JOURNAL OF ALGEBRA 156, 65–75 (1993)

Reducibility of Polynomials in Two Variables

DINO LORENZINI*

Department of Mathematics, Yale University, New Haven, Connecticut 06520

Communicated by Joe Harris

Received March 1, 1990

Let K be an algebraically closed field. Let $f(x, y) \in K[x, y]$ be an irreducible polynomial of degree d. For each $a \in K$, write the factorization of $f(x, y) + a$ in $K[x, y]$ as

$$f(x, y) + a = \prod_{i=1}^{n_a+1} (f_{a,i}(x, y))^{r_{a,i}}.$$

When K has characteristic zero, Y. Stein [Ste] proved the following inequality:

$$\sum_a n_a \leqslant d - 1.$$

Using standard facts about algebraic surfaces, we are able to prove that a sharper formula holds in any characteristic, namely,

$$\sum_{a \in K} n_a \leqslant \min_b \left\{ \sum_i \deg(f_{b,i}) \right\} - 1 \leqslant d - 1.$$

We are also going to consider a slightly more general problem. Let $\alpha(x_0, x_1, x_2)$ and $\beta(x_0, x_1, x_2)$ be two homogeneous polynomials in $K[x_0, x_1, x_2]$ of degree d. These polynomials define, in the projective plane $\mathbf{P}^2$, two closed curves denoted respectively by C_0 and C_∞. We assume that $\gcd(\alpha, \beta) = 1$, so that C_0 and C_∞ intersect only in finitely many points. For $a \in K$, let $C_a \subset \mathbf{P}^2$ denote the curve

$$C_a = \{(x_0, x_1, x_2) \in \mathbf{P}^2 \mid \alpha(x_0, x_1, x_2) - a\beta(x_0, x_1, x_2) = 0\}.$$

Let $n_a + 1$ denote the number of irreducible components of C_a; we are interested in bounding $\sum_{a \in K} n_a$ in terms of d.

* Research supported by a grant from the Fond national suisse de la recherche scientifique.

0021-8693/93 $5.00

The meromorphic function α/β defines a map:

$$\pi\colon \mathbf{P}^2\backslash\{C_0\cap C_\infty\}\to\mathbf{P}^1$$
$$(x_0, x_1, x_2)\mapsto(\alpha(x_0, x_1, x_2), \beta(x_0, x_1, x_2)).$$

We identify the elements of $\mathbf{P}^1(K)$ with $K\cup\{\infty\}$ in such a way that $(a, 1)\in\mathbf{P}^1$ corresponds to $a\in K$ and $(1, 0)\in\mathbf{P}^1$ corresponds to ∞. With this convention, the curve C_a, $a\in\mathbf{P}^1$, is the closure of $\pi^{-1}(a)$ in $\mathbf{P}^2$. There exists (see the Theorem below for more details) a birational map $p\colon\tilde{X}\to\mathbf{P}^2$, obtained by a finite sequence of blow-ups of points, and a map $\tilde{\pi}\colon\tilde{X}\to\mathbf{P}^1$, such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\pi}} & \mathbf{P}^1 \\ {\scriptstyle p}\downarrow & & \uparrow{\scriptstyle \pi} \\ \mathbf{P}^2 & \longleftarrow & \mathbf{P}^2\backslash\{C_0\cap C_\infty\} \end{array}$$

In order to simplify the notations, we shall write π instead of $\tilde{\pi}$ to describe the map $\tilde{X}\to\mathbf{P}^1$. Let $F_a:=\pi^{-1}(a)\subset\tilde{X}$ denote the fiber of π over $a\in\mathbf{P}^1$. It is an effective divisor on $\tilde{X}$ and as such we write it as

$$F_a:=\sum_{i=1}^{m_a+1} s_{a,i}Y_{a,i},$$

where $s_{a,i}$ is the multiplicity of the irreducible component $Y_{a,i}$. Let ρ denote the number of blow-ups needed to describe the map $p\colon\tilde{X}\to\mathbf{P}^2$. Since $\mathrm{Pic}(\mathbf{P}^2)\cong\mathbf{Z}$,

$$\mathrm{Pic}(\tilde{X})\cong\mathbf{Z}^{\rho+1}$$

(see [Har, V, 3.2]). To bound $\sum n_a$ in terms of d, we shall, on one hand, bound ρ in terms of d and, on the other hand, study the subgroup of $\mathrm{Pic}(\tilde{X})$ generated by the irreducible curves $Y_{a,i}$.

