p \nmid c_m$ is also non-trivial. It follows that $\Psi_{K,L} \subseteq V := < \tau_1, \ldots, \tau_{m-1} >$, as desired. $\square$

**Remark 3.5** The conclusion of Theorem 3.4 also holds when the associated graph is of type (a). The proof of this result is longer and more delicate, but uses essentially the same techniques as the proof of 3.4.

**3.6** Having proved in 3.4 that $\Psi_{K,L} = \Phi_{A,K}$ for some Jacobians $A/K$, it is natural to wonder whether we could deduce from this fact that $A/K$ has potentially good reduction. Unfortunately, this implication is not true for all Jacobians, as it may happen for instance that $\Phi_{A,K} = \{0\}$ and $A/K$ has purely multiplicative reduction (e.g., an elliptic curve with reduction $\mathrm{I}_1$; see also 2.2, (3)). However, in the cases considered in 3.4, the $p$-part of the component group is 'maximal', and $t_N = 0$ for any tame extension $N/K$. Our next lemma shows that we are reduced in this case to consider the above question only when $\Psi_{K,L} = \Phi_{A,K}$ and $L/K$ is a $p$-extension.

**Lemma 3.7.** *Let $A/K$ be a Jacobian with $\mathrm{ord}_p(|\Phi_{A,K}|)(p-1) = 2u_K$. Let $L_0/K$ denote the maximal tame subextension of $L/K$ and assume that $t_{L_0} = 0$. If $\Psi_{K,L} = \Phi_{A,K}$, then $\Psi_{L_0,L} = \Phi_{A_{L_0},L_0}$.*

*Proof.* The kernel of the natural map $\Phi_{A,K} \to \Phi_{A_{L_0},L_0}$ is killed by $[L_0 : K]$. Thus, since $p \nmid [L_0 : K]$,

$$2u_K = \mathrm{ord}_p(|\Phi_{A,K}|)(p-1) \leq \sum_{\ell \text{ prime}} \mathrm{ord}_\ell(|\Phi_{A_{L_0},L_0}|)(\ell - 1) \leq 2u_{L_0},$$

where the latter inequality follows from (2.1.2). Since $u_{L_0} \leq u_K$, we find that $u_{L_0} = u_K$ and $\Phi_{A,K} \to \Phi_{A_{L_0},L_0}$ is an isomorphism. It follows that $\Psi_{L_0,L} = \Phi_{A_{L_0},L_0}$. □

**Proposition 3.8.** *Let $\ell \neq p$ be a prime. Let $A/K$ be an abelian variety with $a_K = t_K = 0$. Assume that $\mathrm{Gal}(L/K)$ is an $\ell$-group. Then:*

(1) *The $\ell$-part of $\Phi_{A,K}$ is not trivial.*
(2) *Assume that $t_L > 0$. When $A/K$ has a polarization of degree prime to $\ell$, then the $\ell$-parts of $\Psi_{K,L}$ and $\Phi_{A,K}/\Psi_{K,L}$ are not trivial. In particular, $\ell^2 \mid |\Phi_{A,K}|$.*
(3) *Assume that $A/K$ is principally polarized and that $\Phi_{A,K}$ and $\Psi_{K,L}$ have isomorphic $\ell$-parts. Then $A/K$ has potentially good reduction.*

*Proof.* Let $T_\ell$ denote the Tate module of $A$. The Galois group $I_K := I(\overline{K}/K)$ acts on $T_\ell$. Our assumption is that its pro-$p$-Sylow subgroup $P$ acts trivially on $T_\ell$. Let $\sigma$ denote a topological generator of $I_K/P$, and let $\sigma_\ell$ denote the automorphism of $T_\ell^P = T_\ell$ induced by $\sigma$. The $\ell$-part $\Phi_\ell$ of the group $\Phi_{A,K}$ is isomorphic to the torsion subgroup $F$ of $T_\ell^P/(\sigma_\ell - \mathrm{id})(T_\ell^P)$ (see, e.g., [15], 2.1). To any submodule $X$ of $T_\ell$, we associate a subgroup $s(X)$ of $F$, defined to be the torsion subgroup of $(X + (\sigma_\ell - \mathrm{id})(T_\ell))/(\sigma_\ell - \mathrm{id})(T_\ell)$ ([15], 2.10). Using now our hypothesis that $a_K = t_K = 0$, we find that $T_\ell^{I_K} = (0)$. Fix a polarization on $A/K$, and consider the induced pairing on $T_\ell \times T_\ell$. The orthogonal of $T_\ell^{I_K}$ under this pairing is thus the full module $T_\ell$. Using this fact and the exact sequence in [15], 2.10, we find that

$$F/s(X) = \frac{T_\ell}{X + (\sigma_\ell - \mathrm{id})(T_\ell)}.$$

Thus, $F/s(X)$ is the cokernel of the map $(\sigma_\ell - \mathrm{id}) : T_\ell/X \to T_\ell/X$. Choosing now a submodule $X$ such that $T_\ell/X$ is free and $\sigma_\ell$ has finite order when acting on $T_\ell/X$, we can apply 3.9 to find that $F/s(X)$ is a non-trivial $\ell$-group. When $A$ has potentially good reduction, $X = (0)$ is such a submodule.

