WILD QUOTIENTS OF PRODUCTS OF CURVES

DINO LORENZINI

ABSTRACT. Let k be an algebraically closed field of characteristic $p > 0$. Let B_1/k and B_2/k be two smooth proper connected curves, each endowed with an automorphism $\sigma_i : B_i \to B_i$ of order p. Let $Y := B_1 \times B_2$, and let $\sigma : Y \to Y$ be the automorphism $\sigma_1 \times \sigma_2$. We show that the graph of the resolution of any singularity of $Y/\langle\sigma\rangle$ is a star-shaped graph with three terminal chains when B_2 is an ordinary curve of positive genus. The intersection matrix N of the resolution satisfies $|\det(N)| = p^2$, and can be completely determined when B_1 is also ordinary, or when σ_1 has a unique fixed point. The singularity is rational.

Wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities of surfaces are expected to have resolution graphs which are trees, with associated intersection matrices N satisfying $|\det(N)| = p^r$ for some $r \geq 0$. We show, for any $s > 0$ coprime to p, the existence of resolution graphs with one node, $s+2$ terminal chains, and with intersection matrix N satisfying $|\det(N)| = p^{s+1}$.

KEYWORDS Product of curves, cyclic quotient singularity, rational singularity, wild, intersection matrix, resolution graph, fundamental cycle.

MSC: 14B05, 14G20 (14E15, 14H20, 13H15, 14J17)

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. In this article, we study the wild quotient singularities of the simplest type of surfaces over k, the quotients of products of curves by a 'diagonal' automorphism of order p.

Let B_1/k and B_2/k be two smooth proper connected curves, each endowed with an automorphism $\sigma_i : B_i \to B_i$ of order p. Let $Y := B_1 \times B_2$, and let $\sigma : Y \to Y$ be the automorphism $\sigma_1 \times \sigma_2$. Let P_i be a fixed point of σ_i, $i = 1, 2$. Then $Z := Y/\langle\sigma\rangle$ is singular at the image Q of (P_1, P_2). Our aim is to provide information on the resolution of the singularity Q. Let us introduce the following notation.

Let Z/k be a normal surface. Let $Z_{\rm sing}$ denote the singular locus of Z. A morphism $f : Z^{\rm desing} \to Z$ will be called a *minimal resolution of the singularities of* Z if the following properties hold: The scheme $Z^{\rm desing}/k$ is a smooth proper variety and f is a birational proper morphism whose restriction $Z^{\rm desing} \setminus f^{-1}(Z_{\rm sing}) \to Z \setminus Z_{\rm sing}$ is an isomorphism. Moreover, for each closed point $R \in Z_{\rm sing}$, the divisor $f^{-1}(R)$ is assumed to have smooth components with normal crossings, and to be minimal with this property.

Recall that to a connected curve $\cup_{i=1}^n E_i$ on a regular surface X/k we associate an intersection matrix N and a connected graph $G(N)$ as follows. For each $1 \leq i, j \leq n$, let $(E_i \cdot E_j)_X$ denote the intersection number of E_i and E_j on the regular scheme X. Then $N := ((E_i \cdot E_j)_X)_{1 \leq i,j \leq n}$, and $G(N)$ is the graph whose vertices are denoted by $E_1, \ldots, E_n$, and when $i \neq j$, E_i is linked to E_j by $(E_i \cdot E_j)_X$ edges. For future reference, recall that the *degree* of a vertex E in a graph G is the number of edges attached to E. A vertex of degree at least 3 on a graph is called a *node*. A vertex of degree 1 is a *terminal vertex*. A *chain* is a subgraph of G with vertices $C_0, C_1, \ldots, C_m$, $m \geq 1$, such that C_i is linked

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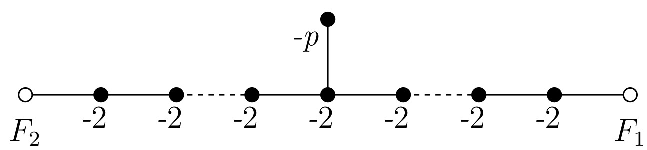
to C_{i+1} by exactly one edge in G when $i = 0, \dots, m-1$, and the degree of C_i is 2 when $i = 1, \dots, m-1$. If the chain contains a terminal vertex (which can only be C_0 or C_m), the chain is called a *terminal chain*.

Consider the curves $B_1 \times \{P_2\}$ and $\{P_1\} \times B_2$ on the surface $Y := B_1 \times B_2$, and let F_1, resp. F_2, be their images in $Z := Y/\langle\sigma\rangle$. Both F_1 and F_2 contain the image Q of (P_1, P_2). In the graphs below, we indicate by open circles the strict transforms of F_1 and F_2 in the desingularization Z^{desing}, and we denote these strict transforms again by F_1 and F_2. The negative integer next to a given vertex of a desingularization graph is the self-intersection $(E_i \cdot E_i)_{Z^{\mathrm{desing}}}$ of the corresponding irreducible curve E_i.

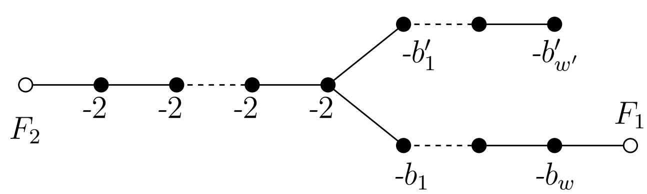
Recall that a curve B/k of genus $g > 0$ is called *ordinary* if its Jacobian $\mathrm{Jac}(B)$ has exactly p^g points of order dividing p. Our most easily stated result is in the case where both B_1 and B_2 are ordinary curves of positive genus.

Theorem 1.1. *Let B_1/k and B_2/k be two smooth projective ordinary curves of positive genus. Let $Y := B_1 \times B_2$, and let $\sigma : Y \to Y$ be as above. Then the singularities of $Z := Y/\langle\sigma\rangle$ all have the following symmetric resolution graph with $2p$ (bold) vertices:*



When only one of the curves B_i/k is ordinary, the graph of the resolution of a singular point Q is already more complicated to describe. Our next theorem shows that this graph can be specified by a single integer parameter s coprime to p.

Theorem 1.2. *Let B_1/k and B_2/k be two smooth projective curves of positive genus. Let $Y := B_1 \times B_2$, and let $\sigma : Y \to Y$ be as above. Assume that B_2 is ordinary. Then the singular point Q of $Z := Y/\langle\sigma\rangle$ has an explicit resolution with intersection matrix N depending on a single positive integer parameter s, coprime to p. The graph $G(N)$ is represented below with bold vertices.*



The graph $G(N)$ has one node and three terminal chains. The number of (-2)-components on the terminal chain on the left of the node, including the node itself, is equal to ps. Let r_1 be the unique integer in $[1, p-1]$ such that $r_1 \equiv -s^{-1} \pmod{p}$. Then the pair (p, r_1) uniquely determines the self-intersections $-b_1, \dots, -b_w$, of the terminal chain linked to F_1, and the pair $(p, p - r_1)$ uniquely determines the self-intersections $-b'_1, \dots, -b'_{w'}$ of the last terminal chain. The Smith Normal Form of N is $\mathrm{diag}(1, \dots, 1, p, p)$.

The self-intersections $-b_1, \dots, -b_w$, are easily determined using a variation on the Euclidean algorithm (2.3) applied to the pair (p, r_1), as described in 2.5. (The same applied to $(p, p - r_1)$ holds for the self-intersection of the third terminal chain.) When $s = 1$, the graph $G(N)$ in Theorem 1.2 gives the symmetric graph appearing in Theorem 1.1.

It would be of interest to be able to further specify the parameter s in terms of data attached to the point Q. In this regard, we propose the following conjecture. As above, let (P_1, P_2) denote the preimage of Q in $Y := B_1 \times B_2$. Consider the morphism $B_1 \to D_1 :=$

$B_1/\langle\sigma_1\rangle$ and let Q_1 be the image of P_1. The valuation of the different of the extension $\mathcal{O}_{B_1,P_1}/\mathcal{O}_{D_1,Q_1}$ is then of the form $(s(P_1)+1)(p-1)$ for some integer $s(P_1)$ coprime to p. We conjecture that the integer s in Theorem 1.2 is equal to $s(P_1)$. We can prove this conjecture in one instance, as follows.

Theorem 1.3. *Let B_1/k and B_2/k be two smooth projective curves of positive genus. Let $Y := B_1 \times B_2$, and let $\sigma : Y \to Y$ be as above. Assume that B_2 is ordinary. Assume also that the morphism $B_1 \to D_1 := B_1/\langle\sigma_1\rangle$ is only ramified at one single point, P_1, with image Q_1. The valuation of the different of the extension $\mathcal{O}_{B_1,P_1}/\mathcal{O}_{D_1,Q_1}$ is then of the form $(s(P_1)+1)(p-1)$ for some integer $s(P_1)$ coprime to p, and the singular point Q of $Y/\langle\sigma\rangle$ has a resolution graph as in Theorem 1.2 with $s = s(P_1)$.*

Theorem 1.3 thus completely describes the intersection matrix of a resolution of the $\mathbb{Z}/p\mathbb{Z}$-singularity Q in terms of p and of the wild ramification of the map $B_1 \to D_1$ at P_1. It is quite possible that the hypothesis in Theorem 1.3 that B_2/k is ordinary can be weakened as follows. Let P_2 denote a ramification point of the morphism $B_2 \to D_2 := B_2/\langle\sigma_2\rangle$, with image Q_2. Write the valuation of the different of the extension $\mathcal{O}_{B_2,P_2}/\mathcal{O}_{D_2,Q_2}$ as $(s(P_2)+1)(p-1)$ for some integer $s(P_2) \geq 1$ coprime to p. When B_2 is ordinary, it is known that $s(P_2) = 1$. In general, when $s(P_2) = 1$, the point P_2 is called *weakly ramified.* It is natural to wonder whether the singular point Q of $Y/\langle\sigma\rangle$ image of (P_1, P_2) has an intersection matrix N as in Theorem 1.2 as soon as $s(P_2) = 1$, without also requiring as we do in Theorem 1.2 that all ramification points of $B_2 \to D_2$ are weakly ramified.

The results on explicit desingularizations in this article are completely uniform in p, and provide evidence that the intersection matrix associated with the resolution of Q depends only on the integer $s(P_1)$ when $s(P_2) = 1$. It would be interesting to determine if this remains the case when both $s(P_1)$ and $s(P_2)$ are bigger than 1. In view of the four graphs presented in [16], 4.9, which could possibly occur with $s(P_2) = 2$ and p is odd, one may wonder whether several different types of intersection matrices may arise in general for a given pair $(s(P_1), s(P_2))$ with $s(P_1) \geq 2$ and $s(P_2) \geq 2$.

Remark 1.4 Let σ be an automorphism of order p on a smooth proper surface Y/k, and consider the singularities of $Y/\langle\sigma\rangle$. The literature on resolutions of the wild quotient singularities of $Y/\langle\sigma\rangle$ is sparse, and very few examples of such resolutions are known explicitly. An example where $|\det(N)| = 1$ is given in [19], Example 10, where it is asserted that a certain $(\mathbb{Z}/2\mathbb{Z})$-quotient singularity has resolution graph E_8. Further $(\mathbb{Z}/2\mathbb{Z})$-quotient singularities are resolved with omitted computations in [1], p. 64. In [19], Example 7, a certain $(\mathbb{Z}/3\mathbb{Z})$-quotient singularity is asserted to have resolution graph E_6, which has determinant $|\det(N)| = p = 3$. Theorems 1.1 and 1.3 above when $p = 2$ and the curves B_1 and B_2 are both elliptic curves with their canonical involution are treated in [8], Theorem C (see also [20]). To further put the above theorems in perspective, we list below some of the few general results in the literature pertaining to the resolution of wild quotient singularities. The singularities of $Y/\langle\sigma\rangle$ have been shown to be rational when $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = (0)$ in [9], Main Theorem, using the results of [19]. The case of $K3$-surfaces is discussed in [5], 2.4. The singularities when $p = 2$ and Y is an abelian surface with its canonical involution are discussed in [8]. The recent preprint [18] generalizes some of the results of this paper using a completely different method. Examples of resolutions of wild cyclic quotient singularities that are not obtained from the product of two curves are given in [17].