LEMMA. *Let $\pi\colon\tilde{X}\to\mathbf{P}^1$ be a morphism defined as above by a meromorphic function. If π has connected fibers, then*

$$\sum_{a\in\mathbf{P}^1} m_a\leqslant\mathrm{rank}(\mathrm{Pic}(\tilde{X}))-2.$$

In particular, the number of reducible fibers is finite.

Proof. Let F be any fiber of π and denote by P the $\mathbf{Q}$-vector space

$$P:=\mathrm{Pic}(\tilde{X})\otimes\mathbf{Q}/(\mathbf{Q}\cdot F).$$

It is clear that $\dim_{\mathbf{Q}} P = \text{rank}(\text{Pic}(\tilde{X})) - 1$. Let P_a be the subvector space of P generated by the set $\{Y_{a,i}, i = 1, ..., m_a + 1\}$. Zariski's lemma states that if $D := \sum_i d_i Y_{a,i}$, $d_i \in \mathbf{Q}$, is any divisor supported on F_a, then

(1) $D^2 \leqslant 0$.

(2) $D^2 = 0$ if and only if $D = qF_a$ for some $q \in \mathbf{Q}$.

(See for instance [BPV, III, 8.2], for a proof when $K = \mathbf{C}$.) It is an easy consequence of Zariski's lemma that

$$P_a \text{ has dimension } m_a$$

and that

$$P_a \cap P_b = \{0\}.$$

To prove the desired inequality, it remains only to show that $\bigoplus_a P_a \neq P$. Let E be any line in $\mathbf{P}^2 \setminus \{C_0 \cap C_\infty\}$ and denote by $\tilde{E}$ its transform in $\tilde{X}$. We claim that $\bigoplus P_a \oplus \mathbf{Q}\tilde{E} \subseteq P$. Assume on the contrary that $\tilde{E} \in \bigoplus P_a$, and write $\tilde{E} = \sum_a D_a + qF$ with $D_a^{\text{red}} \subseteq F_a^{\text{red}}$. Then $(\tilde{E} \cdot \tilde{E}) = \sum D_a^2 \leqslant 0$ by Zariski's lemma, contradicting the fact that $(\tilde{E} \cdot \tilde{E}) = (E \cdot E) = 1$.

Notations. Let $\kappa_a : D_a \to C_a^{\text{red}}$ denote the normalization of C_a^{red}. Let

$$C_0 \cap C_\infty = \{P_1, ..., P_s\}.$$

For each closed point P_i, $i = 1, ..., s$, set

$$\delta_i^{(a)} := |\kappa_a^{-1}(P_i)| \qquad \text{and} \qquad \delta^{(a)} := \sum_{i=1}^{s} \delta_i^{(a)}.$$

THEOREM. *Let α/β be a meromorphic function on $\mathbf{P}^2$. Let $\pi: \tilde{X} \to \mathbf{P}^1$ and $p: \tilde{X} \to \mathbf{P}^2$ be the associated maps. Assume that π has connected fibers. Then*

(1) $\sum_{a \in \mathbf{P}^1} m_a \leqslant d^2 - 1$.

(2) $\sum_{a \in \mathbf{P}^1} n_a \leqslant \sum_{i=1}^{s} (\delta_i^{(0)} + \delta_i^{(\infty)} - 1) - 1$.