When $t_L > 0$, a natural choice for such a module $X$ is to take $X = T_\ell^{I_L}$. Since we assume that $t_L > 0$, we have $T_\ell^{I_L} \neq T_\ell$. Let

$$\Sigma_\ell(A) \subseteq \Sigma_\ell^{[2]}(A) \subseteq \Sigma_\ell^{[3]}(A)$$

denote the filtration[6] of the $\ell$-part $\Phi_\ell$ of $\Phi_{A,K}$ introduced in [15], 2.9. The group $\Sigma_\ell^{[3]}$ is defined in 2.9 as $s(T_\ell^{I_L})$. Thus we have proved above that $\Phi_\ell/\Sigma_\ell^{[3]}$ is a non-trivial $\ell$-group, ending the proof of (1).

We may now use the fact that $A/K$ has a polarization of degree prime to $\ell$ to obtain, using [15], 3.21, that there are three subgroups $\Theta_\ell^{[3]} \subseteq \Theta_\ell^{[2]} \subseteq \Theta_\ell$ of $\Phi_\ell$ such that $\Theta_\ell^{[3]}$ is isomorphic to $\Phi_\ell/\Sigma_\ell^{[3]}$. As we just showed that the latter group is not trivial, we obtain that $\Theta_\ell^{[2]}$ is not trivial. It is shown in 3.22 that $\Theta_\ell^{[2]}$ is the $\ell$-part of the group $\Psi_{K,L}$. It remain to show that the $\ell$-part of $\Phi/\Psi_{K,L}$ is not trivial. This follows from the fact that $\Theta_\ell^{[2]} \subseteq \Sigma_\ell^{[3]}$. This inclusion is not noted explicitly in [15], but follows from [15], 3.21: As the pairing is non-degenerate when restricted to $\Theta_\ell$, and as $\Theta_\ell^{[3]}$ is orthogonal to $\Theta_\ell^{[2]}$ under this pairing, we conclude that the intersection of $\Theta_\ell$ with $\Sigma_\ell^{[3]}$ (the orthogonal of $\Theta_\ell^{[3]}$ under the original pairing) is equal to $\Theta_\ell^{[2]}$. Thus, $\Theta_\ell^{[2]} \subseteq \Sigma_\ell^{[3]}$, completing the proof of (2). Part (3) follows immediately from (2). □

[6] This filtration is expected to be the $\ell$-part of the filtration $\Sigma^3 \subseteq \Sigma^2 \subseteq \Sigma^1$ later introduced by Bosch and Xarles, who chose a reverse ordering.

**Lemma 3.9.** *Let $\ell$ be any prime. Let $G$ be a finite $\ell$-group acting on a free finitely generated $\mathbb{Z}_\ell$-module $X$. If $X^G = (0)$, then $H^1(G, X) \neq (0)$. In particular, if $G =< \sigma >$, then $X/(\sigma - 1)(X)$ is not trivial.*

*Proof.* Since $X$ is torsion free, the sequence

$$0 \longrightarrow X \xrightarrow{\ \ell\ } X \longrightarrow X/\ell X \longrightarrow 0$$

is exact, with associated long exact sequence

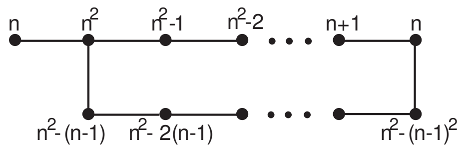
$$0 \longrightarrow X^G \xrightarrow{\ \ell\ } X^G \longrightarrow (X/\ell X)^G \longrightarrow H^1(G, X) \xrightarrow{\ \ell\ } H^1(G, X).$$

By hypothesis, $X^G = (0)$. Since the $\ell$-group $X/\ell X$ is not trivial and is acted upon by the $\ell$-group $G$, $(X/\ell X)^G \neq (0)$ (use [22], IX, §1, Lemme 2).

In case $G$ is cyclic of order $r$, the group $H^1(G, X)$ is isomorphic to $\ker(N)/(\sigma - 1)(X)$, where $N = 1 + \sigma + \cdots + \sigma^{r-1}$, and the latter group is isomorphic to the torsion subgroup of $X/(\sigma - 1)(X)$. □

**Remark 3.10** The condition $t_K = 0$ is needed in 3.8 (1). Indeed, for each $\ell \neq p$, there exists a Jacobian $A/K$ with $\mathrm{Gal}(L/K)$ cyclic of order $\ell^2$ and with $t_K = t_L = 1$, but such that $\Phi_{A,K} = (0)$.

Consider a field $K$ and a curve $X/K$ with a good model $\mathcal{X}/\mathcal{O}_K$ having reduction of the type $(G, M, R)$, with $G$ and $R$ represented below.




