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Jacobians with potentially good ℓ-reduction

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Introduction

Let K be a complete field with a discrete valuation. Let $\mathcal{O}_K$ denote the ring of integers of K. Let k be the residue field of $\mathcal{O}_K$, assumed to be algebraically closed of characteristic $p \geqq 0$. Let A/K be an abelian variety of dimension g. Denote by $\mathcal{A}/\mathcal{O}_K$ the Néron model of A/K. Recall that the special fiber $\mathcal{A}_k/k$ of $\mathcal{A}/\mathcal{O}_K$ is an extension of the finite abelian group of components $\Phi(A)$ by a smooth connected group scheme $\mathcal{A}_k^0/k$, the connected component of zero in $\mathcal{A}_k$. By Chevalley's theorem, the group $\mathcal{A}_k^0$ can be described by an exact sequence

$$0 \to \mathcal{U} \times \mathcal{T} \to \mathcal{A}_k^0 \to \mathcal{B} \to 0,$$

where $\mathcal{B}$ is an abelian variety of dimension a_K, $\mathcal{T}$ is a torus of dimension t_K, and $\mathcal{U}$ is a unipotent group of dimension u_K. We call a_K, t_K, and u_K respectively the abelian, toric, and unipotent ranks of A/K.

Let L/K denote the minimal extension of K such that A_L/L has semi-stable reduction (see for instance [Des], 5.15). For each prime ℓ dividing $[L:K]$, $\ell \neq p$, let K_ℓ/K denote the unique subfield of L with the property that

$$[K_\ell : K] = \ell^{\operatorname{ord}_\ell([L:K])}.$$

We say that A/K has *potentially good ℓ-reduction* if the abelian variety A_{K_ℓ}/K_ℓ has toric rank equal to zero. Clearly, an abelian variety with potentially good reduction has potentially good ℓ-reduction for all prime $\ell \neq p$. Note also that if A/K has potentially good ℓ-reduction for some prime ℓ, then the toric rank of A/K is equal to zero.

Let X/K be a smooth proper geometrically irreducible curve having a K-rational point. Let A/K denote its jacobian. Raynaud [Ray] has described the group of components $\Phi := \Phi(A)$ in terms of the special fiber of a regular model $\mathcal{X}/\mathcal{O}_K$ of the curve X/K. In the

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first section of this paper, we show how the property of having potentially good ℓ-reduction implies a "Condition C_ℓ" on the special fiber of $\mathscr{X}$. In the second section, we show how to compute explicitly the ℓ-part Φ_ℓ of Φ when the special fiber of $\mathscr{X}$ satisfies Condition C_ℓ (Theorem 2.1). In the third section, we use our explicit computations to prove the theorem stated below. Other applications to Fermat curves and to modular curves will appear in forthcoming papers.

Let Φ_ℓ denote the ℓ-part of the group of components Φ. Write

$$\Phi_\ell \cong \mathbb{Z}/\ell^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell^{a_{s(\ell)}}\mathbb{Z}, \quad \text{with} \quad a_i > 0 .$$

Let

$$\operatorname{ord}_\ell(|\Phi|) := \sum_{i=1}^{s(\ell)} a_i .$$

The following theorem sharpens Theorem 2.4 in [Lor2].

Theorem 3.6. *Let X/K be a smooth proper geometrically irreducible curve with a K-rational point. Let A/K be the jacobian of X/K, and assume that it has toric rank equal to zero. Let $\mathscr{L}$ denote the set of primes $\ell \neq p$ such that A/K has potentially good ℓ-reduction. Then*

$$\sum_{\ell \in \mathscr{L}} \left(\sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1) \right) + \sum_{\ell \notin \mathscr{L}} \operatorname{ord}_\ell(|\Phi|)(\ell - 1) \leqq 2u_K .$$

The bound presented in the above theorem imposes very severe restrictions on the possible order and group structure of the group of components Φ. For instance, it follows from this theorem that $|\Phi| \leqq 2^{2u_K}$, and that, if ℓ divides $|\Phi|$, then $\ell \leqq 2u_K + 1$.

The proof of Theorem 3.6 relies on the fact that a rational function attached in a natural way to the special fiber of $\mathscr{X}$ is in fact a polynomial of degree at most equal to $2u_K$ (Theorem 3.1). We show in [Lor3] that, when K is of equicharacteristic zero, this polynomial divides the characteristic polynomial of a monodromy transformation acting on the Tate module $T_\ell A$. The next theorem shows that our bound for $\Phi(A)$ is rather sharp. Recall that an abelian variety A/K is said to have purely additive reduction if $a_K = t_K = 0$.

Theorem 4.5. *Let g be any positive integer. Let $\Phi = \prod_{\ell \neq 2} \Phi_\ell$ be any finite abelian group of odd order. Write $\Phi_\ell = \sum_{i=1}^{s(\ell)} \mathbb{Z}/\ell^{a_i}\mathbb{Z}$. Assume that*

$$\sum_{\ell \neq 2} \left(\sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1) \right) \leqq 2g .$$

Then there exists a field K and an abelian variety A/K of dimension g with purely additive reduction, potentially good reduction, and such that

$$\Phi(A) \cong \Phi .$$

Let A_ℓ denote the ℓ-part of the torsion subgroup of $A(K)$. When $\ell \neq p$ and A/K has purely additive reduction, the reduction map $\pi: A(K) \to \Phi$ induces an isomorphism

$$A_\ell \cong \Phi_\ell = \sum_{i=1}^{s(\ell)} \mathbb{Z}/\ell^{a_i}\mathbb{Z}.$$

Therefore, we immediately obtain the following corollary, which is a sharpening, for jacobians, of a bound obtained by Lenstra and Oort in [L-O], 1.13.