(3) *If the fiber F_b is integral, then $\sum n_a \leqslant \delta^{(b)} - 1$.*

(4) *If π is generically smooth, then $\delta^{(a)}$ is constant for all but finitely many fibers.*

Proof of (1). In view of the previous lemma, we only need to estimate how many blow-ups are necessary to obtain $\tilde{X}$ from $\mathbf{P}^2$: we will show that $\rho \leqslant d^2$. Consider the following situation: let φ be a meromorphic function on a surface X. Let C and D be two effective divisors on X such that $\text{div}(\varphi) = C - D$. The function φ defines a map

$$\pi: X \setminus \{C \cap D\} \to \mathbf{P}^1.$$

We are looking for a finite sequence of blow-ups

$$X_\rho \xrightarrow{p_\rho} X_{\rho-1} \to \cdots \to X_1 \xrightarrow{p_1} X$$

such that the rational map $\pi: X \to \mathbf{P}^1$ extends to a morphism $\pi: X_\rho \to \mathbf{P}^1$. Let $P \in C \cap D$ and let $p: \bar{X} \to X$ be the blow-up at P. As usual, $\mu_P(C)$ denotes the multiplicity of a curve C at a point P. We may assume without loss of generality that $\mu_P(D) \geqslant \mu_P(C)$. We have

$$p^*(C) = \bar{C} + \mu_P(C)E,$$

where $\bar{C}$ is the strict transform of C in $\bar{X}$ and E is the exceptional divisor. We can extend the rational map $\pi: X \to \mathbf{P}^1$ to a rational map $\bar{\pi}: \bar{X} \to \mathbf{P}^1$ using the two linearly equivalent divisors $C' := \bar{C}$ and $D' := \bar{D} + (\mu_P(D) - \mu_P(C))E$. We have

$$(C' \cdot D')_{\bar{X}} = (C \cdot D)_X - \mu_P^2(C).$$

After performing at most $(C \cdot D)_X$ such processes, we obtain a surface X_ρ and two linearly equivalent divisors C_ρ and D_ρ on X_ρ such that

$$C_\rho \cap D_\rho = \varnothing \qquad \text{in } X_\rho.$$

Hence, the associated rational map $X_\rho \to \mathbf{P}^1$ is defined everywhere and extends $\pi: X \to \mathbf{P}^1$. Applying now this discussion to $\varphi = \alpha/\beta$, we conclude that $\rho \leqslant d^2 = (C_0 \cdot C_\infty)_{\mathbf{P}^2}$, and the first part of our theorem is proven.

Before turning to the proof of (2), let us make the following remarks.

Remark 1. We keep the notations introduced in the proof of (1). If the curve C on the surface X is smooth and P is a point of C, then $\mu_P(C) = 1$. Moreover, the curve $\bar{C}$ is also smooth. Hence, X_ρ is obtained after exactly $(C \cdot D)_X$ blow-ups.

EXAMPLES. As the following examples will show, the inequality $\sum m_a \leqslant \rho - 1$ in the above lemma may or may not be strict.

- Assume that $\mathrm{char}(K) \neq 3$, and let $\alpha(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3$ and $\beta(x_0, x_1, x_2) = 3x_0x_1x_2$ define the Hessian family of elliptic curves. The curves C_0 and C_∞ intersect in 9 distinct points. Hence, $\rho = d^2 = 9$. The associated fibration $\pi: \tilde{X} \to \mathbf{P}^1$ has four singular fibers, each consisting of a cycle of three rational curves. Hence, $\sum n_a = \sum m_a = 8 = d^2 - 1$.

- Let $\alpha(x_0, x_1, x_2)$ be a homogeneous polynomial of degree d in $K[x_0, x_1, x_2]$ defining a smooth curve C_0 in $\mathbf{P}^2$. Let L_∞ denote the line $\{x_2 = 0\}$ and let C_∞ be the curve $d \cdot L_\infty$. Assume that C_0 and L_∞ intersect in only one point. One easily checks that the fiber F_∞ has $d^2 = 1 + m_\infty$ components. Therefore, $\sum m_a = m_\infty = d^2 - 1$ and $\sum n_a = 0$.

• Fix an integer $d \geq 3$ prime to the characteristic of K and let $\alpha(x_0, x_1, x_2) = x_0^d - x_2^d$ and $\beta(x_0, x_1, x_2) = x_1^d - x_2^d$. The curves C_0 and C_∞ intersect in d^2 points and, therefore, $\tilde{X}$ is obtained after d^2 blow-ups. Simple computations show that the curve C_a is reducible only when $a = 0, 1, \infty$, and that for these three values, C_a is the union of d lines. Hence, $\sum n_a = \sum m_a = 3(d-1) < d^2 - 1$.