We claim that the component group of the matrix $M$ is trivial. To see this, it suffices to prove that the principal minor $M'$ obtained from $M$ be removing the row and column corresponding to the node of $G$ has determinant $n^4$ ([12], 1.1). The matrix $M'$ has only $-2$'s on its diagonal, except for $-n$ at the place corresponding to the vertex of degree 1, and $-3$ at the place corresponding to the vertex of $G$ of degree 2 and multiplicity $n$. The 'graph' of $M'$ is disconnected, with one (topological) component consisting of the terminal vertex of self-intersection $-n$, and a second component consisting of a chain. Considering this latter chain as two terminal chains attached to the vertex of self-intersection $-3$, the matrix

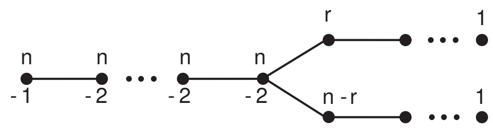
$M'$ can be shown to be row and column equivalent to the matrix $M''$ below plus an identity matrix of the appropriate size, where

$$M'' := \begin{pmatrix} -n & 0 & 0 & 0 \\ 0 & -3 & n^2-n-1 & n-1 \\ 0 & 1 & -(n^2-n) & 0 \\ 0 & 1 & 0 & -n \end{pmatrix}$$

The reader will verify that $\det(M'') = n^4$. Assume that all components of the good model are rational curves. We also leave it to the reader to check that when $p \nmid n$, the curve $X/K$ achieves semi-stable reduction over an extension of order $n^2$, with $a_L = n(n-1)/2$ and $t_L = 1$.

**Remark 3.11** The condition $a_K = 0$ is needed in 3.8 (1). Indeed, for each $\ell \neq p$, there exists a Jacobian $A/K$ with $\mathrm{Gal}(L/K)$ cyclic of order $\ell$ and with $a_K > 0 = t_K$, but such that $\Phi_{A,K} = (0)$.

Consider a field $K$ and a curve $X/K$ with a good model $\mathcal{X}/\mathcal{O}_K$ having reduction of type $(G, M, R)$, with $G$ and $R$ represented below and $p \nmid n$.




The terminal vertex of multiplicity $n$ on the left represents an irreducible component $C$ of genus $g(C) > 0$. All other components are assumed to be of genus 0. The curve $X/K$ has genus $ng(C)$, and has semi-stable reduction after an extension of degree $n$. Either the Jacobian of $X/K$ has potentially good reduction, or $t_L = n-1$ and $2a_L - 2 = n(2g(C) - 2)$.

**Remark 3.12** Let $\ell = p$. The following is an example where $\Phi_{A,K} = (0)$ and $[L : K] = p$, with $t_L = 0$. Thus the statement of 3.8 (1) does not hold when $\ell = p$ (we do not have a similar example where $t_L > 0$). Let $\pi_K$ denote a uniformizer of a field $K$ of characteristic 0. Let $X/K$ denote the smooth hyperelliptic curve given by the affine equation $y^2 = x^m + \pi_K$, with $m$ odd. This curve has genus $(m-1)/2$.

The scheme $\mathrm{Spec}(\mathcal{O}_K[x, y]/(y^2 - x^m - \pi_K))$ is regular, and the minimal regular minimal model of $X/K$ has an irreducible and reduced special fiber. The component group $\Phi_{A,K}$ of $A := \mathrm{Jac}(X/K)$ is thus trivial.

Let $p = 2$. The curve $X/K$ has potentially good reduction. Indeed, over $L = K(\sqrt{\pi_K})$, $X_L/L$ is given by the equation $y^2 = x^m + \pi_L^2$. We rewrite this equation as $\pi_L^2((y/\pi_L)^2 - 1) = x^m$. A translation leads to the equation $\pi_L^2 z(z+2) = x^m$, which we transform into $4\pi_L^2 u(u+1) = x^m$. The curve $u(u+1) = v^m$ has genus $(m-1)/2$ when $p = 2$ and $m$ is odd. In particular, $X$ achieves good reduction over $L(\sqrt[m]{4\pi_L^2})$. Since $p = 2$, we can choose $\pi_L^{2s} = 2$ with $m \mid 2s+1$ to produce an example of a field $K$ such that $[L : K] = 2$ and $X_L/L$ has good reduction.