Corollary. *Let X/K be a smooth proper geometrically irreducible curve with a K-rational point. Let A/K be the jacobian of X/K, and assume that it has purely additive reduction. Let $\mathscr{L}$ denote the set of primes $\ell \neq p$ such that A/K has potentially good ℓ-reduction. Then*

$$\sum_{\ell \in \mathscr{L}} \left(\sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1) \right) + \sum_{\ell \notin \mathscr{L}, \ell \neq p} \operatorname{ord}_\ell(|A_\ell|)(\ell - 1) \leqq 2g.$$

Remark. Let X/K be a smooth proper geometrically irreducible curve of genus g with a K-rational point. Let A/K be the jacobian of X/K and assume that it has purely additive reduction and potentially good reduction. Let $A(K)^{(p)}_{\text{tors}}$ denote the prime-to-p part of the torsion subgroup of $A(K)$. It follows immediately from the above corollary that:

- If $g = 1$, then $A(K)^{(p)}_{\text{tors}}$ is isomorphic to one of the following groups:

$$\{0\}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^2.$$

- If $g = 2$, then $A(K)^{(p)}_{\text{tors}}$ is isomorphic to a subgroup of one of the following groups:

$$\mathbb{Z}/5\mathbb{Z}, \quad (\mathbb{Z}/3\mathbb{Z})^2, \quad (\mathbb{Z}/2\mathbb{Z})^4, \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.$$

We believe that when $g = 2$, the group $A(K)^{(p)}_{\text{tors}}$ cannot contain a point of order 4. We hope to return to this point in a forthcoming paper.

1. Reduction to linear algebra

In this section, we show that, if the jacobian of a curve X/K has potentially good ℓ-reduction, then the graph G associated to a good model of X/K satisfies Condition C_ℓ, stated in 1.5. In the next section, we will show how to compute the group Φ_ℓ when the graph G satisfies Condition C_ℓ.

Let X/K be a smooth proper geometrically irreducible curve. Let $\mathscr{X}/\mathcal{O}_K$ be a regular model of X/K. Its special fiber $\mathscr{X}_k$ is an effective Cartier divisor and, as such, we write it as

$$\mathscr{X}_k = \sum_{i=1}^{n} r_i C_i,$$

where r_i is the multiplicity of the irreducible component C_i. Let

$$M := ((C_i \cdot C_j))$$

be the intersection matrix associated to $\mathscr{X}_k$ and set

$${}^tR := (r_1, \ldots, r_n) .$$

The vector R is in the kernel of the matrix M, or, in other words, $M \cdot R = 0$.

1.1. The integer $\gcd(r_1, \ldots, r_n)$ does not depend on the choice of a regular model of X/K. The fact that X has a K-rational point implies that

$$\gcd(r_1, \ldots, r_n) = 1 .$$

Raynaud [Ray] (see also [BLR], 9.6) has proven that, under this assumption, the group of components Φ of $\mathrm{Jac}(X)/K$ is isomorphic to

$$\mathrm{Ker}({}^tR)/\mathrm{Im}(M) ,$$

where $M: \mathbb{Z}^n \to \mathbb{Z}^n$ and ${}^tR: \mathbb{Z}^n \to \mathbb{Z}$ are the linear transformations associated to the matrices M and ${}^tR$. The group Φ can therefore be computed by performing a row and column reduction of the matrix M (see for instance [Lor 1], 1.4). In this paper, we shall always assume that a curve X/K satisfies the additional hypothesis that $\gcd(r_1, \ldots, r_n) = 1$.

1.2. We call a regular model $\mathscr{X}/\mathcal{O}_K$ of X/K a *good* model if the following additional properties hold:

- The components C_i are smooth of genus $g(C_i)$.
- If $i \neq j$, the intersection number $(C_i \cdot C_j)$ is equal to zero or one.

To the model $\mathscr{X}$ we associate a graph $G(\mathscr{X})$ defined as follows: the vertices of $G(\mathscr{X})$ are the curves C_is, and C_h is linked to C_j by $(C_h \cdot C_j)$ edges. When we will need to emphasize that a vertex C_i has multiplicity r_i, we will denote this vertex by (C_i, r_i). When no confusion may arise, we denote $G(\mathscr{X})$ simply by G. The triple (G, M, R), associated to a good model of X/K as above, is an example of what we called a *simple arithmetical graph* in [Lor 1]. Note that, when $\mathscr{X}$ is not regular, it is still possible to associate a graph $G(\mathscr{X})$ to its special fiber. The vertices of $G(\mathscr{X})$ are the irreducible components of $\mathscr{X}_k$ and two components C_i and C_j are linked in $G(\mathscr{X})$ if and only if they intersect in $\mathscr{X}$.

1.3. We let $\beta(G)$ denote the first Betti number of G. Raynaud (see [BLR], Theorem 4 on page 267 and Propositions 9 and 10 on pages 248–249, or [Lor 2], 1.3) has shown that, if $\mathscr{X}/\mathcal{O}_K$ is a good model of X/K, then:

$$\sum_{i=1}^{n} g(C_i) = a_K ,$$

and

$$\beta(G) = t_K .$$

1.4. The *degree* of a vertex C_i of G is the integer

$$d_i := \sum_{j \neq i} (C_i \cdot C_j) .$$

A *node* of G is a vertex of degree greater than two. A *terminal vertex* is a vertex of degree one. The topological space obtained from G by removing all its nodes is a union of connected components. A *chain* of G is the closure in G of such a connected component. If a chain contains a terminal vertex, we call it a *terminal chain*. It contains exactly one node. The other chains are called *connecting chains*. They contain one or two nodes. We define the *weight* of a chain $\mathscr{D}$ to be the integer

$$w(\mathscr{D}) := \gcd(r_j,\ C_j \text{ a vertex on } \mathscr{D}) .$$

Note that, if $\mathscr{D}$ is a terminal chain and (C_j, r_j) is the terminal vertex on $\mathscr{D}$, then $w(\mathscr{D}) = r_j$ (see for instance [Lor1], 4.2). Note also that, if (C, r) is a node on a chain $\mathscr{D}$ and (C_i, r_i) is a vertex on $\mathscr{D}$ adjacent to C, then $w(\mathscr{D}) = \gcd(r, r_i)$. For a node (C, r) of G, we let

$$w_\ell(C) := \min\{\operatorname{ord}_\ell(w(\mathscr{D})) \mid \mathscr{D} \text{ a chain containing } C\} .$$

Note that the definitions of $w(\mathscr{D})$ and $w_\ell(C)$ make sense even when $\mathscr{X}$ is only a normal scheme.