For the singularities resolved in Theorem 1.2, we further show:

Theorem 4.1. *For each prime p, the singularities resolved in Theorem* 1.2 *have multiplicity p and are rational.*

It follows from [15], 2.6, that we should expect that the intersection matrix N of the resolution graph of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity of surface is such that $|\det(N)| = p^r$ for some $r \geq 0$. (In fact, we expect the stronger result that the Smith group $\Phi_N := \mathbb{Z}^n/\mathrm{Im}(N)$ is killed by p.) One of our initial motivations for our study of the resolution graphs of quotient singularities associated with products of curves was to provide families of examples of wild quotient singularities where the determinant of the matrix N is 'large'. In this respect, we show:

Theorem (see 3.15). *Fix a prime p. For each positive integer s coprime to p, there exists a 2-dimensional regular local ring A of equicharacteristic p endowed with an action of $H := \mathbb{Z}/p\mathbb{Z}$ such that* Spec A^H *is singular exactly at its closed point, and such that the intersection matrix N associated with a minimal resolution of* Spec A^H *has determinant* $|\det(N)| = p^{s+1}$.

We do not know if the statement of Theorem 3.15 also holds when p divides s. In each example presented in this article of a minimal resolution of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity, the associated graph has exactly one node. It is possible, however, that for a fixed p, the set consisting of the number of nodes of the minimal resolution graph of all $\mathbb{Z}/p\mathbb{Z}$-quotient singularities is unbounded. In an earlier version of this article, we asked whether it is possible to exhibit a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity of surface whose resolution graph does not have a node. This question is now addressed in [7].

This paper is organized as follows. In section 2, we review the definition of an intersection matrix, and prove some combinatorial results on intersection matrices needed in the proof of 1.2. We prove Theorems 1.1, 1.2, and 1.3, in section 3. In section 4, we prove that the singularities occurring in 1.2 are rational.

It is my pleasure to thank Qing Liu, Werner Lüktebohmert, and Michel Raynaud, for helpful discussions. Thanks to Jonathan Wahl for bringing to my attention the reference [23], and to Sungkon Chang for sharing with me some code to compute fundamental cycles. Thanks also to the referees for a careful reading of the article.

2. Intersection matrices

In this preliminary section, we first review some terminology pertaining to intersection matrices N. Our main result is Proposition 2.7 below, which will be needed in the proof of Theorem 1.2.

2.1 An $n \times n$ *intersection matrix* $N = (c_{ij})$ is a symmetric negative definite integer matrix with negative coefficients on the diagonal, and non-negative coefficients off the diagonal. The *Smith group* Φ_N of the matrix N is the group $\Phi_N := \mathbb{Z}^n/N(\mathbb{Z}^n)$. It is completely determined by the Smith Normal Form of the matrix N.

We associate a graph $G = G(N)$ to N as follows. Pick n vertices $v_1, \dots, v_n$, and for $i \neq j$ link v_i to v_j in G by exactly c_{ij} edges. We will always assume, unless stated otherwise, that G is connected. When this is the case, N is called *irreducible*.

Recall that if $X, Y \in \mathbb{Z}^n$, we write $X > 0$ (resp., $X \geq 0$) if all coefficients of X are positive (resp., if all coefficients are non-negative). We write $X > Y$ if $X - Y > 0$, and we write $X \geq Y$ if $X - Y \geq 0$.

2.2 Attached to an intersection matrix N is a unique integer vector $Z > 0$ such that $NZ \leq 0$ and such that Z is minimal for this property (i.e., if $Z' > 0$ is an integer vector with $NZ' \leq 0$, then $Z \leq Z'$). This vector is called the *fundamental cycle of* N ([2], p. 132).

The fundamental cycle Z is in general quite a difficult invariant to understand. Therefore, for each $i = 1, \dots, n$, we recall below the definition of a vector R_i associated to N which is an upper bound for Z (i.e., $Z \leq R_i$), and which is quite easy to compute in terms of N.

Let N^* denote the comatrix of N, with $N^*N = NN^* = \det(N)\mathrm{Id}_n$. Let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n$. A symmetric irreducible non-singular positive definite matrix with non-positive off-diagonal coefficients has a comatrix with only positive coefficients ([4], Chapter 6, 2.5-2.7). As the intersection matrix N is non-singular, $-N$ has the above properties, and we find that $(-1)^{n+1}N^*$ has only non-negative coefficients. It follows that if we let $(-1)^{n+1}R_i$ denote the i-th column vector of N^* divided by the greatest common divisor of its coefficients, then R_i has positive coefficients, and $NR_i = -p_ie_i$ for some positive integer p_i.

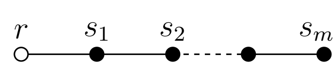
Note that the matrix N can be completely recovered from the following data: the graph $G(N)$, and for some i, the vector R_i and the equality $NR_i = -p_ie_i$. We found it convenient to represent this data by adding a 'virtual' vertex to the graph $G(N)$, as in the graph in 2.3 and at the end of 2.4 below.

2.3 For use below, we recall here the following standard construction. Given an ordered pair of positive integers $r > s$ with $\gcd(r, s) = 1$, we construct an associated intersection matrix $N = N(r, s)$ with vector $R_1 = R_1(r, s)$ and $NR_1 = -re_1$ as follows.

We can find integers $b_1, \dots, b_m > 1$ and $s_1 = s > s_2 > \cdots > s_m = 1$ such that $r = b_1 s - s_2$, $s_1 = b_2 s_2 - s_3$, and so on, until we get $s_{m-1} = b_m s_m$. These equations are best written in matrix form:

$$\begin{pmatrix} -b_1 & 1 & \dots & 0 \\ 1 & -b_2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & \dots & 1 & -b_m \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ \vdots \\ s_m \end{pmatrix} = \begin{pmatrix} -r \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We let N denote the above square matrix, and R_1 the first column matrix. It is well-known that $\det(N) = \pm r$ (see, e.g, [13], 2.6). The matrix N is completely recovered from the data consisting of $G(N)$, R_1, and $NR_1 = -re_1$. We represent this data using the graph $G(N)$, adorning each (bold) vertex of $G(N)$ with the corresponding entry of the vector R_1. To represent the equality $NR_1 = -re_1$, we use an additional open circle, adorned with the integer r, and linked to the vertex of the graph corresponding to the first coefficient of R_1, as follows:



For use in the computation of the arithmetical genus in 4.5, we record here the following easy fact:

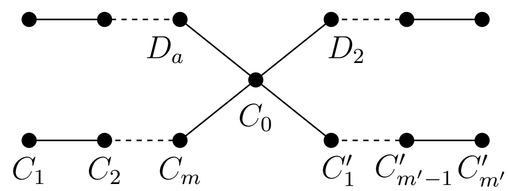
$$\sum_{i=1}^{m} (b_i - 2)s_i = r - s_1 - s_m.$$

2.4 For each prime p, and for any integers $m \geq 1$ and $a \geq 2$, we introduce below a class of intersection matrices N whose associated graphs $G(N)$ are trees with exactly one node

C_0 and $a+1$ terminal chains attached to it. A connected graph with a single node is called *star-shaped*. This class of matrices N is considered in [15], 3.17, with a slight modification of the labeling of the vertices of the graph $G(N)$.

Our goal is to establish a relation in Proposition 2.7 between two of the positive vectors R_i associated to N (notation as in 2.2). For convenience, we will label the n vertices of $G(N)$ so that the vectors of interest will be R_1 and R_n. We start by describing the matrix N with the help of the vector R_1.

Fix $m \geq 1$. Fix $a \geq 2$, and consider positive integers $r_1, \ldots, r_a < p$, such that p divides $r_1 + \cdots + r_a$. Let C_m, C'_1, and $D_2, \ldots, D_a$, denote the neighbors of C_0, that is, the vertices of G linked to C_0 by an edge. We will need the precise labeling of the vertices of only two terminal chains, and we will use the notation $C_1, \ldots, C_m$, and $C'_1, \ldots, C'_{m'}$. We picture below the shape of the graph $G(N)$ with this labeling when $a = 3$.



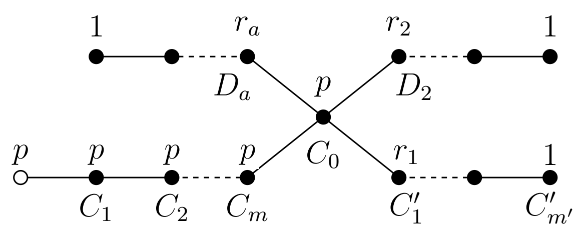
In our labeling of the n vertices of the graph $G(N)$, C_1 corresponds to the first vertex, and $C'_{m'}$ corresponds to the n-th vertex. We now determine N completely by specifying the vector R_1, which will be such that $NR_1 = -pe_1$.

We set the coefficient of R_1 corresponding to C_0 to be p. For $i = 2, \ldots, a$, we set the coefficient corresponding to D_i to be r_i, the coefficient of C'_1 to be r_1, and the coefficient of C_m to be p. The self-intersection of C_0 is $(C_0 \cdot C_0) := -(r_1 + \cdots + r_a + p)/p$.

The matrix N 'restricted' to the chain started by C'_1 is taken to be the matrix constructed in 2.3 using the ordered pair p and r_1. Similarly, the matrix N 'restricted' to the chain started by D_i, $i = 2, \ldots, a$, is taken to be the matrix constructed in 2.3 using the ordered pair p and r_i. The vector R_1 'restricted' to the chain started by D_i is taken to be the corresponding vector described in 2.3. In particular, the coefficient of R_1 corresponding to the terminal vertex of the chain is 1. The terminal chain started by C_m consists of m vertices, all of self-intersection -2. The vector R_1 restricted to this terminal chain has all its coefficients equal to p.

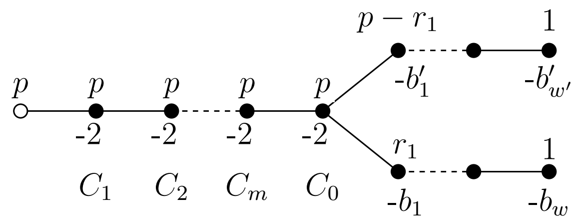
It is easy to check that the vector NR_1 has all its coefficients equal to 0, except for the coefficient corresponding to the vertex C_1, where the coefficient of NR_1 is $-p$. With our labeling, $NR_1 = -pe_1$.

We represent below the data (N, R_1) as follows. The graph $G(N)$ has vertices represented by bullets $\bullet$. A positive number next to a vertex represents the coefficient of this vertex in R_1. Write ${}^tR_1$ for the transpose of R_1. We represent the relation ${}^tR_1 N = (-p, 0, \ldots, 0)$ by attaching a 'virtual' vertex to the terminal vertex of the first terminal chain, represented by an open circle, and we give this virtual vertex the 'multiplicity' p.



The matrix N is completely determined by the shape of its graph and the data $(p, m, r_1, \ldots, r_a)$.

2.5 The special case where $a = 2$ gives the intersection matrices N occurring in Theorem 1.2. Indeed, when $a = 2$, the conditions $r_1, r_2 < p$ and $p \mid (r_1 + r_2)$ imply that $r_1 + r_2 = p$. We picture below for completeness the graph of N along with the data pertaining to R_1.



The negative integer under a vertex is the self-intersection of the vertex, that is, the coefficient on the diagonal of N which corresponds to the vertex. The pair (p, r_1) uniquely determines the self-intersections $-b_1, \ldots, -b_w$ as in 2.3, and the pair $(p, p - r_1)$ uniquely determines the self-intersections $-b'_1, \ldots, -b'_{w'}$. Let $\alpha := m + 1$. The integer α is the number of (-2)-components on the terminal chain on the left of the node, including the node itself. We may denote the intersection matrix N by $N(p, \alpha, r_1)$ and we note that it depends only on the data (p, α, r_1).