Remark 2. By an entirely different method, W. Ruppert [Rup, Satz 6] shows that, in characteristic zero, the number r of reducible fibers in the family $\{C_a\}$ is bounded by $d^2 - 1$. When the generic fiber is assumed to be smooth, he shows that $r \leq 3(d-1)$ and that this bound is achieved.

Proof of (2). A fiber F_a^{red} consists of the strict transform $\tilde{C}_a^{\text{red}}$ of C_a^{red} and of $m_a - n_a$ "exceptional" divisors. By the previous lemma, we have

$$\sum_{a \in \mathbf{P}^1} n_a \leq \rho - \sum_{a \in \mathbf{P}^1} (m_a - n_a) - 1.$$

There are exactly $\rho - \sum (m_a - n_a)$ exceptional divisors that map onto $\mathbf{P}^1$ under π. These divisors form s disjoint subsets S_i, indexed by their image P_i under $p: \tilde{X} \to \mathbf{P}^2$. Fix a point P_ℓ in $C_0 \cap C_\infty$. We are going to show that the order of the corresponding set S_ℓ is bounded by $\delta_\ell^{(0)} + \delta_\ell^{(\infty)} - 1$. In order to achieve this, we are going to construct a tree Γ such that

$$|S_\ell| \leq \text{number of edges of } \Gamma$$

and

$$\delta_\ell^{(0)} + \delta_\ell^{(\infty)} \geq \text{number of vertices of } \Gamma.$$

Our claim will follow from the fact that for any (possibly disconnected) tree Γ,

$$\text{number of edges of } \Gamma \leq (\text{number of vertices of } \Gamma) - 1.$$

Recall that, if $\{D_i\}$ is any family of effective divisors on $\tilde{X}$, then one defines its dual graph $G(\{D_i\})$ as follows: the vertices of $G(\{D_i\})$ are the irreducible components of the divisors D_i, and two vertices are linked by one edge if and only if the corresponding curves intersect in $\tilde{X}$. The graph $G := G(p^{-1}(P_\ell))$ is a connected tree by construction.

Let

$$E_0 := F_0 - \tilde{C}_0$$

and

$$E_\infty := F_\infty - \tilde{C}_\infty.$$

Let $\mathscr{E}^0$ denote the set of irreducible divisors in the support of $p^{-1}(P_\ell) \cap E_0$ that intersect $\tilde{C}_0$. Similarly, let $\mathscr{E}^\infty$ denote the set of irreducible divisors in the support of $p^{-1}(P_\ell) \cap E_\infty$ that intersect $\tilde{C}_\infty$. The sets $\mathscr{E}^0$ and $\mathscr{E}^\infty$ are disjoint since $F_0 \cap F_\infty = \varnothing$. We can always write

$$\mathscr{E}^0 = T_1 \sqcup \cdots \sqcup T_k$$

and

$$\mathscr{E}^\infty = T'_1 \sqcup \cdots \sqcup T'_{k'}$$

in such a way that the graphs $G(T_i)$ and $G(T'_j)$ are connected trees for all $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k'$. A path in G from a vertex in $\mathscr{E}^0$ to a vertex in $\mathscr{E}^\infty$ is called *special* if it is minimal and if, except for its end vertices, no other vertex on the path belongs to $\mathscr{E}^0 \cup \mathscr{E}^\infty$.

We can write the set S_ℓ as a disjoint union of four sets

$$S_\ell = M \sqcup N_0 \sqcup N_\infty \sqcup O,$$

where

$$M = \{E \in S_\ell \mid (E \cdot E_0) \neq 0 \text{ and } (E \cdot E_\infty) \neq 0\},$$

$$N_0 = \{E \in S_\ell \mid (E \cdot E_0) = 0 \text{ and } (E \cdot E_\infty) \neq 0\},$$

$$N_\infty = \{E \in S_\ell \mid (E \cdot E_0) \neq 0 \text{ and } (E \cdot E_\infty) = 0\},$$

and

$$O = \{E \in S_\ell \mid (E \cdot E_0) = 0 \text{ and } (E \cdot E_\infty) = 0\}.$$