To be able to address the case $\ell = p$ in 3.8 in our next proposition, we review the following facts. Except for $\Theta^1$, all subgroups in the two filtrations of $\Phi_{A,K}$ in 3.2 can be defined directly as images of components (sub)groups associated with the rigid analytic uniformization of $A/K$, as follows. Recall that there exist a semi-abelian variety $G/K$ and a lattice $\Lambda$ in $G$ such that the following sequence of rigid analytic groups is exact ([B-X], Theorem 1.2):

$$0 \to \Lambda \to G \to A \to 0. \tag{3.12.1}$$

Moreover, $G/K$ is an (algebraic) extension of an abelian variety $B/K$ with potentially good reduction by a torus $T/K$. Denote by $\mathcal{L}$, $\mathcal{G}$, $\mathcal{T}$, $\mathcal{B}$, and $\mathcal{A}$, the Néron models over $\mathcal{O}_K$ of $\Lambda$, $G$, $T$, $B$ , and $A$, with component groups $\Phi_\Lambda$, $\Phi_G$, $\Phi_T$, $\Phi_B$, and $\Phi_{A,K}$. The exact sequence (3.12.1) induces an isomorphism $\mathcal{G}_k^0 \simeq \mathcal{A}_k^0$ ([6], Theorem 2.3) and an exact sequence

$$0 \to \Phi_\Lambda \to \Phi_G \to \Phi_{A,K}$$

([6], Theorem 4.12). Since $\Lambda$ is a discrete group, $\mathcal{L}$ is locally finite over $\mathcal{O}_K$. Thus $\Lambda(K) \simeq \mathcal{L}_k(k) \simeq \Phi_\Lambda$. In particular, $\Phi_\Lambda$ is torsion free and $(\Phi_G)_{\mathrm{tors}}$ injects into $\Phi_{A,K}$.

Let $T^I/K$ denote the maximal split subtorus of the torus $T/K$. Consider the following diagram of component (sub)groups obtained from the corresponding natural maps of Néron models:

$$\begin{array}{ccccccc}
\{0\} & \longrightarrow & (\Phi_T)_{\mathrm{tors}} & \longrightarrow & (\Phi_G)_{\mathrm{tors}} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Phi_{T^I} & \longrightarrow & \Phi_T & \longrightarrow & \Phi_G & \longrightarrow & \Phi_{A,K}.
\end{array}$$

Taking the images in $\Phi_{A,K}$ of each of these groups produces the diagram of subgroups

$$\begin{array}{ccccccc} \{0\} & \longrightarrow & \Theta^3 & \longrightarrow & \Theta^2 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma^3 & \longrightarrow & \Sigma^2 & \longrightarrow & \Sigma^1 & \longrightarrow & \Phi_{A,K}. \end{array}$$

**Proposition 3.13.** *Let $\ell$ be any prime. Let $A/K$ be an abelian variety with a rigid analytic uniformization $A = G/\Lambda$ as above and $t_L > t_K = 0$.*

(1) *Assume that* $\mathrm{Gal}(L/K)$ *is an $\ell$-group. Then $(\Phi_T)_{\mathrm{tors}} \neq (0)$. If the canonical map $(\Phi_T)_{\mathrm{tors}} \to (\Phi_G)_{\mathrm{tors}}$ is not trivial, then the $\ell$-part of the group $\Theta^3$ is not trivial. If in addition $A/K$ has a polarization of degree prime to $\ell$ and Grothendieck's pairing is perfect for all abelian varieties $B/K$, then the $\ell$-part of $\Phi_{A,K}/\Sigma^1$ is also not trivial. In particular, $\ell^2 \mid |\Phi_{A,K}|$.*
(2) *Assume that $t_L = g$. If $A/K$ has a polarization of degree prime to $\ell$, then* $\mathrm{ord}_\ell(|\Phi_{A,K}|)$ *is even. If* $\mathrm{Gal}(L/K)$ *is an $\ell$-group, then $\ell^2 \mid |\Phi_{A,K}|$.*

*Proof.* (1) It is shown in [1], or [28] 2.18, that the group $(\Phi_T)_{\mathrm{tors}}$ is isomorphic to $H^1(\mathrm{Gal}(L/K), X(T))$, where $X(T)$ is the Galois module of characters of $T/K$. It follows then immediately from 3.9 that the group $(\Phi_T)_{\mathrm{tors}}$ is not trivial. We recalled above that $(\Phi_G)_{\mathrm{tors}}$ injects in $\Phi_{A,K}$. Hence, if the canonical map $(\Phi_T)_{\mathrm{tors}} \to (\Phi_G)_{\mathrm{tors}}$ is not trivial, we find that $\Theta^3 \neq (0)$. By duality ([2], 6.1), the group $\Phi_{A,K}/\Sigma^1$ is not trivial. Since $\Theta^3 \subseteq \Sigma^1$, our claim follows.

(2) When $t_L = g$, $T = G$ and, in particular, $\Theta^3 = \Theta^2$. If $\ell \mid |\Phi_{A,K}|$, then either $\ell \mid |\Theta^2|$ or $|\Phi_{A,K}/\Theta^2|$. But when $A/K$ has a polarization of degree prime to $\ell$, we find by duality that these groups are isomorphic, and $\mathrm{ord}_\ell(|\Phi_{A,K}|)$ is even. To justify that the filtrations are dual to each other, we note that when $t_L = g$, the hypothesis in [2], 6.1, reduces to the perfectness of the pairing for $A/K$, and that the perfectness holds for $A/K$ due to [2], 5.3, (iii). When $T = G$ and $\mathrm{Gal}(L/K)$ is an $\ell$-group, the hypothesis that $\Phi_T \to \Phi_G$ is non-trivial is obviously satisfied, and we conclude using (1) that $\ell \mid |\Theta^3|$. This fact also holds for the dual abelian variety $A'$, and $\ell \mid |\Theta^3(A')|$. By duality, $\ell \mid |\Phi_{A,K}/\Theta^2(A)|$. □