2.6 Let us return to the general case of the matrices N with star-shaped graphs having $a + 1$ terminal chains, as defined in 2.4. It follows from [15], 3.14, that $|\Phi_N| = p^a$. We will assume from now on that

$$m = \alpha - 1 = ps - 1$$

for some integer $s > 0$, so that $\alpha = ps$. Then it follows from [15], 3.20, that Φ_N is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^a$.

We now turn to describing the positive vector R_n when the condition (2) in 2.7 below holds for R_1. Recall that we number the vertices so that $C'_{m'}$ is the n-th vertex. We will for convenience denote R_n by $R_{C'_{m'}}$, with $NR_{C'_{m'}} = -p_{C'_{m'}} e_{C'_{m'}}$. Recall that $p_{C'_{m'}}$ is the order of the class of $e_{C'_{m'}}$ in the Smith group Φ_N ([15], 3.5). Since Φ_N is killed by p because of our hypothesis that $\alpha = ps$, we find that $p_{C'_{m'}} = 1$ or p.

Proposition 2.7. *Fix a prime p, and $a \geq 2$. Let N and $G(N)$ be as in 2.4, with $m = ps-1$ for some integer $s > 0$. Let y_1 denote the coefficient of C'_1 in $R_{C'_{m'}}$. Then the following are equivalent:*

(1) *p divides y_1 and the coefficient of C_1 in $R_{C'_{m'}}$ is equal to 1.*

(2) $r_1 s \equiv -1 \pmod p$.

In particular, when either of these equivalent conditions holds, s is coprime to p.

Proof. Let us assume first that the coefficient of C_1 in $R_{C'_{m'}}$ is equal to 1. Then the coefficients in $R_{C'_{m'}}$ of the vertices on the chain $C_1, C_2, \ldots, C_m, C_0$ must be $1, 2, \ldots, ps-1$, and ps. Let $k_2, \ldots, k_a$ denote the coefficient in $R_{C'_{m'}}$ of the terminal vertices of the chains starting at $D_2, \ldots, D_a$. We know that k_i divides ps, and $k_i < ps$ because no vertex on the terminal chain started by D_i has self-intersection -1. Assume now that $p \mid y_1$, and write $y_1 = px_1$. Since the coefficient of C_0 is also divisible by p, then every vertex on the chain started by C'_1 has coefficient in $R_{C'_{m'}}$ divisible by p. Thus, we find that $p_{C'_{m'}} = p$. We claim then that $k_i = s$ for all $i = 2, \ldots, a$. Indeed, we can compute $|\Phi_N| = p^a$ using $R_{C'_{m'}}$ and the formula [15], 3.14. We find that

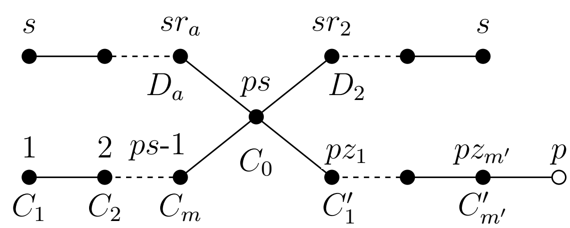
$$\frac{(ps)^{a-1}}{k_2 \cdot \ldots \cdot k_a} p_{C'_{m'}} = p^a.$$

In other words, $s^{a-1} = k_2 \cdot \ldots \cdot k_a$. Since $\frac{ps}{k_2} \cdot \ldots \cdot \frac{ps}{k_a} = p^{a-1}$ and $\frac{ps}{k_i} > 1$, we find that $\frac{ps}{k_i} = p$, so $k_i = s$, as claimed. It follows that the coefficient in $R_{C'_{m'}}$ of any vertex on the chain started by D_i, $i \geq 2$, is simply s times its coefficient in R_{C_1}. We therefore find that

$$\begin{aligned} |C_0 \cdot C_0| p &= p + r_1 + (r_2 + \cdots + r_a), \\ |C_0 \cdot C_0| ps &= ps - 1 + px_1 + (r_2 + \cdots + r_a)s. \end{aligned}$$

It follows that $0 = r_1 s + 1 - px_1$, and (2) holds.

Let us now assume that $r_1 s \equiv -1 \pmod p$, and let $z_1 := \frac{r_1 s+1}{p}$. We claim that the following vector V, given in the diagram below, is equal to $R_{C'_{m'}}$ (so that the diagram below is a representation of the pair $(N, R_{C'_{m'}})$). The integers $z_2, \ldots, z_{m'}$ are specified below.



Let us now describe the coefficients of V. On the chains started by $D_2, \ldots, D_a$, the coefficients of V are those of R_{C_1} multiplied by s. It is clear that

$$|C_0 \cdot C_0| ps = (ps - 1) + pz_1 + (r_2 + \cdots + r_a)s.$$

It remains to describe the integers $z_2, \ldots, z_{m'}$. Write $-a_i := (C'_i \cdot C'_i)$. We have

$$\begin{pmatrix} -a_1 & 1 & \ldots & 0 \\ 1 & -a_2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & \ldots & 1 & -a_{m'} \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let A denote the above square $(m' \times m')$-matrix. Recall that the determinant of A equals $\pm p$, and that the determinant of the bottom right $(m'-1)$-minor of A equals $\pm r_1$ (see, e.g., [13], 2.6). It follows that after reduction modulo p, the rank of A is $m' - 1$. There exist positive integers $1 = h_1 < h_2 \cdots < h_{m'}$ such that

$$(1 = h_1, h_2, \ldots, h_{m'})A = (0, 0, \ldots, -p).$$

In particular, the vector $(1 = h_1, h_2, \ldots, h_{m'})$ generates the kernel of A. Set

$$(x_1, \ldots, x_{m'}) := s(r_1, \ldots, 1) + (1, h_2, \ldots, h_{m'})$$

so that

$$(x_1, \ldots, x_{m'})A = (-ps, 0, \ldots, 0, -p).$$

It follows that modulo p, either $(x_1, \ldots, x_{m'})$ is the trivial vector, or it is a non-zero multiple of $(1, h_2, \ldots, h_{m'})$. Since $x_1 = r_1 s + 1$ and p divides x_1 by hypothesis, we find that $(x_1, \ldots, x_{m'})$ is the trivial vector, so that p divides x_i for all $i = 1, \ldots, m'$. Thus, $x_i/p \in \mathbb{Z}$ for all $i = 1, \ldots, m'$, and we define

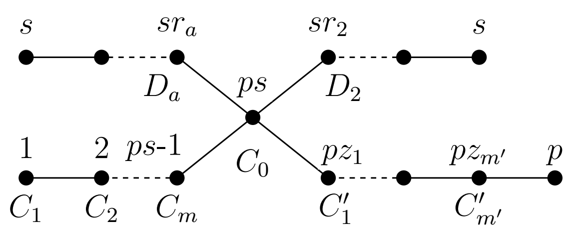
$$z_i := x_i/p. \tag{2.7.1}$$

Since the greatest common divisor of the coefficients of V is 1 and $NV = -pe_n$, we have $V = R_{C'_{m'}}$. □

2.8 We conclude this section by recalling some facts about arithmetical graphs which will be applied to the arithmetical graph defined below. This result is needed in the proof of 1.2. Let N be as in Proposition 2.7, with the vertices numbered so that $C'_{m'}$ is the last vertex listed. Assume that Condition (1) in 2.7 holds, that is, assume that p divides y_1 and that the coefficient of C_1 in $R_{C'_{m'}}$ is equal to 1. Recall the definition of $z_{m'}$ in (2.7.1) and consider the matrix M given by

$$M := \begin{pmatrix} & & & & 0 \\ & & N & & \vdots \\ & & & & 0 \\ & & & & 1 \\ 0 & \dots & 0 & 1 & -z_{m'} \end{pmatrix}$$

This symmetric matrix is only semi-definite negative, since it has a kernel R whose transpose is given by $({}^tR_{C'_{m'}}, p)$. It follows that (M, R) defines an arithmetical graph $(G(M), M, R)$ (in [11], page 481, it is $(G(M), -M, R)$ which is called an arithmetical graph). The graph $G(M)$ is represented below, along with the coefficients of the vector R on top of the corresponding vertices.



2.9 For use in the next section, let us recall how one associates an arithmetical graph to any regular model of a curve. Let $\mathcal{O}_K$ denote a discrete valuation ring, with field of fractions K and residue field k. Let X/K be any smooth, proper, geometrically connected, curve of genus g. Let $\mathcal{X}/\mathcal{O}_K$ be a regular model of X/K. Let $\mathcal{X}_k := \sum_{i=1}^{v} r_i C_i$ denote the special fiber of $\mathcal{X}$, where C_i is an irreducible component and r_i is its multiplicity. Let $M := ((C_i \cdot C_j)_{\mathcal{X}})_{1 \leq i,j \leq v}$ be the associated symmetric matrix. Denote by $G(M)$ the associated graph, with vertices C_ℓ, $\ell = 1, \dots, v$, and where C_i is linked to C_j with $j \neq i$ by exactly $(C_i \cdot C_j)$ edges. Let ${}^tR := (r_1, \dots, r_v)$, so that $MR = 0$, and assume that $\gcd(r_1, \dots, r_v) = 1$. Then the triple $(G(M), M, R)$ is an *arithmetical graph*.

Let d_i denote the degree of the vertex C_i in $G := G(M)$. Recall that the first Betti number $\beta(G)$ of the graph G is given by the formula $2\beta(G) - 2 = \sum_{i=1}^{v}(d_i - 2)$. The main combinatorial invariant associated with an arithmetical graph $(G(M), M, R)$, which plays a role analogous to the genus of a curve, is denoted by $g_0(M)$. It is given by the formula

$$2g_0(M) = 2\beta(G) + \sum_{i=1}^{v}(r_i - 1)(d_i - 2).$$

It is shown in [11], 4.7, that $g_0(M) \geq \beta(G)$.

2.10 Let us return to the arithmetical graph $(G(M), M, R)$ described in 2.8. It is completely straightforward to verify that

$$2g_0(M) = (as - s - 1)(p - 1).$$

3. Explicit desingularizations

We are now ready to prove Theorems 1.1, 1.2, and 1.3. Most of the work will be done in the proof of Theorem 1.2, and we now recall our notation in this theorem. Let k be an algebraically closed field of characteristic $p > 0$. Let B_1/k and B_2/k be two smooth proper connected curves, each endowed with an automorphism $\sigma_i : B_i \to B_i$ of order p. Let $Y := B_1 \times B_2$, and let $\sigma : Y \to Y$ be the automorphism $\sigma_1 \times \sigma_2$. Let $(P_1, P_2) \in Y$ be a fixed point of σ. Let $Z := Y/\langle\sigma\rangle$. Assume that B_2 is an ordinary curve of positive genus. Our goal is to show that a singular point Q in Z, image of $(P_1, P_2) \in Y$, has a resolution with an intersection matrix $N = N(p, \alpha, r_1)$, with $\alpha = ps$ and $r_1 \equiv -s^{-1} \mod p$ for some integer $s \geq 1$ coprime to p (notation as in 2.5).

We now begin the proof of Theorem 1.2. For $i = 1, 2$, let L_i/k denote the function field of B_i/k. Let $D_i := B_i/\langle\sigma_i\rangle$ denote the quotient curve, and let K_i/k denote the function field of D_i. We have the natural maps:

$$\begin{array}{ccc} Y := B_1 \times B_2 & \longrightarrow & Z \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & D_i. \end{array}$$

Let P_i be a ramification point of $f_i : B_i \to D_i$, with image $Q_i \in D_i$. We note without proof the following well-known fact.