Consider the following graph $\Gamma(M)$: its vertices are the graphs $G(T_i)$ and $G(T'_j)$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k'$. Two vertices $G(T_i)$ and $G(T'_j)$ are linked in $\Gamma(M)$ if and only if there exists a special path in G between an element of T_i and an element of T'_j. Clearly, there can be at most one special path in G between an element of T_i and an element of T'_j because the graph G is a tree and the graphs $G(T_i)$ and $G(T'_j)$ are connected. Therefore, the graph $\Gamma(M)$ is a tree. Since the images of the connected fibers F_0 and F_∞ under the blow-down of an element of S_ℓ, must intersect, the elements of M define $|M|$ distinct special paths from $\mathscr{E}^0$ to $\mathscr{E}^\infty$ and, therefore, define $|M|$ distinct edges of $\Gamma(M)$.

Let E be an element of N_0. Since the images of the connected fibers F_0 and F_∞ under the blow-down of an element of S_ℓ must intersect, there exists an index j and a path in G from E to an element of T'_j. Denote by v_E the vertex corresponding to $G(T'_j)$ in $\Gamma(M)$. Let $\Gamma(M, E)$ denote the tree constructed from the graph $\Gamma(M)$ by adding to it an extra vertex and linking this new vertex only to the vertex v_E of $\Gamma(M)$. Let $\Gamma(M, N_0)$ be the tree obtained from $\Gamma(M)$ by performing the above construction for all elements of N_0. Construct then a tree $\Gamma(M, N_0, N_\infty)$, obtained from $\Gamma(M, N_0)$ in an analogous manner by adding a vertex and an edge for each element of N_∞. Finally, let $\mathscr{G}$ denote the unique connected graph having two vertices and one edge. Let Γ denote the union of $\Gamma(M, N_0, N_\infty)$ with $|O|$ disjoint copies of the graph $\mathscr{G}$. This new graph Γ is a possibly disconnected tree with at least $|S_\ell|$ edges and at most $k+k'+|N_0|+|N_\infty|+2\,|O|$ vertices.

The key point is to note that $k+|N_0|+|O| \leqslant \delta_\ell^{(0)}$ and that $k'+|N_\infty|+|O| \leqslant \delta_\ell^{(\infty)}$. Let $p_0: \tilde{C}_0^{\text{red}} \to C_0^{\text{red}}$ denote the restriction of p to $\tilde{C}_0^{\text{red}}$. To show that $k+|N_0|+|O| \leqslant \delta_\ell^{(0)}$, it is sufficient to show that

$$k+|N_0|+|O| \leqslant |p_0^{-1}(P_\ell)|.$$

First, note that if $E \in \mathscr{E}_0$, then

$$E \cap \tilde{C}_0^{\text{red}} = E \cap p_0^{-1}(P_\ell).$$

Since by construction an element of T_i does not intersect an element of T_j if $i \neq j$, it follows that

$$k \leqslant |E_0^{\text{red}} \cap p_0^{-1}(P_\ell)|.$$

We claim now that if E and F are elements of S_ℓ, then $E \cap F = \varnothing$. Indeed, let

$$X_p \to \cdots \to X_E \xrightarrow{p_E} \cdots \to X_F \xrightarrow{p_F} \cdots \to \mathbf{P}^2$$

be a sequence of blow-ups such that π extends to X_p and such that p_E contracts E and p_F contracts F. By construction, $(E \cdot E)_{X_p} = (F \cdot F)_{X_p} = -1$. If $E \cap F \neq \varnothing$, then $(F \cdot F)_{X_F} \geqslant 0$, a contradiction.

Note now that if E is an element of $N_0 \cup O$, then

$$E \cap \tilde{C}_0^{\text{red}} = E \cap p_0^{-1}(P_\ell).$$

Therefore, since $E \cap \tilde{C}_0^{\text{red}} \neq \varnothing$ by construction, it follows that

$$|N_0|+|O| \leqslant |p_0^{-1}(P_\ell)| - |E_0^{\text{red}} \cap p_0^{-1}(P_\ell)|.$$

This concludes the proof of (2).