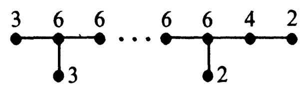
1.5. Condition C_ℓ. Let (G, M, R) be a simple arithmetical tree. Pick a node C_i of G with $w_\ell(C_i) = 0$. Such a node always exists because $\gcd(r_1, \ldots, r_n) = 1$. We say that Condition C_ℓ holds for G if, for any node C_j of G, the following equality holds:

$$w_\ell(C_j) = \operatorname{ord}_\ell(w(\mathscr{D}_{j,i})) ,$$

where $\mathscr{D}_{j,i}$ is the unique chain of G containing the node C_j and contained in the unique shortest path on G from C_j to C_i.

If Condition C_ℓ holds with respect to a choice C_i of node with $w_\ell(C_i) = 0$, then it holds with respect to any such choice. Indeed, if $C_h \neq C_i$ is a node with $w_\ell(C_h) = 0$, then Condition C_ℓ with respect to C_i implies that $w_\ell(C_s) = 0$ for any node C_s on the unique shortest path of G from C_h to C_i.

Example 1.6. The following graph satisfies both Conditions C_2 and C_3.



Proposition 1.7. *Let X/K denote a smooth proper geometrically irreducible curve. Let A/K denote its jacobian. If A/K has potentially good ℓ-reduction for some prime $\ell \neq p$, then the graph G associated to a good model $\mathscr{X}/\mathcal{O}_K$ of X/K is a tree satisfying Condition C_ℓ.*

Proof. The fact that G is a tree follows from a theorem of Raynaud [Ray], recalled in 1.3, which implies that a jacobian variety has toric rank equal to zero if and only if the graph associated to a good model is a tree. It also follows from Raynaud's results that, if M/K is any extension with $[M:K] = \ell^a$ for some $a \geqq 1$ and if A/K has potentially good ℓ-reduction, then the graph G associated to a good model of X_M/M must be a tree. We are going to show that, if Condition C_ℓ is not satisfied, then there exists such an extension M/K and a good model of X_M/M whose graph is not a tree. We first need to recall a description of a good model of X_M/M.

1.8. Let $\mathscr{X}/\mathcal{O}_K$ be a good model of X/K. Fix a prime $q \neq p$. Let M_q/K denote the unique extension of K of degree q. Let $\mathscr{Y}/\mathcal{O}_{M_q}$ denote the normalization of the scheme $\mathscr{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_{M_q})$. Let $\pi : \mathscr{Y} \to \mathscr{X}$ denote the composition of the natural maps

$$\mathscr{Y} \to \mathscr{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_{M_q}) \to \mathscr{X}.$$

Let

$$p : \mathscr{Z} \to \mathscr{Y}$$

denote the minimal desingularization of $\mathscr{Y}$. To recall the descriptions of the maps p and π, we need the following definition. Let $\mathscr{X}/\mathcal{O}_K$ be any regular model of X/K. Let $C_1, \ldots, C_m$ be irreducible components of the special fiber $\mathscr{X}_k$. The divisor

$$C = \sum_{i=1}^{m} C_i$$

is said to be a *Hirzebruch-Young string* if:

- $g(C_i) = 0 \quad \forall i = 1, \ldots, m.$
- $(C_i \cdot C_i) \leqq -2 \quad \forall i = 1, \ldots, m.$
- $(C_i \cdot C_j) = 1 \quad \text{if} \quad |i-j| = 1.$
- $(C_i \cdot C_j) = 0 \quad \text{if} \quad |i-j| > 1.$

The following facts are well known; we state them without proofs (see for instance [BPV], Theorem 5.2, when $\mathscr{X}/\mathbb{C}$ is a surface).

Facts 1.9. Let $\mathscr{X}/\mathcal{O}_K$ be a good model of X/K and q be a prime, $q \neq p$.

- The map $\pi : \mathscr{Y} \to \mathscr{X}$ is ramified over the divisor

$$R := \sum_{\gcd(q, r_i) = 1} C_i.$$

In particular, $R \subset \mathscr{X}$ has normal crossings. A point $P \in \mathscr{Y}$ is singular if and only if $\pi(P)$ is a singular point of R.

• If $P \in \mathscr{Y}$ is a singular point, then the divisor $p^{-1}(P) := \sum_{i=1}^{m(P)} E_i$ is a Hirzebruch-Young string. Let $P \in D_i \cap D_j$, where D_i and D_j are irreducible components of $\mathscr{Y}_k$. Write $\tilde{D}_i$ for the strict transform of D_i in $\mathscr{Z}$. Then:

$$(p^{-1}(P) \cdot \tilde{D}_i) = (E_1 \cdot \tilde{D}_i) = 1 = (E_{m(P)} \cdot \tilde{D}_j) = (p^{-1}(P) \cdot \tilde{D}_j).$$

Moreover,

$$(p^{-1}(P) \cdot D) = 0 \quad \text{if } D \neq \tilde{D}_i, \tilde{D}_j, \text{ is an irreducible component of } \mathscr{Z}_k.$$

1.10. Using Facts 1.9, one easily proves that, if $\mathscr{X}/\mathcal{O}_K$ is a good model for X/K, then $\mathscr{Z}/\mathcal{O}_{M_q}$ is a regular model for X_{M_q}/M_q whose special fiber is a divisor with smooth components and normal crossings. After blowing up some singular points in the special fiber of $\mathscr{Z}/\mathcal{O}_{M_q}$ if necessary, we obtain a good model $\mathscr{Z}_{\mathrm{good}}/\mathcal{O}_{M_q}$ of X_{M_q}/M_q. It is clear that $G(\mathscr{Z}_{\mathrm{good}})$ is a tree if and only if $G(\mathscr{Z})$ is a tree. It follows immediately from 1.9 that $G(\mathscr{Z})$ is a tree if and only if $G(\mathscr{Y})$ is a tree.