Lemma 3.1. *Any singular point of the quotient $Z := Y/\langle\sigma\rangle$ is the image Q of a point of the form (P_1, P_2), where P_i is a ramification point of $f_i : B_i \to D_i$.*

It is clear that to resolve a singular point Q of Z, it is sufficient to resolve the singularity of the local scheme $\mathrm{Spec}(\mathcal{O}_{Z,Q}) \to Z$. We do not know how to study the singularity of the scheme $\mathrm{Spec}(\mathcal{O}_{Z,Q})$ directly. Instead, we explain below how to perform only two 'localizations in one direction' to obtain two schemes $\mathcal{Z}_1$ and $\mathcal{Z}_2$ with

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{Z,Q}) & \longrightarrow & \mathcal{Z}_1 \\ \downarrow & & \downarrow \\ \mathcal{Z}_2 & \longrightarrow & Z. \end{array}$$

The results of [16] can then be applied to obtain information on the resolutions of the singularities of $\mathcal{Z}_1$ and $\mathcal{Z}_2$.

3.2 The curve X_2/K_1. Denote by X_2/K_1 the base change by $\mathrm{Spec}(K_1) \to D_1$ of the natural map $Z \to D_1$, so that the following diagram is Cartesian:

$$\begin{array}{ccc} X_2 & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K_1) & \longrightarrow & D_1. \end{array}$$

Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(L_1) \times B_2 & \longrightarrow & B_1 \times B_2 & \longrightarrow & Z \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(L_1) & \longrightarrow & \mathrm{Spec}(K_1) & \longrightarrow & D_1. \end{array}$$

which induces the commutative diagram:

$$\begin{array}{ccccc} \mathrm{Spec}(L_1)\times B_2 & \longrightarrow & X_2 & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L_1) & \longrightarrow & \mathrm{Spec}(K_1) & \longrightarrow & D_1. \end{array}$$

Lemma 3.3. *The square on the left of the latter diagram is Cartesian. In particular, the curve X_2/K_1 and the curve $(B_2)_{K_1}/K_1$ become isomorphic over the extension L_1/K_1, and the curve X_2/K_1 is smooth.*

Proof. Let $\mathrm{Spec}\, A_1$ and $\mathrm{Spec}\, A_2$ be two affine open subsets of B_1 and B_2, respectively, and invariant under the action of σ_1 and σ_2, respectively. Then we have the commutative diagram

$$\begin{array}{ccccc} L_1\otimes A_2 & \longleftarrow & L_1^{\langle\sigma_1\rangle}\otimes_{A_1^{\langle\sigma_1\rangle}}(A_1\otimes A_2)^{\langle\sigma\rangle} & \longleftarrow & (A_1\otimes A_2)^{\langle\sigma\rangle} \\ \uparrow & & \uparrow & & \uparrow \\ L_1 & \longleftarrow & L_1^{\langle\sigma_1\rangle} & \longleftarrow & A_1^{\langle\sigma_1\rangle}. \end{array}$$

Since B_1/k and B_2/k are irreducible and smooth, all rings above can be identified with subrings of the function field $L_1\otimes_k L_2$ of $B_1\times B_2$. We first note that

$$L_1^{\langle\sigma_1\rangle}\otimes_{A_1^{\langle\sigma_1\rangle}}(A_1\otimes A_2)^{\langle\sigma\rangle}\longrightarrow(L_1\otimes A_2)^{\langle\sigma\rangle}$$

is an isomorphism. Surjectivity follows here from the fact that every element of L_1 can be written with a numerator in A_1 and a denominator in $L_1^{\langle\sigma_1\rangle}$. Then we are reduced to considering the natural inclusions

$$\begin{array}{ccc} L_1\otimes A_2 & \longleftarrow & (L_1\otimes A_2)^{\langle\sigma\rangle} \\ \uparrow & & \uparrow \\ L_1 & \longleftarrow & L_1^{\langle\sigma_1\rangle}. \end{array}$$

By hypothesis, L_1 has prime degree p over $L_1^{\langle\sigma_1\rangle}$, and we find that $L_1\otimes A_2$ is also free of degree p over $(L_1\otimes A_2)^{\langle\sigma\rangle}$, and $(L_1\otimes A_2)^{\langle\sigma\rangle}$ and L_1 together generate $L_1\otimes A_2$. □

3.4 The model $\mathcal{Z}_2/\mathcal{O}_{K_1}$ of X_2/K_1. Choose a ramification point P_1 of $f_1: B_1\to D_1$, with image $Q_1\in D_1$. Let $\mathcal{O}_{K_1}:=\mathcal{O}_{D_1,Q_1}$, and $\mathcal{O}_{L_1}:=\mathcal{O}_{B_1,P_1}$. Consider the base change $\mathcal{Y}_2$ of $B_1\times B_2\to B_1$ by the morphism $\mathrm{Spec}(\mathcal{O}_{L_1})\to B_1$. The scheme $\mathcal{Y}_2$ is a smooth model of $(B_2)_{L_1}$ over $\mathcal{O}_{L_1}$:

$$\begin{array}{ccccc} (B_2)_{L_1} & \longrightarrow & \mathcal{Y}_2 & \longrightarrow & B_1\times B_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L_1) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{L_1}) & \longrightarrow & B_1. \end{array}$$

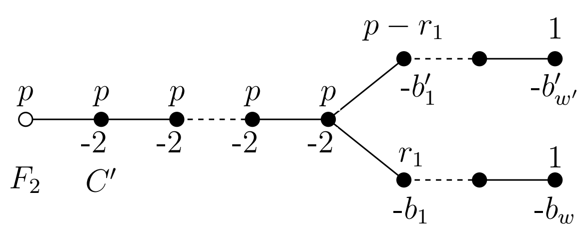
The quotient $\mathcal{Z}_2$ of this model by the action of the Galois group of L_1/K_1 is nothing but the base change of $Z\to D_1$ by the map $\mathrm{Spec}(\mathcal{O}_{K_1})\to D_1$:

$$\begin{array}{ccccc} X_2 & \longrightarrow & \mathcal{Z}_2:=\mathcal{Y}_2/(\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K_1) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{K_1}) & \longrightarrow & D_1. \end{array}$$

Let P_2 be a ramification point of $f_2 : B_2 \to D_2$, with image $Q_2 \in D_2$. Let Q denote the singular point of the quotient $Z := Y/\langle\sigma\rangle$ image of the point (P_1, P_2). Resolving this singularity Q on Z is equivalent to resolving the corresponding one on $\mathcal{Z}_2$.

We have now reduced the study of the singularities of Z to those of $\mathcal{Z}_2/\mathcal{O}_{K_1}$. We are going to obtain the needed information on the singularities of $\mathcal{Z}_2/\mathcal{O}_{K_1}$ by using Theorem 6.8 of [16]. Our set-up is as follows. The curve X_2/K_1 obtains good reduction after the extension L_1/K_1 of degree p. The reduction of $(X_2)_{L_1}/L_1$ is ordinary since the special fiber of the smooth model of $(X_2)_{L_1}/L_1$ is nothing but the curve B_2/k, and in Theorem 1.2 we assume that B_2/k is an ordinary curve. We also assume that B_2/k has positive genus. When $g(B_2) \geq 1$, Lemma 3.7 (a) shows that X_2/K_1 has a K_1-rational point. When $g(B_2) = 1$, the existence of a K_1-rational point is needed to apply Theorem 6.8 of [16] to our situation (this hypothesis is not stated explicitly in Theorem 6.8 but appears as a standing hypothesis in section 6 of [16]). To apply Theorem 6.8 of [16] to our situation, it remains only to address the fact that in this theorem, the base field is *complete* with respect to its discrete valuation. For this, we use the following general argument. Suppose that R is a discrete valuation ring with field of fractions K, and that we have a regular model $\mathcal{X}/R$ of a curve X/K. Let $\widehat{R}$ denote the completion of R with respect to its maximal ideal. Then the model $\mathcal{X} \times_R \widehat{R}$ is again regular, and is thus a regular model of the curve $X_{\widehat{K}}/\widehat{K}$. Moreover, the special fibers of $\mathcal{X} \times_R \widehat{R}$ and of $\mathcal{X}$ are isomorphic over the residue field k (see, e.g., [10], 8.3.49). Thus, any information on the special fiber of the regular model $\mathcal{X} \times_R \widehat{R}$ can be readily transfered back to the special fiber of $\mathcal{X}$. When applying results from [16] in the remainder of this proof, we may use this argument without further mention of it.

Consider the curves $B_1 \times \{P_2\}$ and $\{P_1\} \times B_2$ on the surface $B_1 \times B_2$, and let F_1 and F_2 be their images in Z, respectively. Both F_1 and F_2 contain Q. Let $f : Z^{\mathrm{desing}} \to Z$ denote the desingularization of Q minimal with the property that all irreducible components of $f^{-1}(Q)$ are smooth and intersect normally. Let $\mathcal{X}_2 \to \mathcal{Z}_2$ denote the resolution of the singularity Q of $\mathcal{Z}_2$ obtained as $\mathcal{X}_2 := Z^{\mathrm{desing}} \times_{D_1} \mathrm{Spec}(\mathcal{O}_{K_1})$. It is also minimal with the property that the components of the exceptional divisor are smooth with normal crossings. It follows from [16], Theorem 6.8, that the intersection matrix associated with the resolution $\mathcal{X}_2$ of any singularity of $\mathcal{Z}_2$ is of type $N(p, \alpha, r_1)$ with $\alpha = ps$ for some $s > 0$, and some $1 \leq r_1 < p$ (notation as in 2.5). The special fiber of the model $\mathcal{X}_2/\mathcal{O}_{K_1}$ contains the strict transform of the unique irreducible component of $(\mathcal{Z}_2)_k$, which is of multiplicity p in $(\mathcal{X}_2)_k$. We denote this component on $(\mathcal{X}_2)_k$ again by F_2. We picture below the component F_2 along with the desingularization of Q. The positive integer on a vertex denotes the multiplicity of the corresponding component in $(\mathcal{X}_2)_k$.



We denote by C' the irreducible component of the resolution which intersects the component F_2. We use the same notation for the component C' when viewed as a curve on Z or on $\mathcal{Z}_2$. Note that [16], Theorem 6.8, shows that C' intersects F_2 with intersection number 1 (the model on which the graph G_{Q_i} lies in 6.8 is introduced in [16], 5.2, and has a special fiber with smooth components and normal crossings).

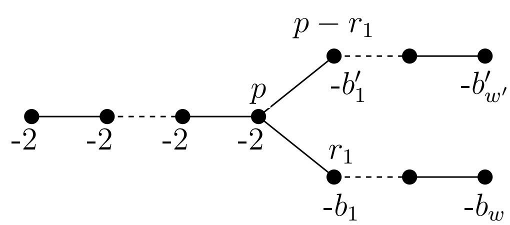
To complete the proof of Theorem 1.2, it remains to show that $r_1 s \equiv -1 \mod p$. This is done in 3.9. We also need to determine the value of s in Theorems 1.1 and 1.3, and this is done in 3.11. The remainder of our proofs exploits the fact that the roles played by B_1 and B_2 on the surface $B_1 \times B_2$ are symmetric. Switching the roles leads us to define the following objects.