Proof of (3) *and* (4). If the fiber F_b is integral, then it is the strict transform of the curve C_b. The map p restricts to a morphism $p_b: F_b \to C_b$ and clearly $|p_b^{-1}(P_\ell)| \leqslant \delta_\ell^{(b)}$. Let again S_ℓ denote the set of exceptional divisors contracting to P_ℓ under p and which map onto $\mathbf{P}^1$ under π. Each element E of S_ℓ intersects F_b in a point of $p_b^{-1}(P_\ell)$ and each point in $p_b^{-1}(P_\ell)$ is in the intersection of F_b with an element of S_ℓ. As we pointed out in the proof of (2), the elements of S_ℓ do not intersect and, therefore, $|S_\ell| \leqslant |p_b^{-1}(P_\ell)|$, which shows (3). Note that the inequality $|S_\ell| \leqslant |p_b^{-1}(P_\ell)|$ may be strict as, for instance, in the family $x_1^2 x_2 - a x_0^2(x_0 + x_2)$ at the node $P = (0, 0, 1)$ for $b = 1$.

Assume now that the map is generically smooth. There exists then an open U in $\mathbf{P}^1$ such that F_b is smooth for all $b \in U$. Therefore, it follows from the definitions that $|p_b^{-1}(P_\ell)| = \delta_\ell^{(b)}$ for all $b \in U$. Let E be a divisor in S_ℓ for some $\ell \in \{1, ..., s\}$. The restricted map $\pi: E \to \mathbf{P}^1$ may be ramified; but since the number of such ramification points and the number of such divisors E are finite, there exists a dense open set $V \subseteq U$ such that, for each E fixed, the value of $|F_b \cap E|$ is constant for all $b \in V$. In particular, for $b \in V$, we have $\delta_\ell^{(b)} = |p_b^{-1}(P_\ell)| = \sum_{E \in S_\ell} |F_b \cap E|$.

Remark 3. Assume that $\delta^{(a)}$ is constant on a dense open set of $\mathbf{P}^1$. The following examples show that the finitely many exceptional values of $\delta^{(a)}$ can be both bigger and smaller than the generic value. We also give an example to show that (3) is false if the fiber F_b is not assumed to be integral.

- Consider the Legendre family of elliptic curves given by

$$y^2 = x(x-1)(x-a).$$

The curves C_0 and C_1 are nodal cubics that intersect in three points: at the two nodes and at (0, 1, 0). We have $\delta^{(0)} = \delta^{(1)} = 4$ and $\delta^{(\infty)} = 5$; all other curves C_a have $\delta^{(a)} = 3$.

- Assume that $\operatorname{char}(K) \geqslant d \geqslant 3$, and consider the polynomial

$$f(x, y) + a = x(x+1)(x+2) \cdot \ldots \cdot (x+d-2)y + x + a.$$

We show in Remark 5 below that the associated map π has connected fibers. Since this family has $d-1$ reducible curves at $a = 0, 1, ..., d-2$, the inequality (3) in the above theorem shows that the generic values of $\delta^{(a)}$ must at least equal d. The curves C_a intersect on the line at infinity in two points, so that $\delta^{(\infty)} = 2$. In particular, $\sum n_a > \delta^{(\infty)} - 1$.

Let us now prove the inequality stated at the beginning of this article. We say that a polynomial $f(x, y) \in K[x, y]$ of degree d is *not a composite polynomial* if $f(x, y)$ cannot be written as $h(g(x, y))$ with $g(x, y) \in K[x, y]$ and $h(z) \in K[z]$ of degree at least 2. In particular, an irreducible polynomial is not a composite polynomial. For each $a \in K$, write

$$f(x, y) + a = \prod_{i=1}^{n_a+1} (f_{a,i})^{r_{a,i}}.$$

Note that, if $f(x, y)$ is not a composite polynomial, then $n_a > 0$ for all values of a where $f(x, y) + a$ is reducible.