Facts 1.11. Let $\mathscr{X}/\mathcal{O}_K$ be a good model of X/K and let $q \neq p$ be a prime. The map π can be described as follows.

• If $q \nmid r_i$, then $\pi^{-1}(C_i) =: D_i$ is irreducible and the restricted map

$$\pi|_{D_i} : D_i \to C_i$$

is an isomorphism. The curve D_i has multiplicity r_i in $\mathscr{Y}_k$.

• If $q | r_i$ and $C_i \cap R_q \neq \emptyset$, then $\pi^{-1}(C_i) =: D_i$ is irreducible and the restricted map

$$\pi|_{D_i} : D_i \to C_i$$

is a morphism of degree q ramified over $|C_i \cap R_q|$ points of C_i. The curve D_i has multiplicity r_i/q in $\mathscr{Y}_k$. Its genus is computed using the Riemann-Hurwitz formula.

• If $q | r_i$ and $C_i \cap R_q = \emptyset$, then

$$\pi : \pi^{-1}(C_i) \to C_i$$

is an etale map and each irreducible component of $\pi^{-1}(C_i)$ has multiplicity r_i/q in $\mathscr{Y}_k$. If $\pi^{-1}(C_i)$ is not irreducible, then it is equal to the disjoint union $D_1 \sqcup \cdots \sqcup D_q$ of q irreducible curves, and each restricted map

$$\pi|_{D_j} : D_j \to C_i$$

is an isomorphism.

1.12. We are now ready to begin the proof of Proposition 1.7. Let G denote the graph associated to a good model $\mathscr{X}/\mathcal{O}_K$ of X/K. Suppose that Condition C_ℓ does not hold for G.

We claim that there exist two nodes C_j and C_h on G such that, if we let $\mathscr{P}$ denote the unique shortest path between C_j and C_h on G, and if we let

$$w(\mathscr{P}) := \gcd(\text{multiplicities of all vertices on } \mathscr{P}),$$

then

$$\mathrm{ord}_\ell(w(\mathscr{P})) = w_\ell(C_s), \text{ for all nodes } C_s \in \mathscr{P} \text{ with } s \neq j, h,$$

and

$$\max(w_\ell(C_j), w_\ell(C_h)) < \mathrm{ord}_\ell(w(\mathscr{P})).$$

We call such a path $\mathscr{P}$ a *bad* path.

To show our claim, we proceed as follows: without loss of generality, we may assume that the node

$$C_n \text{ is such that } w_\ell(C_n) = 0,$$

and that the node

$$C_1 \text{ is such that } w_\ell(C_1) < \mathrm{ord}_\ell(w(\mathscr{D}_{1,n})).$$

Upon renumbering the nodes (if necessary), we may assume that $\{C_1, C_2, \ldots, C_s, C_n\}$ is the ordered set of nodes on the unique shortest path on G from C_1 to C_n. Let $j \in \{1, \ldots, s\}$ denote the largest integer such that

$$w_\ell(C_j) < \mathrm{ord}_\ell(w(\mathscr{D}_{j,n}))$$

Let $h \in \{j+1, \ldots, n\}$ denote the smallest integer such that

$$w_\ell(C_h) < \mathrm{ord}_\ell(w(\mathscr{D}_{h-1,n})).$$

We then set $\mathscr{P} = \bigcup_{i=j}^{h-1} \mathscr{D}_{i,n}$.

1.13. If $\mathscr{P}$ is a bad path with end nodes C_j and C_h, we let

$$a(\mathscr{P}) := \max(w_\ell(C_j), w_\ell(C_h)).$$

Let $\mathscr{P}$ be a bad path in $G(\mathscr{X})$ with $a(\mathscr{P}) > 0$. Let M/K be an extension of degree $\ell^{a(\mathscr{P})}$. Let $\mathscr{V}/\mathcal{O}_M$ be the normalization of

$$\mathscr{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_M)$$

and let π_M denote the composition of the natural maps

$$\mathscr{V} \to \mathscr{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_M) \to \mathscr{X}.$$

Let

$$\mathscr{W} \to \mathscr{V}$$

denote the minimal desingularization of $\mathscr{V}$. It follows from 1.10 that $\mathscr{W}$ is a regular model of X_M/M whose special fiber is a divisor with smooth components and normal crossings. After blowing up some singular points in its special fiber if necessary, we obtain a good model $\mathscr{W}_{\text{good}}/\mathcal{O}_M$ of X_M/M. Consider the sequence of maps:

$$\mathscr{W}_{\text{good}} \to \mathscr{W} \to \mathscr{V} \xrightarrow{\pi_M} \mathscr{X}.$$

We claim that the graph $G(\mathscr{W}_{\text{good}})$ contains a bad path $\mathscr{P}''$ with $a(\mathscr{P}'') = 0$. Let D_j and D_h denote preimages under π_M of C_j and C_h. By 1.11, we have

$$w_\ell(D_j) = w_\ell(D_h) = 0.$$

We can certainly choose D_j and D_h in such a way that there is a path $\mathscr{P}'$ in $G(\mathscr{V})$ between D_j and D_h mapping bijectively onto $\mathscr{P}$ in $G(\mathscr{X})$. Note that, since every vertex on $\mathscr{P}$ other than C_j and C_h has multiplicity divisible by $\ell^{a(\mathscr{P})}$, it follows from 1.9 that each curve in $\mathscr{X}$ corresponding to a vertex of $\mathscr{P}$ other than C_j and C_h does not intersect the branch locus of π_M. Therefore the preimages in $\mathscr{V}$ of such curves are in the regular locus of $\mathscr{V}$. Hence the path $\mathscr{P}'$ in $G(\mathscr{V})$ can be considered as a path in $G(\mathscr{W})$. It follows from 1.11 that the multiplicity of any vertex on the path $\mathscr{P}'$ other than D_j and D_h is divisible by $\ell^{\operatorname{ord}_\ell(w(\mathscr{P}))-a(\mathscr{P})}$. Therefore, the path $\mathscr{P}'$ in $G(\mathscr{W})$, from D_j to D_h, is a bad path with $a(\mathscr{P}') = 0$. Let now $\mathscr{P}''$ be the preimage of $\mathscr{P}'$ in $G(\mathscr{W}_{\text{good}})$. Our claim is proved.