3.5 The model $\mathcal{Z}_1/\mathcal{O}_{K_2}$ of X_1/K_2. Let K_2 denote the function field over k of the curve $D_2 := B_2/\langle\sigma_2\rangle$, and let $L_2 := k(B_2)$, the function field of B_2/k. Making the base change by $\mathrm{Spec}(K_2) \to D_2$ of the natural map $Z \to D_2$ produces a smooth complete curve X_1/K_2 which becomes isomorphic over L_2 to $(B_1)_{L_2}$. Let $\mathcal{O}_{K_2} := \mathcal{O}_{D_2,Q_2}$, and $\mathcal{O}_{L_2} := \mathcal{O}_{B_2,P_2}$. Consider the base change $\mathcal{Y}_1$ of $B_1 \times B_2 \to B_2$ by the morphism $\mathrm{Spec}(\mathcal{O}_{L_2}) \to B_2$. This is a smooth model of $(B_1)_{L_2}$ over $\mathcal{O}_{L_2}$. The quotient $\mathcal{Z}_1$ of this model by the action of the Galois group of L_2/K_2 is nothing but the base change of $Z \to D_2$ by the map $\mathrm{Spec}(\mathcal{O}_{K_2}) \to D_2$.

$$\begin{array}{ccccc} X_1 & \longrightarrow & \mathcal{Z}_1 := \mathcal{Y}_1/(\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K_2) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{K_2}) & \longrightarrow & D_2. \end{array}$$

Let $\mathcal{X}_1 \to \mathcal{Z}_1$ denote the resolution of the singularity Q of $\mathcal{Z}_1$ obtained as $\mathcal{X}_1 := Z^{\text{desing}} \times_{D_2} \mathrm{Spec}(\mathcal{O}_{K_2})$. It is also minimal with the property that the components of the exceptional divisor are smooth with normal crossings.

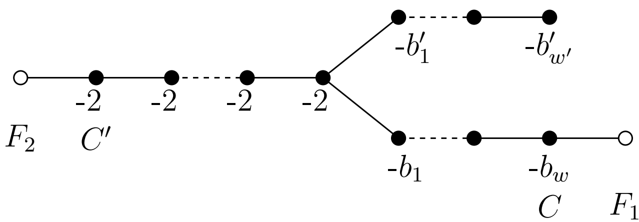
3.6 Let $Q \in Z$ be the image of the point (P_1, P_2) under the quotient map $Y \to Z$. Let $f : Z^{\text{desing}} \to Z$ denote the desingularization minimal with the property that all irreducible components of $f^{-1}(Q)$ are smooth and intersect normally. Let N denote the intersection matrix associated with the resolution $f : Z^{\text{desing}} \to Z$ of the singularity Q. At this point, we know that the desingularization on Z^{desing} has a graph $G(N)$ of the following form:



where the number of -2 components on the left is equal to $\alpha = ps$ for some s, and the self-intersections $-b_1, \ldots, -b_w$, are completely determined by the pair (p, r_1). Similarly, the self-intersections $-b'_1, \ldots, -b'_{w'}$, are completely determined by the pair $(p, p - r_1)$.

Since we have specified two curves F_1 and F_2 in Z containing Q, we find that the graph $G(N)$ has two marked vertices, corresponding to the components of $f^{-1}(Q)$ which meet the strict transforms of the specified curves in Z (which are denoted again by F_1 and F_2). We use now one of the two marked vertices to distinguish between the two terminal chains determined by (p, r_1) and $(p, p - r_1)$. First, recall that we call C' the component of the resolution which meets the curve F_2, and we have $(C' \cdot F_2)_{Z^{\text{desing}}} = 1$. Let us call C the component of the resolution which meets the curve F_1. Then Lemma 3.7 (a) shows that the component C must have multiplicity 1 in the special fiber $(\mathcal{X}_2)_k$. There are exactly two components on $(\mathcal{X}_2)_k$ with multiplicity 1, both terminal vertices on the associated graph (see the graph in 3.4). We choose to call r_1 the multiplicity of the first vertex after the node on the terminal chain ending in C. Lemma 3.7 (a) also shows that

$(C \cdot F_1)_{Z^{\text{desing}}} = 1$. *The above discussion shows that we have the following configuration of curves on* Z^{desing}*:*



Before we proceed with the proof of Theorem 1.2, we establish the following lemma used above.

Lemma 3.7. *Keep the above notation.*

(a) *The curve F_1 defines a K_1-rational point of X_2/K_1. In particular, the closure of this K_1-rational point in the regular model $\mathcal{X}_2/\mathcal{O}_{K_1}$ meets $(\mathcal{X}_2)_k$ at a smooth point and, thus, on a component C of multiplicity 1 in $(\mathcal{X}_2)_k$. Moreover, the closure of this K_1-rational point meets C with normal crossings.*

(b) *A similar statement is true for F_2 when the roles of B_1 and B_2 are reversed. The curve F_2 defines a K_2-rational point of X_1/K_2. In particular, the closure of this K_2-rational point in $\mathcal{X}_1$ meets $(\mathcal{X}_1)_k$ at a smooth point and, thus, on a component C' of multiplicity 1 in $(\mathcal{X}_1)_k$. Moreover, the closure of this K_2-rational point meets C' with normal crossings.*

Proof. We prove Part (a) only, since Part (b) is similar. Consider $\langle\sigma_1\rangle \times \langle\sigma_2\rangle$ as a subgroup of automorphisms of $B_1 \times B_2$. The natural quotient of this action is $D_1 \times D_2$. Consider also the natural commutative diagram of intermediate natural quotient maps of degree p:

$$\begin{array}{ccc} B_1 \times B_2 & \longrightarrow & D_1 \times B_2 \\ \big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ Z & \xrightarrow{\ \gamma\ } & D_1 \times D_2 \end{array}$$

Note that γ and β are both morphisms over D_1. The curve $D_1 \times \{Q_2\}$ is the image by γ of the curve F_1. Consider the pull-back of the above diagram by $\operatorname{Spec}(K_1) \to D_1$, to obtain

$$\begin{array}{ccc} & & (B_2)_{K_1} \\ & & {\scriptstyle\beta'}\big\downarrow \\ X_2 & \xrightarrow{\ \gamma'\ } & (D_2)_{K_1}. \end{array}$$

The curve $D_1 \times \{Q_2\}$ is in the branch locus of the map β, and $Q_2 \in (D_2)_{K_1}$ is a branch point of β'. We claim that $Q_2 \in (D_2)_{K_1}$ is also a branch point of γ'. This claim follows from 3.8 below, where X takes the role of B_2, and X' is then the twist X_2. Assuming this claim, we find that F_1 pulls back by $\operatorname{Spec}(K_1) \to D_1$ to a ramification point of the map $\gamma' : X_2 \to (D_2)_{K_1}$. Since this latter morphism has degree p, and the branch point $Q_2 \in (D_2)_{K_1}$ is K_1-rational by construction, so is its preimage in X_2, as desired.

3.8 Let K be a field of characteristic p with a cyclic Galois extension L/K of degree p. Let X/K denote a smooth projective curve with an automorphism σ of order p. Let $X \to X_0 = X/\langle\sigma\rangle$ denote the quotient map. We denote again by σ the induced automorphism of $K(X)$ over $K(X_0)$. Let X'/K denote the smooth projective curve obtained as follows: Choose a nontrivial automorphism $\tau : L \to L$ fixing K, and let

$K(X')$ be the field of elements of $L \otimes_K K(X)$ fixed by the automorphism $\tau \otimes \sigma$. *Then there is a natural morphism $X' \to X_0$ over K, and the branch loci of $X \to X_0$ and $X' \to X_0$ are equal in X_0.*

Indeed, we can write explicitly the Galois extension $K(X)/K(X_0)$ as an Artin-Schreier extension, with $K(X)$ isomorphic to

$$K(X) = K(X_0)[y]/(y^p - y + f)$$

for some $f \in K(X_0)^*$. The automorphism σ is then of the form $y \mapsto y + i$ for some $i \in \mathbb{F}_p^*$. Similarly, the Galois extension L/K is given by an Artin-Schreier extension with

$$L = K[z]/(z^p - z + g)$$

for some $g \in K^*$. The automorphism τ is of the form $z \to z + j$ for some $j \in \mathbb{F}_p^*$. It follows that in the field $L \otimes_K K(X)$, the element $Y := (1 \otimes y) - j^{-1}i(z \otimes 1)$ is fixed by the action of $\tau \otimes \sigma$. It is clear that

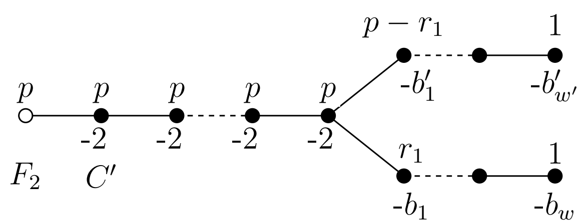
$$Y^p = Y - ((1 \otimes f - j^{-1}i(g \otimes 1)).$$

The fixed field of $L \otimes_K K(X)$ by $\tau \otimes \sigma$ also contains $1 \otimes K(X_0)$, and we find that with the appropriate identifications, we can write that

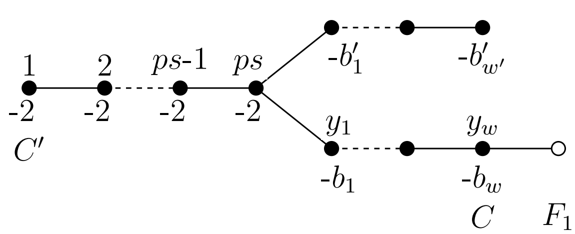
$$K(X') = K(X_0)[Y]/(Y^p - Y + f - j^{-1}ig),$$

with the natural inclusion $K(X_0) \to K(X')$ giving the morphism $X' \to X_0$. The Artin-Schreier morphism $X \to X_0$ is ramified at a place above $P \in X_0$ if and only if f has a pole at P. Similarly, the Artin-Schreier morphism $X' \to X_0$ is ramified at a place above P if and only if $f - j^{-1}ig$ has a pole at P. Since by construction $g \in K$, we find that f and $f - j^{-1}ig$ have exactly the same set of poles. □

3.9 We are now ready to prove that $r_1 s \equiv -1 \mod p$ in Theorem 1.2. For this, we will use Proposition 2.7. Let us assume that the vertices of $G(N)$ are numbered from 1 to n, with the vertex C_1 being C' and the vertex C_n being C. The use of the special fiber of the model $\mathcal{X}_2/\mathcal{O}_{K_1}$ allows us to describe the vector R_1 on the matrix N:



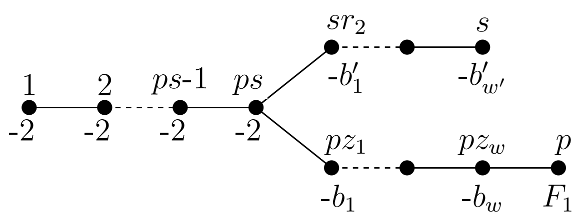
The use of the special fiber of the model $\mathcal{X}_1/\mathcal{O}_{K_2}$ will provide us with the information needed on the vector R_n to apply 2.7. Indeed, we know that on $(\mathcal{X}_1)_k$, we have the following configuration (the positive integer above a vertex is the corresponding coefficient of R_n):



The fact that the component C' has multiplicity 1 in $(\mathcal{X}_1)_k$ comes from Lemma 3.7 (b). The self-intersections on the terminal chain ending with C' are all equal to -2, and there are $ps - 1$ vertices on the chain before the node; this forces the multiplicities to increase regularly as 1, 2, ..., $ps-1$, and the node to have multiplicity ps. The component F_1 has multiplicity p since it is the strict transform of the reduced special fiber of $\mathcal{Z}_1$, and on this fiber the unique component has multiplicity p, $\mathcal{Z}_1$ being a quotient by $\mathbb{Z}/p\mathbb{Z}$ ([16], 5.1). Theorem 5.3 in [16] then shows that p divides y_w. If follows from the fact that p divides the multiplicity of two consecutive components on a chain that all multiplicities on the chain containing C must be divisible by p (this follows easily using the intersection matrix, as in 4.2 in [16]). In particular, y_1 is divisible by p. This is the last condition needed to be able to apply Proposition 2.7. It follows then from 2.7 that $r_1 s \equiv -1 \mod p$.

To complete the proof of Theorem 1.2, we only need to show that the Smith Normal Form of N is $\mathrm{diag}(1, \ldots, 1, p, p)$. This follows immediately from the fact that Φ_N is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, a fact that we noted already in 2.6. This concludes the proof of Theorem 1.2.

3.10 For use in Theorem 3.15, we explicitly write down below the arithmetical graph associated with the special fiber $(\mathcal{X}_1)_k$, where as above F_1 denotes the strict transform of the reduced special fiber of $\mathcal{Z}_1$:



This arithmetical graph is a special case in the class of arithmetical graphs introduced in 2.8.