**Corollary 3.14.** *Let $A/K$ be a Jacobian, and $\ell$ a prime. If $a_K = t_K = 0$ and* $\mathrm{ord}_\ell(|\Phi_{A,K}|)$ *is odd, then $a_L > 0$.*

*Proof.* If $a_L = 0$, then $t_L = g$, and we can apply 3.13 (2) to find that $\mathrm{ord}_\ell(|\Phi_{A,K}|)$ is even, a contradiction. □

**Remark 3.15** Let $K$ be a field of mixed characteristic, so that Grothendieck's pairing is known to be perfect. We construct now examples of abelian varieties uniformized as $G/\Lambda$ where $\Phi_T \to \Phi_G$ is the trivial map. Using the duality of the filtration [2], the dual of these abelian varieties will produce examples of abelian varieties with $a_K = t_K = 0$, $t_L > 0$, and with $\Psi_{K,L}$ and $\Phi_{A,K}$ having equal $p$-parts.

We choose a non-split torus $T/K$ with a point $P \in T(K)$ of prime order $\ell$ such that $P$ reduces to a generator of $\Phi_T$. (For the existence of such a torus, see, e.g., [10], 4.18, and 4.4.) We also choose an abelian variety $B/K$ with potentially good reduction and with a point $P' \in B(K)$, also of prime order $\ell$, but such that $P'$ reduces to the trivial element in $\Phi_B$. Note that when $\ell \neq p$, this latter condition can only happen if the abelian rank of $B/K$ is positive. Choose a lattice $\Lambda \subset T \times B$, and let $A_0 := (T \times B)/\Lambda$. Let $A/K$ denote the abelian variety quotient of $A_0$ by the subgroup generated by the image of the point $(P, P')$. The abelian variety $A/K$ is uniformized by $G := T \times B/ < (P, P') >$, with lattice $\Lambda'$, image of $\Lambda$ under the quotient $T \times B \to G$. By construction, the closed immersion $T \to G$ produces the trivial map $\Phi_T \to \Phi_G$.

Assuming that we constructed an example with $t_K(A) = 0$ and with the map $\Phi_T \to \Phi_G$ being trivial, we find that the group $\Theta^3_{A,K}$ is trivial. By duality, the dual abelian variety $A'/K$ has the property that $\Phi_{A',K}/\Theta^2(A') = (0)$, so $\Theta^2(A') = \Psi_{K,L}(A') = \Phi_{A',K}$. Since $A'$ is isogenous to $A$, $t_K(A') = 0$ and $t_L(A') > 0$.

**Remark 3.16** The condition $t_L > t_K = 0$ may possibly imply that $\Psi_{K,L} \neq \Phi_{A,K}$ when $\Phi_{A,K}$ is 'as large as possible', that is, when $\mathrm{ord}_p(|\Phi_{A,K}|)(p-1) = 2g$. Indeed, it is shown in [6], 5.10, that

$$\mathrm{ord}_p(|\Theta^3_{A,K}|)(p-1) \leq t_L - t_K, \text{ and } \mathrm{ord}_p(|\Phi_{A,K}/\Theta^2_{A,K}|)(p-1) \leq t_L - t_K.$$

It is natural, in view of (2.1.2), to ask whether (2.1.2) holds for any abelian variety with $t_K = 0$. If such were the case, it would follow from [6], 5.6 (iii), that

$$\mathrm{ord}_p(|\Theta^2_{A,K}|/|\Theta^3_{A,K}|)(p-1) \leq 2(a_L - a_K)$$

holds in general. Assume now that $t_L > t_K$ and that $\mathrm{ord}_p(|\Phi_{A,K}|)(p-1) = 2u_K$. Then the condition $\Psi_{K,L} \neq \Phi_{A,K}$ is implied by the conditions $\Theta^2_{A,K} = \Psi_{K,L}$ and $\mathrm{ord}_p(|\Theta^2|/|\Theta^3|)(p-1) \leq 2(a_L - a_K)$.

## 4. On the extension $[L:K]$

**Proposition 4.1.** *Let $\mathcal{X}/\mathcal{O}_K$ be a regular model of a curve $X/K$. Let $C$ be a component of $\mathcal{X}_k$ of multiplicity $r$, and let $C_i$, $i = 1, \dots, d$, denote the irreducible components of $\mathcal{X}_k$ meeting $C$. Let $r_i$ denote the multiplicity of $C_i$.*