Without loss of generality, we may assume now that $G(\mathscr{X})$ contains a bad path $\mathscr{P}$ with $a(\mathscr{P}) = 0$. To conclude the proof of our Proposition, we are going to show that, after an extension M_ℓ/K of degree ℓ, there exists a good model of X_{M_ℓ}/M_ℓ whose associated graph is not a tree. We keep the notations introduced in 1.8:

$$\mathscr{Z}_{\text{good}} \to \mathscr{Z} \to \mathscr{Y} \xrightarrow{\pi} \mathscr{X}.$$

We claim that the graph associated to the good model $\mathscr{Z}_{\text{good}}$ of X_{M_ℓ}/M_ℓ is not a tree. Since the end nodes C_j and C_h of $\mathscr{P}$ are such that

$$w_\ell(C_j) = w_\ell(C_h) = 0 = a(\mathscr{P}),$$

both curves C_j and C_h intersect the branch locus of π. Let D_j and D_h denote the preimages of C_j and C_h under π. Since

$$\operatorname{ord}_\ell(w(\mathscr{P})) \geqq 1,$$

D_j and D_h are linked in $G(\mathscr{Y})$ by at least ℓ distinct paths, the preimages under π of $\mathscr{P}$. Indeed, if the preimage of each curve of $\mathscr{P}$ is the disjoint union of ℓ components of $\mathscr{Y}$, then D_j and D_h are linked in $G(\mathscr{Y})$ by exactly ℓ distinct paths. On the other hand, if the preimage of some curve of $\mathscr{P}$ is not the disjoint union of ℓ components of $\mathscr{Y}$, then D_j and D_h are linked in $G(\mathscr{Y})$ by more than ℓ paths. This shows that $G(\mathscr{Y})$ is not a tree. Since the curves corresponding to the vertices of $\mathscr{P}\backslash\{C_j, C_h\}$ do not intersect the branch locus of π, their preimages under π are in the regular locus of $\mathscr{Y}$. Hence the preimages in $G(\mathscr{Y})$ of the path $\mathscr{P}$ can be considered as paths on $G(\mathscr{X})$. This implies that $G(\mathscr{X})$ is not a tree, and hence $G(\mathscr{X}_{\text{good}})$ is not a tree. □

2. Explicit computation of Φ

We prove in this section the following theorem.

Theorem 2.1. *Let (G, M, R) be a simple arithmetical tree satisfying Condition C_ℓ. For each node (C_i, r_i) of G of degree d_i, let*

$$w_{i,1}, \ldots, w_{i,d_i}$$

denote the weights of the chains of G containing C_i, ordered in such a way that

$$\operatorname{ord}_\ell(w_{i,1}) \geqq \ldots \geqq \operatorname{ord}_\ell(w_{i,d_i}).$$

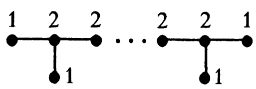
Let $\Phi_\ell(G)$ denote the ℓ-part of $\Phi(G)$. Then

$$\Phi_\ell(G) \cong \prod_{\text{all nodes } C_i} \left(\prod_{j=1}^{d_i - 2} \mathbb{Z}/\ell^{\operatorname{ord}_\ell(r_i/w_{i,j})}\mathbb{Z} \right).$$

Corollary 2.2. *Let X/K be a curve and let A/K denote its jacobian. If A/K has potentially good ℓ-reduction at some prime $\ell \neq p$, then the ℓ-part of its group of components can be computed, as in Theorem 2.1, using the arithmetical graph (G, M, R) associated to a good model of X/K.*

Proof. The corollary follows immediately from 1.1 and 1.7. □

Remark 2.3. Obviously, if X/K has a good model whose graph satisfies Condition C_p, then the group Φ_p can be computed using Theorem 2.1. Note, however, that even when A/K has potentially good reduction, it is not true in general that Condition C_p holds. For instance, there are elliptic curves over $\mathbb{Q}_2^{\text{unr}}$, the maximal unramified extension of $\mathbb{Q}_2$, having potentially good reduction and whose minimal model have graphs of the form:



Before proving our theorem, let us first recall the following facts.

Definition/Construction 2.4. We call a connected subgraph T of G a *terminal string* of G if it has the following properties:

• There exists a vertex C of G such that $T \backslash \{C\}$ is equal to a connected component of $G \backslash \{C\}$.

- T contains a terminal vertex of G.

- T does not contain any node of G.

Given a pair of integers (r, r_1), we can construct a terminal string

$$(C, r), (C_1, r_1), \ldots, (C_m, r_m)$$

of a simple arithmetical graph, with multiplicities r_i and "self-intersection" $c_i := |(C_i \cdot C_i)|$ defined using Euclid's algorithm:

$$\begin{array}{lll} \text{(i)}\ r \geqq r_1: & r = c_1 r_1 - r_2 & \text{with } r_2 < r_1, \\ & r_1 = c_2 r_2 - r_3 & \text{with } r_3 < r_2, \\ & \vdots & \vdots \\ & r_{m-1} = c_m r_m & \text{with } r_m = \gcd(r, r_1). \end{array}$$

$$\begin{array}{lll} \text{(ii)}\ r < r_1: & r = r_1 - (r_1 - r) & \text{with } (r_1 - r) < r_1, \\ & \text{and case (i)} & \text{with } r_1, (r_1 - r). \end{array}$$

Let (C_i, r_i) and (C_j, r_j) be two vertices on a graph G linked by an edge e. By "break the graph G at e and complete," we will mean the following construction:

- Remove e from G.

- Construct a terminal string T_i using the pair (r_i, r_j). Construct a terminal string T_j using the pair (r_j, r_i).