3.11 We may now prove Theorems 1.1 and 1.3. Assume first as in Theorem 1.1 that both B_1 and B_2 are ordinary of positive genus. Then using the resolution $\mathcal{X}_1 \to \mathcal{Z}_1$, we find that the self-intersections $-b_1, \ldots, -b_w$ can be completely determined: they must all equal -2. Moreover, we have that p divides $w+1$. Returning to the model $\mathcal{X}_2 \to \mathcal{Z}_2$, we find that the bottom terminal chain on the right (with initial vertex of multiplicity equal to $r_1 < p$) can have at most $p-1$ vertices (since the multiplicities are decreasing on the chain). Since all self-intersections are -2 on the chain, we find that we must have $w = p-1$ and $r_1 = p-1$. Repeating the same argument with the model $\mathcal{X}_1$, we find that $\alpha = p$, and so $s = 1$. This concludes the proof of Theorem 1.1.

Assume now the hypotheses of Theorem 1.3. Consider the resolution $\mathcal{X}_1 \to \mathcal{Z}_1$. The generic fiber of $\mathcal{X}_1$ has genus equal to $g(B_1)$, which can be computed using the Riemann-Hurwitz formula for $B_1 \to D_1$ as follows. Let $\delta(P_1) = (s(P_1)+1)(p-1)$ denote the valuation of the different. Then, since σ_1 has a unique fixed point by hypothesis, $2g(B_1) - 2 = p(2g(D_1)-2) + (s(P_1)+1)(p-1)$, so that

$$2g(B_1) = 2g(D_1)p + (s(P_1)-1)(p-1). \tag{3.11.1}$$

We can also compute $g(B_1)$ using the adjunction formula applied to the curve $(\mathcal{X}_1)_k$. Let (G, M, R) denote the arithmetical graph associated with $(\mathcal{X}_1)_k$ (as in 3.10). Then using 2.10 with $a = 2$, we find that

$$2g(B_1) = 2g(D_1)p + 2g_0(M) = 2g(D_1)p + (s-1)(p-1). \tag{3.11.2}$$

It immediately follows from (3.11.1) and (3.11.2) that $s = s(P_1)$. This concludes the proof of Theorems 1.1 and 1.3. □

Remark 3.12 Examples of curves with an automorphism of degree p in characteristic p can be given in Artin-Schreier form. Consider the smooth complete curve B/k given by the equation

$$y^p - y = \prod_{i=1}^{d} (x - a_i)^{-n_i},$$

where $a_1, \ldots, a_d \in k$ are distinct, and the n_i are positive integers coprime to p. The automorphism σ, which sends $x \mapsto x$ and $y \mapsto y + 1$, has order p. The genus g of B is given by the Riemann-Hurwitz formula

$$2g - 2 = -2p + (p - 1)(\sum_{i=1}^{d} (n_i + 1))$$

(see [21], page 8). Each point $(a_i, 0)$ is a branch point Q_i of $B \to B/\langle\sigma\rangle$, whose corresponding ramification point P_i in B is such that the valuation of the different of $\mathcal{O}_{B,P_i}/\mathcal{O}_{B/\langle\sigma\rangle,Q_i}$ is equal to $(n_i + 1)(p - 1)$. The curve B/k is ordinary if and only if $n_i = 1$ for all $i = 1, \ldots, d$.

Corollary 3.13. *Let p be prime. Let $m \geq 1$ be any integer. Then there exists a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity of surface whose minimal desingularization has a graph with more than m vertices.*

Proof. Let s be any positive integer coprime to p. Let B_1/k denote the Artin-Schreier curve given by the equation $y^p - y = x^{-s}$, with automorphism σ_1 sending y to $y + 1$. Then the quotient map $B_1 \to \mathbb{P}^1$ is ramified only at the unique point P_1 above ∞, with $s(P_1) = s$. Choose B_2/k to be any ordinary curve of positive genus, and apply 1.3. In view of 1.3, the graph of the minimal resolution of a singular point on $Y/\langle\sigma\rangle$ has at least ps vertices. □

Remark 3.14 Theorem 1.2 exhibits examples of intersection matrices occurring as desingularizations of $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in the equicharacteristic case. For instance, when $p = 2$, the singularity in 1.2 is nothing but a classical D_n-singularity, with $n := ps+2$ being the total number of vertices of the associated resolution graph. In particular, Theorem 1.2 exhibits D_n as a resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity only in the equicharacteristic case, and only when $n \equiv 0 \mod 4$, since s is coprime to p (see also [1], p. 64). Examples of D_n-resolution with $n \equiv 2 \mod 4$ are obtained in [15], 4.1, in the mixed characteristic case.

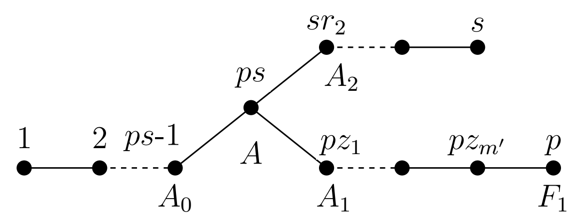
Recall that the determinant of the intersection matrix N of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity is expected to be always a power of p ([15], 2.6). We show below that for any fixed prime p, all powers p^{s+1} with s coprime to p can arise as determinants in the context of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities.

Theorem 3.15. *Fix a prime p. For each positive integer s coprime to p, there exists a 2-dimensional regular local ring $\mathcal{A}$ of equicharacteristic p endowed with an action of $H := \mathbb{Z}/p\mathbb{Z}$ such that* Spec $\mathcal{A}^H$ *is singular exactly at its closed point, and such that the graph associated with a minimal resolution of* Spec $\mathcal{A}^H$ *has exactly one node and $s+2$ terminal chains, and its associated intersection matrix N has determinant $|\det(N)| = p^{s+1}$.*

Proof. Let B_1/k be a curve with an automorphism σ_1 of order p having only one fixed point P_1 (as, e.g., in the proof of 3.13). Let $s(P_1)$ denote the integer coprime to p such that $(s(P_1) + 1)(p - 1)$ is equal to the valuation of the different at P_1. Let B_2/k be an ordinary curve of positive genus with an automorphism σ_2 of order p. We keep our standard notation where $Y := B_1 \times B_2$, and $\sigma := \sigma_1 \times \sigma_2$.

Choose a ramification point P_2 on B_2, with image Q_2 in the quotient $D_2 := B_2/ < \sigma_2 >$. Let K_2 denote the function field of D_2 and let K denote the completion of K_2 at the valuation corresponding to Q. Then $\mathcal{O}_K$ is the completion of $\mathcal{O}_{D_2,Q}$. Consider the curve X_1/K_2 introduced in 3.5, and base change the whole data appearing in 3.5 by Spec K or Spec $\mathcal{O}_K$, as needed. To simplify our notation, we do not change notation when passing from K_2 to K: we now have a normal model $\mathcal{Z}_1/\mathcal{O}_K$ for X_1/K.

Theorem 1.3 allows us to determine the intersection matrix of the exceptional divisor of the desingularization $\mathcal{X}_1 \to \mathcal{Z}_1$ at the unique singular point of $\mathcal{Z}_1$, corresponding to the point of $Y/\langle\sigma\rangle$ image of (P_1, P_2). We let F_1 denote the strict transform in $\mathcal{X}_1$ of the reduced special fiber $(\mathcal{Z}_1)_k^{red}$ (which is isomorphic to the quotient D_1). We describe below the arithmetical graph associated with the special fiber $(\mathcal{X}_1)_k$ of the regular model $\mathcal{X}_1/\mathcal{O}_K$ of the curve X_1/K, following 3.10. (As we will refer to this graph later, we have labeled the most important vertices in $(\mathcal{X}_1)_k$ for convenience.)



Implicit in our notation is that

$$ps|A \cdot A|_{\mathcal{X}_1} = (ps-1) + pz_1 + sr_2.$$

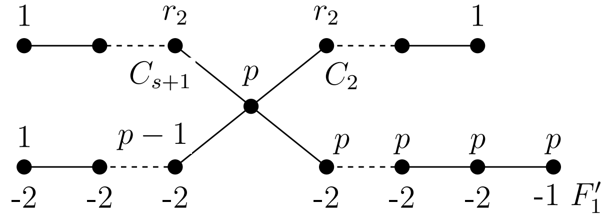
Hence, we have $sr_2 \equiv 1 \mod p$. Moreover, since s is coprime to $(ps-1)$, we find that s is coprime to pz_1. Note that due to our construction with an ordinary curve B_2, we also know that $|A \cdot A|_{\mathcal{X}_1} = 2$.

As in 3.5, let $L_2 := k(B_2)$ and let $L := L_2 \otimes K$. The curve X_1/K has good reduction over L. We now fix an extension F/K of degree s which is totally ramified (and tamely ramified since s is coprime to p). Consider the curve $(X_1)_F/F$. It achieves good reduction over the extension FL of degree p over F. Denote by $\mathcal{Y}$ the smooth model over $\mathcal{O}_{FL}$ of the curve $(X_1)_{FL}/FL$. The group $H := \mathrm{Gal}(FL/L)$ acts on $\mathcal{Y}$, and we let $\mathcal{Z}$ denote the quotient by this action. Then $\mathcal{Z}/\mathcal{O}_F$ has a unique singular point (with local ring which we denote by $\mathcal{A}^H$), and the proof of Theorem 3.15 consists in showing that the resolution of this singular point is as in the statement of Theorem 3.15.

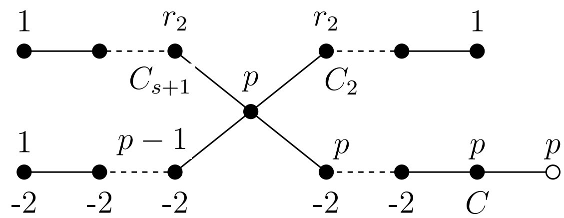
Let $\mathcal{X} \to \mathcal{Z}$ denote the minimal resolution of the singularity of $\mathcal{Z}$. Denote by F_1' the reduced irreducible component $(\mathcal{Z})_k^{red}$, and also its strict transform in $\mathcal{X}$. Clearly, the graph of the resolution of the singularity of $\mathcal{Z}$ is obtained by removing the vertex F_1' from the graph of the special fiber $\mathcal{X}_k$. We will denote by N the matrix associated with the resolution of the singularity.

Recall that we have fixed an integer s coprime to p, and that the choices made so far also fix a second positive integer r_2. We will show below that to prove Theorem 3.15, it suffices to have the following information of the special fiber $\mathcal{X}_k$: *We claim that the arithmetical graph of the special fiber $\mathcal{X}_k$ has a single node, of multiplicity p, with $s+2$ terminal chains attached to it. The node is attached to s vertices of multiplicity r_2, and to two more vertices, of multiplicity $p-1$ and p, respectively. The terminal chain started by the vertex of multiplicity $p-1$ consists of vertices of self-intersection -2, while the terminal chain started by the vertex of multiplicity p consists of vertices of self-intersection*

-2 *except for the terminal vertex on the chain, which has self-intersection* -1. The graph of the special fiber $\mathcal{X}_k$ can be represented[1] as follows[2] (in the case $s = 2$):



We prove the above claim in 3.16, and use it now to deduce the statement of Theorem 3.15. Let us denote by n the size of the matrix N (i.e., N is a $n \times n$-matrix) and label the distinguished vertex C below as the last vertex C_n in the enumeration of the vertices of $G(N)$. Then we have determined exactly the pair (N, R_n) with $NR_n = -pe_n$, and this data is represented below when $s = 2$.



This data is exactly what is needed to apply [15], Theorem 3.14, to obtain that $|\det(N)| = p^{s+1}$. Note that since we do not specify α, it could happen that C is in fact the node of the graph, but [15], Theorem 3.14, applies just as well to this case.