COROLLARY 1. *Let K be an algebraically closed field. If $f(x, y) \in K[x, y]$ is not a composite polynomial and has degree d, then*

$$\sum_{a \in K} n_a \leq \min_b \left\{\sum_i \deg(f_{b,i})\right\} - 1 \leq d - 1.$$

Proof. Let $\alpha(x_0, x_1, x_2) = x_2^d f(x_0/x_2, x_1/x_2)$ and $\beta(x_0, x_1, x_2) = x_2^d$. Let L_∞ be the line $\{x_2 = 0\}$ in $\mathbf{P}^2$. The divisor C_∞ is equal to $d \cdot L_\infty$. In particular, $\delta_i^{(\infty)} = 1$, $\forall i = 1, ..., s$. Let $\pi: \tilde{X} \to \mathbf{P}^1$ denote the morphism associated to α/β. We shall show below that the hypothesis "$f(x, y)$ is not a composite polynomial" implies that π has connected fibers. Once this fact is ascertained, we can conclude the proof of the corollary as follows: let

$$C_0^{\text{red}} := \sum_{i=1}^{n_0+1} C_{0,i}.$$

The linear system defining the normalization map $\kappa_0: D_0 \to C_0^{\text{red}} \subseteq \mathbf{P}^2$ has degree at most $(\sum \deg C_{0,i})$. In particular,

$$\sum_{j=1}^{s} \delta_j^{(0)} \leq \sum_i \deg(C_{0,i}) = \sum_i \deg(f_{0,i}).$$

Hence, it follows from part (2) of the previous theorem that

$$\sum_a n_a \leq \sum_{j=1}^{s} (\delta_j^{(0)} + \delta_j^{(\infty)} - 1) - 1 = \left(\sum_j \delta_j^{(0)}\right) - 1 \leq \sum_i \deg(f_{0,i}) - 1.$$

Let $\tilde{X} \xrightarrow{g} C \xrightarrow{h} \mathbf{P}^1$ be the Stein factorization of π (see [Har, III, 11.5]). Since h is a finite morphism, C is a (smooth) curve. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & C \\ \downarrow & & \downarrow \\ \mathrm{Alb}(\tilde{X}) & \longrightarrow & \mathrm{Alb}(C) \end{array}$$

We claim that C is a smooth rational curve. First, note that $\dim \mathrm{Alb}(\tilde{X}) \leq \dim_K H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ (see [Ses, exposé 8, Thms. 1 and 3]). Since $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}) = \{0\}$ (see [Har, V, 3.4, and III, 5.1]), the abelian variety $\mathrm{Alb}(\tilde{X})$ must be trivial. Hence C is isomorphic to $\mathbf{P}^1$, because otherwise C would inject into $\mathrm{Alb}(C)$ (see [Mil, 6.1]) and the diagram above would not be commutative.

By restriction, we have a map

$$\mathbf{P}^2 \setminus \{C_0 \cap L_\infty\} \xrightarrow{g} \mathbf{P}^1 \xrightarrow{h} \mathbf{P}^1.$$

Without loss of generality, we may assume that $g(L_\infty \backslash \{C_0 \cap L_\infty\})$ is the point ∞ in $\mathbf{P}^1$. The map

$$\pi\colon P^2 \backslash L_\infty \to \mathbf{P}^1 \backslash \{\infty\}$$

factors then into

$$\mathbf{P}^2 \backslash L_\infty \xrightarrow{g} \mathbf{P}^1 \backslash \{\infty\} \xrightarrow{h} \mathbf{P}^1 \backslash \{\infty\}.$$

This sequence of morphisms induces the maps

$$K[x, y] \xleftarrow{g_*} K[z] \xleftarrow{h_*} K[t],$$

with $h_*(t) := h(z) \in K[z]$ and $g_*(z) := g(x, y) \in K[x, y]$. Since the composition $h \circ g$ equals π, we have

$$f(x, y) = h(g(x, y)).$$

Hence, $h(z)$ must be a linear polynomial because $f(x, y)$ is not a composite. Therefore, the morphism h is an isomorphism and the fibers of π are connected because the fibers of g are connected by construction.