- Attach T_i at C_i and T_j at C_j.

One constructs in this way one or two graphs depending on whether $G \setminus \{e\}$ is connected. When $G \setminus \{e\}$ is connected, the new graph obtained by breaking G at e and completing is an arithmetical graph. When $G \setminus \{e\}$ is not connected, let G_1 and G_2 denote the two new graphs obtained by the above construction. Let s_i, $i = 1, 2$, denote the greatest common divisor of the multiplicities of the vertices of G_i. It may occur that s_i is greater than one. The graph G_i' obtained from G_i by dividing all its multiplicities by s_i is an arithmetical graph.

The following lemma is needed in the proof of Theorem 2.1. Its proof is easy and is omitted.

Lemma 2.5. *Let G be a simple arithmetical graph. Let (C, r), $(C_1, r_1), \ldots, (C_m, r_m)$ be consecutive vertices on a terminal string of G, with C_m being the terminal vertex. Let $c_i := |(C_i \cdot C_i)|$. The intersection matrix M associated to G, written in the form:*

$$\begin{pmatrix} * & * & \vdots & \vdots & & & & & \\ * & * & \vdots & 0 & & & & & \\ \cdots & \cdots & * & 1 & & & & & \\ \cdots & 0 & 1 & -c_1 & 1 & & & & \\ & & & 1 & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & 1 & & \\ & & & & & 1 & -c_{m-1} & 1 \\ & & & & & & 1 & -c_m \end{pmatrix} \begin{pmatrix} \vdots \\ * \\ r \\ r_1 \\ \vdots \\ \vdots \\ r_{m-1} \\ r_m \end{pmatrix} = 0$$

can be reduced over $\mathbb{Z}$ *to*:

$$\begin{pmatrix} * & * & \vdots & \vdots & & & & \\ * & * & \vdots & 0 & & & & \\ \cdots & \cdots & * & r_1/r_m & & & & \\ \cdots & 0 & 1 & -r/r_m & 0 & & & \\ & & & 0 & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ * \\ r \\ r_m \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = 0 . \quad \square$$

Lemma 2.6. *Let G be a simple arithmetical graph and let ℓ be a prime number.*

a) *At least one of the chains of G has weight prime to ℓ and, therefore, there exists a node (C, r) of G such that $w_\ell(C) = 0$.*

b) *Let (C, r) be a node of G. Then there exist at least two distinct chains $\mathscr{D}$ and $\mathscr{D}'$ of G containing C such that*

$$\operatorname{ord}_\ell(w(\mathscr{D})) = \operatorname{ord}_\ell(w(\mathscr{D}')) = w_\ell(C) .$$

c) *If G is a tree, then there exist at least two terminal chains of G whose weight is prime to ℓ.*

Proof. a) follows immediately from the fact that $\gcd(r_1, \ldots, r_n) = 1$.

b) Let (C, r) be a node of degree d. Let $(C_1, r_1), \ldots, (C_d, r_d)$ denote the vertices of G adjacent to (C, r). The weight of the chain containing C and C_i is equal to $\gcd(r, r_i)$. In case $\operatorname{ord}_\ell(r) = w_\ell(C)$, the equality $\operatorname{ord}_\ell(w(\mathscr{D})) = \operatorname{ord}_\ell(r)$ holds for all chains $\mathscr{D}$ containing C. When $\operatorname{ord}_\ell(r) > w_\ell(C)$, one of the integers $r_i \ell^{-w_\ell(C)}$ is prime to ℓ. The equality

$$|(C \cdot C)| \cdot r = r_1 + \cdots + r_d$$

shows that a second one of the integers $r_i \ell^{-w_\ell(C)}$ is also prime to ℓ. Hence, for two distinct values of i, we have

$$\operatorname{ord}_\ell(\gcd(r, r_i)) = w_\ell(C) ,$$

which proves part b).

To prove c), consider the subgraph G' of G defined as the union of all the connecting chains of G whose weight is prime to ℓ. If G' is not empty, it is a tree and therefore it has at least two terminal vertices C and C''. By construction of G', the vertex C is a node of G and only one connecting chain of G containing C has weight prime to ℓ. Therefore, by b), there must exist a terminal chain of G containing C whose weight is also prime to ℓ. The same argument also applies to C' and, therefore, c) is proven in the case $G' \neq \emptyset$. If G' is empty, then by a) there exists one node with $w_\ell(C) = 0$. Since all connecting chains containing C have weight divisible by ℓ, it follows from b) that at least two terminal chains of G containing C have weight prime to ℓ. □

Proof of Theorem 2.1. The group Φ may be computed using a row and column reduction of the intersection matrix M (see 1.1). We prove our theorem by induction on the number of nodes of G. If G has no nodes, then $\Phi = \{0\}$.

2.7. Let (C, r) denote a node of G of degree d. Let $(C_1, r_1), \ldots, (C_d, r_d)$ denote the vertices adjacent to C on G. Let $-c = (C \cdot C)$. Since $M \cdot R = 0$, we find that

$$cr = r_1 + \cdots + r_d .$$

We may assume that the vertices C_i are numbered in such a way that

$$\mathrm{ord}_\ell(\gcd(r, r_1)) \geqq \cdots \geqq \mathrm{ord}_\ell(\gcd(r, r_{d-1})) = \mathrm{ord}_\ell(\gcd(r, r_d)) .$$

The equality on the right follows from Lemma 2.6. b). To ease our notations, we denote the weight $\gcd(r, r_i)$ (see 1.4) of the chain containing C and C_i by s_i.