3.16 Let us now turn to proving our claim. For this, we will compute the model $\mathcal{X}$ as follows:

(a) Compute the scheme $\mathcal{X}_1 \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\, \mathcal{O}_F$.
(b) Compute the normalization $\mathcal{N}$ of $\mathcal{X}_1 \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\, \mathcal{O}_F$.
(c) Compute the desingularization $\mathcal{Z}' \to \mathcal{N}$ of $\mathcal{N}/\mathcal{O}_F$.
(d) Construct a scheme $\mathcal{Z}''/\mathcal{O}_F$ and a morphism $\mathcal{Z}' \to \mathcal{Z}''$ obtained as a series of contractions of smooth rational curves of self-intersection (-1), in such a way that no component of $(\mathcal{Z}'')_k$ is smooth rational of self-intersection -1, except possibly a component D'', image of a component D' on $\mathcal{Z}'$, with D' mapping to a component over $(\mathcal{Z}_1)_k^{red}$. We will show that $\mathcal{Z}'' = \mathcal{X}$ is a minimal desingularization of the quotient $\mathcal{Z}/\mathcal{O}_F$.

Because the extension F/K is tame, every step in the above process can be done in an explicit enough fashion allowing for the complete determination of the combinatorics of special fiber of $(\mathcal{Z}'')_k$. In particular, every singularity of $\mathcal{N}$ is a tame cyclic quotient singularity and is resolved by a chain of rational curves (see, e.g., [6]).

[1]Up to permuting the bottom two terminal chains, the matrix N obtained by removing the vertex F_1' is a star-shaped matrix introduced in 2.4 and is determined by $(p, \alpha, r_1, \dots, r_{s+1})$ with $r_1 = p-1$, and $r_2 = \cdots = r_{s+1}$ with $r_2 s \equiv 1 \mod p$.

[2]Since a terminal chain of an arithmetical graph is completely specified when the vertex linked to the node is of multiplicity coprime to the multiplicity of the node and the terminal chain does not contain any vertex of self-intersection -1, we find that in our case, the arithmetical graph would be completely specified once the number α of components of self-intersection (-2) on the bottom right terminal chain is given. There is no need for our purpose to specify α further, but let us note that one can show that α is divisible by p.

To prove our claim, it suffices to prove it first when s is prime, and then apply repeatedly the prime case to all prime divisors of the given s. Let us thus assume from now on that s is prime. Since the base change is tame, and its degree divides the multiplicity of A, we know that the normalization $\mathcal{N}$ is *regular* above any point of the component A. The preimage B of A in $\mathcal{N}$ is a smooth rational curve of multiplicity p, branched exactly over the points where A meets A_0 and A_1 (these are the only components meeting A whose multiplicities are coprime to s). The preimages B_0 and B_1 of A_0 and A_1 in $\mathcal{N}_k$ are irreducible of multiplicities $ps-1$ and pz_1, respectively. The preimage of A_2 consists of s rational curves $C_2, \ldots, C_{s+1}$, each of multiplicity r_2 in $\mathcal{N}_k$. We deduce from this that on $\mathcal{N}$,

$$p|B \cdot B|_{\mathcal{N}} = (ps-1) + pz_1 + sr_2,$$

so that $|B \cdot B|_{\mathcal{N}} = s + z_1 + (sr_2 - 1)/p$.

It also follows from the fact that every curve on the terminal chain started by A_2 has multiplicity divisible by s, that the preimage of the whole chain in the normalization $\mathcal{N}$ is in the regular locus of $\mathcal{N}$, and simply consists of s copies of the original chain found on $(\mathcal{X}_1)_k$ (same intersection numbers and self-intersections).

We now turn to understanding the chains in $\mathcal{Z}''$ started in $\mathcal{N}$ by B_0 and B_1. Since the multiplicity of B_0 is larger than the multiplicity of B, and since every singularity of $\mathcal{N}$ is resolved by a chain of rational curves, we find that the curve B_0 will be contracted in the morphism $\mathcal{Z}' \to \mathcal{Z}''$. The same argument applies to the chain started by B_1 if $pz_1 > p$. Since $ps-1$ is coprime to p, we find that the terminal vertex on the chain in $\mathcal{Z}'$ started by B_0 has multiplicity 1. Similarly, since pz_1 has greatest common divisor p with the multiplicity of B, we find that the terminal vertex on the chain in $\mathcal{Z}'$ started by B_1 has multiplicity p. It follows that in $\mathcal{Z}''$, the image of the component B (again denoted by B) meets a component of multiplicity p (the component corresponding to B cannot be contracted in $\mathcal{Z}''$ since at any stage of the contracting morphism, the image of B meets at least three other components, one of them of multiplicity a multiple of p. The self-intersection of the image of B can then never equal -1 since its multiplicity is p).

Say that the terminal chain started by B_0 in $\mathcal{Z}'$ corresponds in $\mathcal{Z}''$ to a terminal chain started by a component of multiplicity x, with $x < p$ since x is coprime to p. From

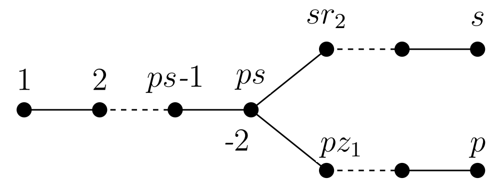
$$p|B \cdot B|_{\mathcal{Z}''} = x + p + sr_2,$$

we conclude that $x = p-1$. Then the terminal chain of $\mathcal{Z}''$ started by the component of multiplicity $p-1$ is completely understood, and consists of $p-1$ components of self-intersection (-2). It remains to discuss the chain of $\mathcal{Z}''$ started by the component of multiplicity p. Clearly, every component of this chain is then also of multiplicity p and self intersection (-2), except for the last one, which has multiplicity p and self-intersection (-1). This concludes the proof of Theorem 3.15. □

Let p be prime, and let (G, M, R) be an arithmetical graph. It is natural to wonder whether there exists a discrete valuation field K of residue characteristic p and ring of integers $\mathcal{O}_K$, and a curve Y/K with a regular model $\mathcal{Y}/\mathcal{O}_K$ whose special fiber has M as its associated intersection matrix. We note here such a statement which does not follow from the general existence results of Viehweg [24] or Winters [25].

Corollary 3.17. *Fix a prime p and an integer $s > 1$ coprime to p. Denote by r_2 the unique integer in $[1, p-1]$ such that $sr_2 \equiv 1 \mod p$. Define $z_1 > 0$ by the equality $ps = sr_2 - 1 + pz_1$. Consider the arithmetical tree (G, M, R) specified as follows: the tree G has a single node, of multiplicity ps, and three vertices linked to it, with multiplicities $ps-1$, sr_2, and pz_1, respectively. The self-intersection of the node is -2, with the relation*

$2ps = (ps-1) + sr_2 + pz_1$. *This data completely specifies* (G, M, R)*, which we represent as follows:*



Then there exists a discrete valuation field K *of equicharacteristic* p*, and a curve* Y/K *with a regular model* $\mathcal{Y}/\mathcal{O}_K$ *whose special fiber has* M *as its associated intersection matrix. The genus of* Y *is* $(s-1)(p-1)/2$*. Moreover, there exists a totally ramified extension* L/K *of degree* p *such that* Y_L/L *has good reduction.*

Proof. Such a curve is exhibited at beginning of the proof of Theorem 3.15, where it is called X_1/K.

Remark 3.18 Let A/K denote the Jacobian of the curve Y/K whose existence is asserted in 3.17. By construction, A/K has purely additive reduction over $\mathcal{O}_K$, and achieves good reduction after a wildly ramified extension of degree p. Using the arithmetical graph (G, M, R) associated with the regular model $\mathcal{Y}/\mathcal{O}_K$, we compute the group of components $\Phi_{A,K}$ of the Néron model $\mathcal{A}/\mathcal{O}_K$ of A/K to be $\Phi_{A,K} = (0)$ (see, e.g., [12], 1.5). Proposition 3.8 in [14] states that when an abelian variety A/K with purely additive reduction achieves good reduction over a *tame* extension of prime power degree, then $\Phi_{A,K}$ is not trivial. The Jacobian A/K is an example which shows that the hypothesis that the extension is tame cannot be removed from the statement of [14], 3.8.

Note also that in this example, the extension L/K minimal with the property that Y_L/L has semi-stable reduction has a degree which strictly divides the multiplicity of the unique node of the graph of the minimal regular model of Y/K over $\mathcal{O}_K$; see Footnote 5 of [14], page 46, for a related discussion.

Remark 3.19 In Theorem 3.15, we proved the existence of $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in equicharacteristic p such that the graph associated with a minimal resolution of Spec $\mathcal{A}^H$ has an intersection matrix N with $|\det(N)| = p^{s+1}$, where s is a positive integer coprime to p. It is natural to wonder whether a similar result could be obtained when p divides s.

4. Rational singularities

Denote by $\mathcal{Z}$ the spectrum of a normal two-dimensional local ring A with algebraically closed residue field k. Assume that the closed point Q of $\mathcal{Z}$ is singular, and let $f : \mathcal{X} \to \mathcal{Z}$ be a resolution of the singularity at Q. Let N be the associated intersection matrix, and let $\mathbf{Z} > 0$ be its fundamental cycle (2.2). Artin showed in [2], Thm. 3, that for the surface singularity Q to be rational, it suffices that its fundamental cycle $\mathbf{Z}$ have arithmetic genus $p_a(\mathbf{Z}) = 0$ (where $p_a(\mathbf{Z}) = 1 - \chi(\mathbf{Z})$). Moreover, when the singularity is rational, then its multiplicity is equal to $|\mathbf{Z}^2|$ ([2], Cor. 6). We will use this criterion to prove the following theorem. (We denote the fundamental cycle by the boldface letter $\mathbf{Z}$ to distinguish it from the model Z on which the singularities in our next theorem lie.)

Theorem 4.1. *For each prime* p*, the singularities resolved in Theorem* 1.2 *have multiplicity* p *and are rational.*

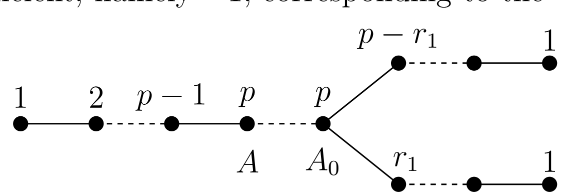
Proof. The resolution of the singularities in Theorem 1.2 have an intersection matrix of the form $N = N(p, \alpha, r_1)$ as in 2.5. We denote by $\mathbf{Z}$ the fundamental cycle of this matrix. As we shall see in 4.3, it is not hard in the case of N to write down a positive vector Z_0 which could be the fundamental cycle $\mathbf{Z}$ of N. Proposition 4.6 proves that $\mathbf{Z} = Z_0$ and provides the necessary facts on $\mathbf{Z}$ allowing us to immediately conclude that the singularities are rational of multiplicity p using Artin's results on rational singularities quoted above. □

Denote as in the above proof by $\mathbf{Z}$ the fundamental cycle of the matrix $N = N(p, \alpha, r_1)$ associated with the singularities in Theorem 1.2. Denote as in 4.3 by Z_0 the positive vector with $Z_0 \geq \mathbf{Z}$. We did not find a satisfactory way of proving directly, using combinatorial tools only, that this 'candidate' is indeed the fundamental cycle of N. We will instead rely on general results of [23], which themselves rely on results in [22]. Both of these papers use geometric tools. The paper [23] studies normal surfaces singularities, and in view of the use of the terminology 'holomorphic' in the introduction to [23], we infer that the author is working in the category of surfaces over $\mathbb{C}$. To be able to apply the results of [23] in our context, we need the following proposition, whose statement is probably classical, but we did not find it stated as such in the literature.