(1) *If $g(C) > 0$, then $r \mid [L:K]$.*
(2) *Assume that $\mathcal{X}$ is a good model and that $d \geq 3$. If there exists a prime $\ell \neq p$ such that $\ell \mid r$, $\ell \nmid r_1r_2$, and $\mathrm{ord}_\ell(r_3) < \mathrm{ord}_\ell(r)$, then $p^{\mathrm{ord}_p(r)} \mid [L:K]$.*
(3) *Assume that $\mathcal{X}$ is a good model whose associated graph is a tree, and that $d \geq 3$. Suppose that $\mathrm{Jac}(X)$ has potentially good reduction. Assume that $p \mid r$ and $p \nmid r_1r_2$. Let $m := \min\{\mathrm{ord}_p(\gcd(r_i, r)), i = 3, \dots, d\}$. Then $p^{\mathrm{ord}_p(r)-m} \mid [L:K]$.*

*Proof.* (1) This statement is well-known. We recall its proof for later use. Let $\mathcal{X}/\mathcal{O}_K$ be a regular model of $X/K$. Let $F/K$ be any (totally ramified) finite extension of degree $e$. Let $\mathcal{W} \to \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_F)$ denote the normalization of the base change, and $\rho : \mathcal{W} \to \mathcal{X}$ denote the normalization map followed by the natural projection map. The scheme $\mathcal{W}/\mathcal{O}_F$ is a normal model of $X_F/F$. Let $\mu : \mathcal{V} \to \mathcal{W}$ denote the minimal desingularization of $\mathcal{W}$. Let $\mathcal{V} \to \mathcal{V}^{min}$ denote the morphism from $\mathcal{V}$ to the minimal regular model of $X_F/F$. Let $C$ be a component of $\mathcal{X}_k$ of multiplicity $r$, with generic point $\xi \in \mathcal{X}$. Let $C'$ be an irreducible component of $\rho^{-1}(C)$, with generic point $\eta \in \mathcal{W}$. Consider the associated local rings, as in the diagram below:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X},\xi} & \longrightarrow & \mathcal{O}_{\mathcal{W},\eta} \\ \uparrow & & \uparrow \\ \mathcal{O}_K & \longrightarrow & \mathcal{O}_F. \end{array}$$

Let $\pi_K$ and $\pi_F$ be uniformizers of $K$ and $F$, and $\pi_\xi$ and $\pi_\eta$ be uniformizers of $\mathcal{O}_{\mathcal{X},\xi}$ and $\mathcal{O}_{\mathcal{W},\eta}$. By hypothesis, $(\pi_K) = (\pi_F)^e$, and $(\pi_K) = (\pi_\xi)^r$. Suppose now that $F = L$. In this case, we claim that $C'$ has multiplicity 1 in $\mathcal{W}_k$. Indeed, the model $\mathcal{V}^{min}$ being semi-stable by hypothesis, is then reduced. The component $C'$ is of positive genus since $g(C) > 0$. Thus, the strict transform of $C'$ in $\mathcal{V}$

is not contracted in $\mathcal{V}^{min}$. Hence, we find that $(\pi_L) = (\pi_\eta)$. It follows that $(\pi_K) = (\pi_\eta)^{[L:K]} = (\pi_\xi)^r$ in $\mathcal{O}_{\mathcal{W},\eta}$. Thus, $r \mid [L:K]$.

(2) Assume first that $\mathrm{ord}_\ell(r_3) = 0$. Let $F/K$ be an extension of degree $\ell$. Consider the normalization of the base change $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_F$. The preimage $C'$ of $C$ in this normalization has multiplicity $r/\ell$, and the induced morphism $C' \to C$ has degree $\ell$ and is ramified in at least three points by assumption. Thus $g(C') > 0$. It follows that $X_F$ has a model with a component of multiplicity $r/\ell$ and of positive genus. Thus, we can apply (1) to this component to obtain that $r/\ell \mid [LF:F]$. It follows that $p^{\mathrm{ord}_p(r)} \mid [L:K]$.

If $\mathrm{ord}_\ell(r_3) > 0$, we reduce the situation to the case $\mathrm{ord}_\ell(r_3) = 0$ by first making the base change by the tame extension $F'/K$ of degree $\ell^{\mathrm{ord}_\ell(r_3)}$. The details are left to the reader.

(3) Write $r = p^{\mathrm{ord}_p(r)}\rho$. Let $K'/K$ denote the tame extension of order $\rho$. Consider the regular model $\mathcal{X}'/\mathcal{O}'_K$ obtained as the minimal desingularization of the normalization of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{K'}$. Let $D$ denote a component of $\mathcal{X}'_k$ mapping surjectively to the component $C$ in $\mathcal{X}_k$. We leave it to the reader to check, using the facts recalled in 2.4, that $D$ has multiplicity $p^{\mathrm{ord}_p(r)}$, that at least two components $D_i$, $i = 1, 2$, of $\mathcal{X}'_k$ meeting $D$ transversally have multiplicity $r'_i$ coprime to $p$, and that one component $E$ of $\mathcal{X}'_k$ meeting $D$ transversally has multiplicity $r'$ with $\mathrm{ord}_p(\gcd(r, r')) \leq m$. Thus, it suffices now to prove the proposition when $r = p^{\mathrm{ord}_p(r)}$.