COROLLARY 2. *Let $f(x, y) = \prod_{i=1}^{n_0+1} L_i(x, y)^{r_i}$ be a product of linear polynomials. Assume that $n_0 > 0$ and that the lines in K^2, defined by the equations $L_1(x, y) = 0$ and $L_2(x, y) = 0$, do intersect. Assume also that $gcd(r_1, ..., r_{n_0+1}) = 1$. Then $f(x, y)$ is not a composite polynomial and the polynomial $f(x, y) + a$ is irredubible if $a \neq 0$.*

Proof. The fact that $f(x, y) + a$ is irreducible follows immediately from the previous corollary once we have shown that $f(x, y)$ is not a composite polynomial. Suppose ab absurdo that $f(x, y)$ is a composite polynomial and factor it as

$$f(x, y) = (g(x, y) - \alpha_1) \cdot \ldots \cdot (g(x, y) - \alpha_s),$$

with $s > 1$ and $\alpha_i \in K$ for all $i = 1, ..., s$. Without loss of generality, we may assume that $g(x, y)$ is not a composite polynomial. Since $f(x, y)$ is a product of linear terms, each polynomial $(g(x, y) - \alpha_i)$ must factor into a product of linear terms. The polynomial $g(x, y)$ cannot have degree one; otherwise the lines defined by $L_1(x, y) = 0$ and $L_2(x, y) = 0$ would not intersect in K^2. The previous corollary applied to $g(x, y)$ implies then that all the constants α_i must be equal. Hence, s must divide $\gcd(r_1, ..., r_{n_0+1})$, which is a contradiction since $s > 1$.

Remark 4. We may construct polynomials $f(x, y) + a$ that factor for several values of a as follows. Let $\ell(x) \in K[x]$ be any polynomial of degree m. For any constant a, let

$$\ell(x)+a=\prod_{j=1}^{m} L_j^{(a)}(x)$$

denote the factorization of $\ell(x)+a$ in $K[x]$. Pick k distinct elements of K^*, say $a_1, ..., a_k$. Choose a polynomial $r(x, y)$ in two variables and k subsets $I_1, ..., I_k$ of $\{1, ..., m\}$. Set

$$t(x, y)=1+\left(\prod_{j\in I_1} L_j^{(a_1)}(x)\right)\cdot \ldots \cdot\left(\prod_{j\in I_k} L_j^{(a_k)}(x)\right)\cdot r(x, y).$$

Finally, set

$$f(x, y)=\ell(x)\cdot t(x, y).$$

The polynomial $f(x, y)+a$ is then reducible for $a=0, a_1, ..., a_k$.

Note that we may construct in this way a polynomial $f(x, y)$ such that there exist two distinct constants b and c with $\sum_i \deg(f_{b,i}) \neq \sum_i \deg(f_{c,i})$.

Remark 5. The polynomial $f(x, y) = x(x+1)\cdot \ldots \cdot(x+d-2)y+x$ has degree d and is not a composite polynomial. Indeed, the monomial $x^{d-1}y$ is the only monomial of degree d in $f(x, y)$. If $f(x, y)$ could be written as $h(g(x, y))$ for some polynomial $h(z)$ of degree $h \geq 2$, then $x^{d-1}y$ would be the hth power of a homogeneous polynomial of degree d/h. This is clearly impossible. The polynomial $f(x, y)$ provides an example where the bound for $\sum n_a$ given in Corollary 1 is achieved. Note also that in this example the polynomial $f(x, y)$ has integer coefficients. This seems to indicate that the bound for $\sum n_a$ cannot be improved when K is not assumed to be algebraically closed.

References

[BPV] W. Barth, C. Peters, and A. Van de Ven, "Compact Complex Surfaces," Springer-Verlag, Berlin/New York, 1984.

[Har] R. Hartshorne, "Algebraic Geometry," Springer-Verlag, Berlin/New York, 1977.

[Mil] J. Milne, Jacobian varieties, *in* "Arithmetic Geometry" (G. Cornell and J. Silverman, Eds.), Springer-Verlag, Berlin/New York, 1986.

[Rup] W. Ruppert, Reducibilität ebener Kurven, *J. reine angew. Math.* **369** (1986), 167–191.

[Ses] C. Seshadri, Exposé 8 in the Séminaire Chevalley, 1958/59 (École Normale Supérieure).

[Ste] Y. Stein, The total reducibility order of a polynomial in two variables, *Israel J. Math.* **68** (1989), 109–122.