Proposition 4.2. *Let N be any intersection matrix as in* 2.1. *Then there exists a complex surface $\mathcal{Z}/\mathbb{C}$, singular at a single point $z \in \mathcal{Z}$, whose resolution of singularities $f : \mathcal{X} \to \mathcal{Z}$ over $\mathbb{C}$ has a fiber $f^{-1}(z)$ whose intersection matrix is equal to N (up to equivalence).*

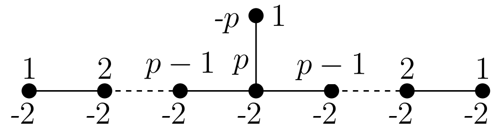
Proof. Choose any ordering of the vertices of $G(N)$. Pick a vertex C_i, and consider the associated positive vector R_i. Then it is always possible to complete the data $(G(N), N, R_i)$ into an arithmetical graph $(G(M), M, R)$ (see [15], proof of 3.14). The main theorem of [25] proves the existence of a smooth surface $\mathcal{X}/\mathbb{C}$ and a smooth curve $W/\mathbb{C}$ with a morphism $g : \mathcal{X} \to W$ such that for some $w \in W$, the arithmetical graph associated with $g^{-1}(w)$ is the given graph $(G(M), M, R)$. By construction, $G(N)$ is a subgraph of $G(M)$. We let $\mathcal{Z}/\mathbb{C}$ denote the surface obtained from $\mathcal{X}$ by contracting the components of $G(N)$. Such a contraction always exists by a theorem of Grauert in the category of complex analytic spaces. (Such a contraction exists in the category of algebraic spaces by a theorem of Artin ([3], 6.12, p. 125).) □

4.3 Consider the star-shaped graph as in 2.5, with intersection matrix $N = N(p, \alpha, r_1)$, and associated vector R_1. Assume from now on that $\alpha \geq p$. Consider the vector Z_0 described as follows. On the second and third terminal chains of the graph of $N(p, \alpha, r_1)$ (both on the right of the node), the coefficient of Z_0 on a vertex D is equal to the corresponding coefficient of R_1 at that vertex. On the first terminal chain to the left of the node, where the coefficients of R_1 are all equal to p, the coefficients of Z_0 are $(1, 2, \ldots, p-1, p, p, \ldots, p)$. For convenience, we call A_0 the node of $G(N)$, and we denote by A the p-th vertex of $G(N)$, counting from the left of the graph. Since we assume that $p \leq \alpha$, A is on the terminal chain on the left, and when $\alpha = p$, we have $A = A_0$. The diagram below describes the vector Z_0 in the case where $\alpha > p$. The vector NZ_0 has only one non-zero coefficient, namely -1, corresponding to the vertex A. Thus, $Z_0^2 = -p$.

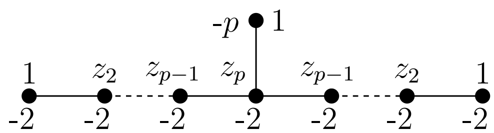


It follows that Z_0 is an upper bound for the fundamental cycle $\mathbf{Z}$ of N. Note that the fact NZ_0 has only one non-zero coefficient shows that Z_0 is in fact a vector of type R_i associated to N in 2.2. The fact that this non-zero coefficient is -1 shows that the class in $\mathbb{Z}^n/\mathrm{Im}(N)$ of the basis vector corresponding to the vertex A is trivial. Our goal is to show that $\mathbf{Z} = Z_0$ when $\alpha \geq p$. We first do so in a completely elementary way for the matrix appearing in Theorem 1.1, which we recall below.

Lemma 4.4. *Consider the matrix N appearing in Theorem* 1.1*, which is a special case of the above matrix $N(p, \alpha, r_1)$; its graph is represented below, and the positive coefficient next to a vertex is the coefficient of the vector Z_0 corresponding to the vertex. Then $\mathbf{Z} = Z_0$.*



Proof. Since the vector Z_0 is an upper bound for $\mathbf{Z}$ and has terminal vertices of multiplicity 1, the coefficients in $\mathbf{Z}$ of the terminal vertices of $G(N)$ must be equal to the corresponding coefficients of Z_0. Since the vector $\mathbf{Z}$ is unique, it is easy to check that the fundamental cycle must be 'symmetric' along the central vertical edge of the graph, as depicted in the picture below (the integer z_i in the graph next to a vertex is the coefficient of $\mathbf{Z}$ corresponding to this vertex.)



It follows from $N\mathbf{Z} \leq 0$ evaluated at the line corresponding to the node that

$$-2z_p + z_{p-1} + 1 + z_{p-1} \leq 0.$$

Thus, we find that $z_p - z_{p-1} \geq 1$. We also must have

$$-2z_i + z_{i-1} + z_{i+1} \leq 0$$

for all $i = 2, \ldots, p-1$. This latter inequality implies that $z_{i+1} - z_i \leq z_i - z_{i-1}$. Therefore, for all $i = 2, \ldots, p-1$,

$$1 \leq z_p - z_{p-1} \leq z_i - z_{i-1}.$$

Since $\mathbf{Z} \leq Z_0$, it follows that $\mathbf{Z} = Z_0$. In particular, $(\mathbf{Z} \cdot \mathbf{Z}) = -p$. □

4.5 Let us return now to the set-up of Theorem 1.1, where the resolution of the singularities of the normal surface Z have intersection matrices as in 4.4. We now show that for the fundamental cycle $\mathbf{Z}$ of these resolutions, we have $p_a(\mathbf{Z}) = 0$. Recall that

$$2p_a(\mathbf{Z}) - 2 = \mathbf{Z} \cdot (\mathbf{Z} + \Omega),$$

where Ω is the relative canonical sheaf. For each irreducible component C_i, which we know to be rational, we also have

$$2p_a(C_i) - 2 = -2 = C_i \cdot (C_i + \Omega),$$

so that $C_i \cdot \Omega = |C_i \cdot C_i| - 2$. Since all but one component C_i have $|C_i \cdot C_i| = 2$, we find that $2p_a(\mathbf{Z}) - 2 = (\mathbf{Z} \cdot \mathbf{Z}) + (\mathbf{Z} \cdot \Omega) = -p + (p-2)$, which shows that $p_a(\mathbf{Z}) = 0$, as desired.

The computation of the arithmetic genus $p_a(Z_0)$ in the case of the intersection matrix $N = (c_{ij})$ appearing in Theorem 1.2 is also straightforward. Let us define the invariant $p_a(Z_0)$ associated with the matrix N and the vector Z_0 by the formula $2p_a(Z_0) - 2 := Z_0 \cdot Z_0 + \sum_{i=1}^{n}(|c_{ii}| - 2)z_i$. In the case of the intersection matrix $N = N(p, \alpha, r_1)$ with vector Z_0 as in 4.3, we find using 2.3 that the sum in the definition of $2p_a(Z_0) - 2$ has a contribution from each of the three terminal chains, namely, 0, $p-r_1-1$, and $p-(p-r_1)-1$, giving $2p_a(Z_0)-2=-2$, so that $p_a(Z_0)=0$.

We now prove a much more general form of Lemma 4.4 using results from [23].

Proposition 4.6. *Assume that $\alpha \geq p$. Then Z_0 is the fundamental cycle $\mathbf{Z}$ of $N(p,\alpha,r_1)$, with $Z_0^2=-p$ and $p_a(Z_0)=0$. In particular, a singularity with intersection matrix $N(p,\alpha,r_1)$ and $\alpha \geq p$, is rational of multiplicity p.*

Proof. We use 4.2 to exhibit $N(p,\alpha,r_1)$ as the intersection matrix associated with the resolution of a complex surface singularity. We are now free to use the relevant results in [23]. Recall that for any positive real number r, the symbols $\{r\}$ denote the smallest integer m such that $m \geq r$.

We find, in [23], (3.4) on page 282, that the coefficient $z(A_0)$ (in the fundamental cycle $\mathbf{Z}$) associated to the node A_0 is equal to the least positive integer k such that

$$k|A_0\cdot A_0| \geq \{kr_1/p\}+\{k(p-r_1)/p\}+\{k(\alpha-1)/\alpha\}. \tag{4.6.1}$$

Indeed, a divisor $[kD]$ on A_0 is defined in [23], (3.2) on page 281 (the $\pm$ sign occurring there should be the $=$ sign). For each terminal chain of the graph, a ratio d_i/e_i is defined on page 281. For the matrix N, these ratios are $\alpha/(\alpha-1)$, p/r_1, and $p/(p-r_1)$, respectively. By definition, the divisor D is in the class of the conormal bundle of A_0, and so its degree is $|A_0\cdot A_0|$. Then (4.6.1) follows immediately from [23], (3.4).

We are going to show that $z(A_0)=p$, using (4.6.1). Recall that $|A_0\cdot A_0|=2$ in our case. The intersection matrix N at the node A_0 gives the relation $2p=r_1+(p-r_1)+p$. Thus,

$$2k=\frac{kr_1}{p}+\frac{k(p-r_1)}{p}+\frac{kp}{p}.$$

By definition, $\frac{kr_1}{p}\leq\{\frac{kr_1}{p}\}$ and $\frac{k(p-r_1)}{p}\leq\{\frac{k(p-r_1)}{p}\}$. Clearly, $\{k(\alpha-1)/\alpha\}=k$ when $k<\alpha$. It follows that when $k<\alpha$,

$$2k\leq\{kr_1/p\}+\{k(p-r_1)/p\}+\{k(\alpha-1)/\alpha\}$$

and we have equality when $k=p$. Since kr_1/p and $k(p-r_1)/p$ cannot be integers when $k<p$, we find that for $k<p$:

$$\frac{kr_1}{p}+\frac{k(p-r_1)}{p}=k<\{kr_1/p\}+\{k(p-r_1)/p\}.$$

Hence, $z(A_0)=p$.

We now compute the full fundamental cycle $\mathbf{Z}$ of N using [23], (3.5), page 283. Let us list consecutively the vertices of the i-th terminal chain of $G(N)$ attached to A_0 by $A_0, D_i, \dots, T_i$, so that D_i is attached to A_0, and T_i is the terminal vertex of the chain. Let N_i denote the intersection matrix associated with the vertices $D_i,\dots,T_i$, with $(D_i\cdot D_i)$ in the top left corner, and $(T_i\cdot T_i)$ in the bottom right corner. Let n_i denote the length of this chain, so that N_i is a square $(n_i\times n_i)$-matrix. Starting with $\tau_{n_i}:=1$, we define inductively the integer vector $(\tau_1,\dots,\tau_{n_i})$ such that $(\tau_1,\dots,\tau_{n_i})N_i=(-\tau_0,0,\dots,0)$. The coefficients $(z_1,\dots,z_{n_i})$ of the vector $\mathbf{Z}$ on the vertices of the chain $D_i,\dots,T_i$ are computed as follows in [23], (3.5), page 283. Let $z_0:=z(A_0)$. Then

$$z_j=\{\frac{\tau_j z_{j-1}}{\tau_{j-1}}\},$$

for $j=1,\dots,n_i$. As proved in [23], Lemma 3.2, if $z(A_0)=a\tau_0$ for some positive integer a, then $z_j=a\tau_j$.

This latter lemma applies to both the second and third terminal chains of $G(N)$ (those on the right of the node A_0). Indeed, we computed above that $z(A_0) = p$, and we find that $z(A_0) = \tau_0$ for both terminal chains. Thus on these chains, the coefficients of $\mathbf{Z}$ are as predicted in Proposition 4.6. On the first chain, we find that $\tau_0 = \alpha$, and $(\tau_1, \dots, \tau_{n_i}) = (\alpha-1, \alpha-2, \dots, 2, 1)$. Thus, $z_1 = \{(\alpha-1)p/\alpha\} = p$, since $\alpha \geq p$. Moreover, for $j < \alpha - p$,

$$z_{j+1} = \{(\alpha - (j+1))p/(\alpha - j)\} = \{p - p/(\alpha - j)\} = p.$$

We leave it to the reader to verify that $(z_1, \dots, z_{\alpha-1}) = (p, \dots, p, p-1, \dots, 2, 1)$, as desired. □

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Department of mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: lorenzin@uga.edu