Write now $[L:K] = p^{\mathrm{ord}_p([L:K])}\lambda$. Let $K''/K$ denote the tame extension of order $\lambda$. The normalization of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{K''}$ is singular at a preimage of a point of the form $C \cap C_i$ if the multiplicity of $C_i$ is not divisible by $\lambda$. If necessary, use a sequence of blow-ups as in 2.5 to obtain a regular model $\mathcal{X}_1 \to \mathcal{X}$ for $X/K$ such that the strict transform of $C$ only meets components with multiplicities divisible by $\lambda$. Consider the regular model $\mathcal{X}'_1/\mathcal{O}_{K''}$ obtained as the minimal desingularization of the normalization of $\mathcal{X}_1 \times_{\mathcal{O}_K} \mathcal{O}_{K''}$. Let $D$ denote a component of $(\mathcal{X}'_1)_k$ mapping surjectively to the strict transform of the component $C$ in $(\mathcal{X}_1)_k$. We leave it to the reader to check, using the facts recalled in 2.4, that $D$ has multiplicity $p^{\mathrm{ord}_p(r)}$, that at least two components $D_i$, $i = 1, 2$, of $(\mathcal{X}'_1)_k$ meeting $D$ transversally have multiplicity $r'_i$ coprime to $p$, and that one component $E$ of $(\mathcal{X}'_1)_k$ meeting $D$ transversally has multiplicity $r'$ with $\mathrm{ord}_p(\gcd(r, r')) \leq m$. Thus, it suffices now to prove the proposition when $r = p^{\mathrm{ord}_p(r)}$ and $[L:K] = p^{\mathrm{ord}_p([L:K])}$.

Upon renumbering if necessary, we may assume that $r_3$ is such that $m = \mathrm{ord}_p(\gcd(r, r_3))$. Assume that $p^{\mathrm{ord}_p(r)-m} > [L : K]$. Consider the model $\mathcal{V}/\mathcal{O}_L$ obtained as the minimal desingularization of the normalization $\mathcal{W}$ of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_L$. Let $\mathcal{Y}/\mathcal{O}_L$ denote the minimal regular (semi-stable) model of $X_L/L$, and denote by $\mathcal{V} \to \mathcal{Y}$ the natural contraction map. Let $D$ be a component of $\mathcal{V}_k$ whose image in $\mathcal{X}$ is $C$. Our hypothesis implies that the multiplicity $r_D$ of $D$ in $\mathcal{V}_k$ is a power of $p$ divisible by $p^{m+1}$ (use the formalism recalled in the proof of (1)). Thus, this component must be contracted under the morphism $\mathcal{V} \to \mathcal{Y}$. Since $r_1$ and $r_2$ are coprime to $[L : K]$, the preimages $E_1$ and $E_2$ in $\mathcal{W}$ of the components $C_1$ and $C_2$ are irreducible, of multiplicities $r_1$ and $r_2$, respectively. Since $\mathrm{Jac}(X)$ has potentially good reduction, we find that the preimage of $C$ in $\mathcal{W}$ is also irreducible, otherwise the graph of $\mathcal{V}$ contains loops. Hence, there exists a unique component $D$ of $\mathcal{V}_k$ whose image in $\mathcal{X}$ is $C$.

Consider the connected components of the topological space $G(\mathcal{V}) \setminus \{D\}$. We claim that at least two such components contain a vertex of multiplicity coprime to $p$ and, hence, coprime to $r_D$, and that a third component contains a vertex of multiplicity not divisible by $r_D$. Lemma 2.6 can then be applied to show that this is a contradiction. To prove our claim, consider an irreducible component $E_i$ in the preimage in the normalization of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_L$ of the component $C_i$ of multiplicity $r_i$. For $i = 1, 2$, the multiplicity of $E_i$ is $r_i$. The multiplicity of $E_3$ cannot be divisible by $p^{m+1}$ since $p^{m+1} \nmid r_3$ by hypothesis. Clearly, since $G(\mathcal{X})$ is a tree by hypothesis, $E_1$, $E_2$, and $E_3$ are in three different connected components of the topological space $G(\mathcal{V}) \setminus \{D\}$. □

The simplest example of reduction type where neither of the above results can be applied to obtain an affirmative answer to Question 1.4 is the case of a tree with a unique node, of degree 3 and multiplicity $p^n$, and with terminal multiplicities $p^{n-1}$, 1, and 1. When $g = 1$, there are two such types[7], III and III$^*$ when $p = 2$, which we consider in our next lemma.

**Lemma 4.2.** *Let $p = 2$ and let $E/K$ be an elliptic curve with reduction of type* III *or* III$^*$. *Then* 4 *divides* $[L : K]$. *In fact, given any extension $M/K$ of degree* 2, *$E_M/M$ has reduction $I_n^*$ for some $n > 0$.*

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[7]The Kodaira type III does not have normal crossings. A sequence of two blow-ups leads to a good model to which Question 1.4 applies.