Assume now that G is a simple arithmetical tree and that (C, r) is the unique node on G. Condition C_ℓ is always satisfied in this case. Applying Lemma 2.5 to all terminal chains of G, we can reduce the intersection matrix to the following one; we have omitted all of the "diagonal 1 s" in the reduced matrix since they do not contribute to Φ_ℓ. We shall do so in the future without mentioning it.

$$\begin{pmatrix} -c & r_1/s_1 & r_2/s_2 & \cdots & r_d/s_d \\ 1 & -r/s_1 & 0 & \cdots & 0 \\ 1 & 0 & -r/s_2 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -r/s_d \end{pmatrix} \begin{pmatrix} r \\ s_1 \\ s_2 \\ \vdots \\ s_d \end{pmatrix} = 0 .$$

This matrix further reduces over $\mathbb{Z}$ to:

$$\begin{pmatrix} r_1/s_1 & \cdots & r_{d-1}/s_{d-1} & r_d/s_d - cr/s_d \\ -r/s_1 & 0 & \cdots & r_d/s_d \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & -r/s_{d-1} & r_d/s_d \end{pmatrix} .$$

Note that we have omitted a null row and a null column to simplify our notations. We continue to reduce this matrix over $\mathbb{Z}_\ell$, using the fact that

$$\operatorname{ord}_\ell(r/s_i) \leqq \operatorname{ord}_\ell(r/s_d) \quad \forall i = 1, \ldots, d.$$

We obtain a new reduced matrix:

$$S := \begin{pmatrix} r_1/s_1 & \cdots & r_{d-1}/s_{d-1} & 0 \\ -r/s_1 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & -r/s_{d-1} & 0 \end{pmatrix}.$$

Assume now that $\operatorname{ord}_\ell(r_{d-1}/s_{d-1}) = 0$. The ℓ-adic number $u := r_{d-1}/s_{d-1}$ is a unit. We can then further reduce S to:

$$\begin{pmatrix} 0 & \cdots & 0 & u \\ -r/s_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & -r/s_{d-2} & 0 \\ u^{-1} r_1 r/s_1 s_{d-1} & \cdots & u^{-1} r_{d-2} r/s_{d-2} s_{d-1} & -r/s_{d-1} \end{pmatrix}.$$

Note that we have omitted a null column on the right. Since $\operatorname{ord}_\ell(r/s_i) \leqq \operatorname{ord}_\ell(r/s_{d-1})$ for all $i = 1, \ldots, d-2$, we obtain a diagonal matrix

$$\operatorname{diag}(r/s_1, \ldots, r/s_{d-2}, u).$$

This completes the proof of our theorem in this case.

Assume now that $\operatorname{ord}_\ell(r_{d-1}/s_{d-1}) > 0$. Then

$$0 = \operatorname{ord}_\ell(r/s_{d-1}) \geqq \operatorname{ord}_\ell(r/s_i) \geqq 0,$$

the first equality holding because $s_{d-1} = \gcd(r, r_{d-1})$. Hence, all the ℓ-adic numbers r/s_i are units and the matrix S reduces to

$$\operatorname{diag}(r/s_1, \ldots, r/s_{d-2}, r/s_{d-1}),$$

which shows that $\Phi_\ell = \{0\}$. Our theorem is therefore proven when G has only one node.

Suppose now that G is a simple arithmetical tree with k nodes, $k \geqq 2$. We claim that we can always pick a node (C, r) of G such that:

- (C, r) is a node of G contained in only one connecting chain $\mathscr{D}$ of G.
- There exists a node C_0 of G, with $C_0 \neq C$ and such that $w_\ell(C_0) = 0$.

Indeed, let $C_1, \ldots, C_m$ be the nodes of G satisfying the first condition stated above. Since G is a tree, $m \geqq 2$. If $w_\ell(C_i) \neq 0$ for all $i = 1, \ldots, m$, then the existence of C_0 follows from Lemma 2.6. a). If $w_\ell(C_i) = 0$ for some i, then let $C = C_j$ for some $j \neq i$ and let $C_0 = C_i$.

Fix now a node (C, r) and a node C_0 as above. We keep the notations introduced in 2.7, and in particular we assume that

$$\mathrm{ord}_\ell(s_1) \geqq \cdots \geqq \mathrm{ord}_\ell(s_{d-1}) = \mathrm{ord}_\ell(s_d)\,.$$

Since Condition C_ℓ is satisfied,

$$\mathrm{ord}_\ell(w(\mathscr{D})) = \mathrm{ord}_\ell(s_{d-1}) = \mathrm{ord}_\ell(s_d)\,.$$

We may assume without loss of generality that $C_d \in \mathscr{D}$. Break the graph G at the edge linking C to C_d and complete, as in 2.4. Let G' denote the new graph containing C_d. This graph may not be an arithmetical graph, i.e., the greatest common divisor s of the multiplicities of the vertices of G' may not equal 1. We therefore let G'' denote the graph obtained from G' by dividing all its multiplicities by s. Note that the existence of $C_0 \in G'$ implies that ℓ is prime to s.

The graph G'' does satisfy Condition C_ℓ and hence, by induction, our theorem is true for G''. Therefore, in order to complete the proof of our theorem, we only need to show that

$$\Phi_\ell(G) \cong \prod_{i=1}^{d-2} (\mathbb{Z}/\ell^{\mathrm{ord}_\ell(r/s_i)}\mathbb{Z}) \times \Phi_\ell(G'')\,.$$

Applying Lemma 2.5 to all terminal chains of G containing C, we reduce the intersection matrix M of G to the matrix:

$$\begin{pmatrix} * & * & \vdots & \vdots & & & \\ * & * & \vdots & 0 & & & \\ \cdots & \cdots & * & 1 & & & \\ \cdots & 0 & 1 & -c & r_1/s_1 & \cdots & r_{d-1}/s_{d-1} \\ & & & 1 & -r/s_1 & & 0 \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & 0 & & -r/s_{d-1} \end{pmatrix} \begin{pmatrix} \vdots \\ * \\ r_d \\ r \\ s_1 \\ \vdots \\ s_{d-1} \end{pmatrix} = 0\,.$$

Note that all of the "diagonal 1 s" have been omitted in the above matrix. Using operations on the rows and colums similar to the ones performed in the case where G had only one node, we can further reduce this matrix to the matrix:

$$\begin{pmatrix} * & * & \vdots & \vdots & & & \\ * & * & \vdots & 0 & & & \\ \cdots & \cdots & * & r/s_{d-1} & & & \\ \cdots & 0 & 1 & -r_d/s_{d-1} & 0 & \cdots & 0 \\ & & & 0 & -r/s_1 & \cdots & 0 \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & 0 & \cdots & -r/s_{d-2} \end{pmatrix}.$$