*Proof*: In the case of type III, there exists a Weierstrass equation for $E/K$ of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $v(a_1), v(a_2), v(a_3) \geq 1$, $v(a_4) = 1$, and $v(a_6) \geq 2$. After a translation $x = X + b$, the equation becomes

$$y^2+a_1xy+(a_3+a_1b)y = x^3+(a_2+3b)x^2+(a_4+3b^2+2a_2b)x+(b^3+a_2b^2+a_4b+a_6).$$

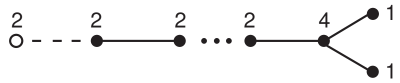
Consider any ramified extension $M/K$ of degree 2, and choose $b \in \mathcal{O}_M$ with $v_M(b) = 1$ and $v_M(b^2+a_4) > 2$ (such a $b$ exists since $k$ is algebraically closed and $v_M(a_4) = 2$). Then $v_M(a_4+3b^2+2a_2b) > 2$ and $v_M(b^3+a_2b^2+a_4b+a_6) > 3$. Since $v_M(a_2 + 3b) = 1$, we find that the translated equation satisfies the conditions of Tate's algorithm, Step 7 on page 367 of [24], and that the reduction is of type $\mathrm{I}_n^*$ for some $n > 0$.

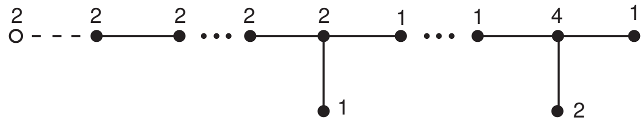
Let us now consider the case of type III$^*$. Then there exists a Weierstrass equation for $E/K$ with $v(a_1) \geq 1$, $v(a_2) \geq 2$, $v(a_3) \geq 3$, $v(a_4) = 3$, and $v(a_6) \geq 5$. Consider any ramified extension $M/K$ of degree 2. The equation is not minimal anymore, and we can make the change of variables $X := x/\pi_M^2$, and $Y := y/\pi_M^3$. The new equation has $v_M(a_1) \geq 1$, $v_M(a_2) \geq 2$, $v_M(a_3) \geq 3$, $v_M(a_4) = 2$, and $v_M(a_6) \geq 4$. Choose again $b \in \mathcal{O}_M$ with $v_M(b) = 1$ and $v_M(b^2 + a_4) > 2$. Then $v_M(a_4 + 3b^2 + 2a_2b) > 2$ and $v_M(b^3 + a_2b^2 + a_4b + a_6) > 3$. Since $v_M(a_2 + 3b) = 1$, we find that the translated equation satisfies the conditions of Tate's algorithm, Step 7 on page 367 of [24], and that the reduction is of type $\mathrm{I}_n^*$ for some $n > 0$.

Since the component group of the reduction $\mathrm{I}_n^*$ has order 4, we find that 2 divides $[ML : M]$ and, hence, 4 divides $[L : K]$. □

**Remark 4.3** A more easily generalizable proof of the above proposition could possibly be obtained if one had a better general understanding of $\mathbb{Z}/p\mathbb{Z}$-quotient singularities. Indeed, $E/K$ has good reduction over an extension $L/K$ (minimal with this property). Let $\mathcal{Y}/\mathcal{O}_L$ denote the smooth model of $E_L/L$, endowed with the action of $H := \mathrm{Gal}(L/K)$. Let $\mathcal{Z}/\mathcal{O}_K$ denote the quotient $\mathcal{Y}/H$. This is a normal model of $E/K$, with quotient singularities. A regular model $\mathcal{X}/\mathcal{O}_K$ of $E/K$ is obtained by resolving these singularities.

Assume now that $[L : K] = 2$, with reduction of type III or III$^*$. Then $\mathcal{Z}$ has a single singular point $P$ since $\mathcal{Y}_k$ is supersingular, and $\mathcal{Z}_k$ has multiplicity 2 ([18], 6.1). Say that the reduction of $E/K$ is of type III, which we blow up to get a model with normal crossings. Then the resolution of $P$ has the following possible shapes, where the open circle represents the strict transform of $\mathcal{Z}_k$:




The first possibility can be eliminated because the intersection matrix $N$ of the resolution has always a component group $\Phi_N$ cyclic of order 8, and $[L:K]$ kills $\Phi_N$ ([18], 3.4). The second possibility can be eliminated for the same reason when the number $\alpha$ of components of multiplicity 2 on the initial chain on the left is odd: in this case $\Phi_N = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. However, when $\alpha$ is even, then $\Phi_N = (\mathbb{Z}/2\mathbb{Z})^3$. An easy computation shows that the fundamental cycle $Z$ has $Z^2 = -[L:K]$, providing no indication that $N$ cannot be the resolution of a cyclic quotient singularity ([18], 3.3). It would be interesting to know whether the latter matrix $N$ with $\alpha$ even can ever occur as the intersection matrix associated with the resolution of a $\mathbb{Z}/2\mathbb{Z}$-singularity. If such were not the case, we would obtain a second proof of 4.2 when the reduction is III.

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