Hence we have shown that

$$\Phi_\ell(G) \cong \prod_{i=1}^{d-2} (\mathbb{Z}/\ell^{\mathrm{ord}_\ell(r/s_i)}\mathbb{Z}) \times \Phi',$$

where Φ' is computed using the matrix

$$\begin{pmatrix} * & * & \vdots & \vdots \\ * & * & \vdots & 0 \\ \cdots & \cdots & * & r/s_{d-1} \\ \cdots & 0 & 1 & -r_d/s_{d-1} \end{pmatrix}.$$

We use now in an essential way the fact that Condition C_ℓ is satisfied:

$$s_d/s_{d-1} \quad \text{is an} \quad \ell\text{-adic unit}.$$

Therefore, we can divide the last column of the above matrix by s_d/s_{d-1} without changing its elementary divisors. Hence, Φ' can also be computed using the matrix

$$N := \begin{pmatrix} * & * & \vdots & \vdots \\ * & * & \vdots & 0 \\ \cdots & \cdots & * & r/s_d \\ \cdots & 0 & 1 & -r_d/s_d \end{pmatrix}.$$

By construction, the graph G'' has a terminal string T of the form

$$(C_d, r_d/s), (C, r/s), \ldots, (C_\alpha, r_\alpha := s_d/s),$$

where C_α is the terminal vertex on the chain. Applying Lemma 2.5 to the terminal chain T, we see that $\Phi(G'')$ can be computed using the matrix N, which shows that $\Phi_\ell(G'') = \Phi'$. This concludes the proof of Theorem 2.1. □

2.8. We prove now a rather technical proposition that will be used later in the course of this paper. Fix a prime ℓ. We keep the notations introduced in Theorem 2.1. In particular, G is a simple arithmetical graph. Let

$$W_\ell(G) := \{\mathrm{ord}_\ell(w_{i,j}) \mid C_i \text{ a node of } G, \quad 1 \leqq j \leqq d_i - 2\},$$

where the weights $w_{i,j}$ have been numbered such that

$$\operatorname{ord}_\ell(w_{i,1}) \geqq \cdots \geqq \operatorname{ord}_\ell(w_{i,d_i-1}) = \operatorname{ord}_\ell(w_{i,d_i}).$$

Define

$$e_G : W_\ell(G) \to \mathbb{N}$$

as follows:

$$e_G(x) := \#\{(i,j),\ C_i \text{ a node},\ 1 \leqq j \leqq d_i - 2, \text{ and } \operatorname{ord}_\ell(w_{i,j}) = x\}.$$

It is clear that

$$\sum_{x \in W_\ell(G)} e_G(x) = \sum_{\text{nodes } C_i} (d_i - 2).$$

Let

$$\delta := \sum_{\text{node } C_i} (d_i - 2).$$

We assume from now on that G is a tree. It is easy to check that in this case

$$\delta = (\text{number of terminal vertices of } G) - 2.$$

By Lemma 2.6. c), we can choose two terminal vertices (C', r') and (C'', r'') such that ℓ does not divide $r'r''$. Let

$$t_1, \ldots, t_\delta$$

denote the multiplicities of all terminal vertices different from C' and C''. Set

$$T_\ell(G) := \{\operatorname{ord}_\ell(t_i), \quad i = 1, \ldots, \delta\},$$

Define

$$\varepsilon_G : T_\ell(G) \to \mathbb{N}$$

as follows:

$$\varepsilon_G(x) := \#\{i,\ \operatorname{ord}_\ell(t_i) = x\}.$$

Note that the set $T_\ell(G)$ and the function ε_G are independent of the choice of C' and C''.

Proposition 2.9. *Let G be a simple arithmetical tree. Assume that Condition C_ℓ holds for G. Then the sets $W_\ell(G)$ and $T_\ell(G)$ are equal. Moreover, the functions e_G and ε_G are also equal.*

Proof. If G has only one node, our proposition follows immediately from the definitions. Therefore, we may assume that G has more than one node. Let $\mathscr{C}$ denote the set of node of G that are contained in only one connecting chain of G. Since G is a tree, the set $\mathscr{C}$ contains at least two elements.

Let us assume first that $w_\ell(C) = 0$ for all nodes C in $\mathscr{C}$. Since Condition C_ℓ holds for G, we find that $\operatorname{ord}_\ell(\mathscr{D}) = 0$ for all connecting chains $\mathscr{D}$ of G. Therefore, it follows that

$$W_\ell(G) \backslash \{0\} = T_\ell(G) \backslash \{0\},$$

and that

$$e_{G\,|\,W_\ell(G)\setminus\{0\}} = \varepsilon_{G\,|\,T_\ell(G)\setminus\{0\}}\,.$$

Since

$$\sum_{x\in W_\ell(G)} e_G(x) = \sum_{x\in T_\ell(G)} \varepsilon_G(x)\,,$$

it follows that $e_G(0) = \varepsilon_G(0)$.

We assume now that there exists a node (C, r) in $\mathscr{C}$, of degree d, with $w_\ell(C) > 0$. We proceed by induction on the number k of nodes of G. As in the proof of the previous theorem, we let

$$(C_1, r_1), \ldots, (C_d, r_d)$$

denote the vertices of G adjacent to C. Let

$$s_i := \gcd(r, r_i)$$

denote the weight of the chain containing C and C_i. Assume that

$$\mathrm{ord}_\ell(s_1) \geqq \cdots \geqq \mathrm{ord}_\ell(s_{d-1}) = \mathrm{ord}_\ell(s_d) > 0\,.$$

Since G satisfies Condition C_ℓ, we may assume that C_d is on the unique connecting chain containing C. Break G between C and C_d and complete, as in 2.4. Denote by G' the new graph containing C_d. This new graph has $k-1$ nodes. Note that since $w_\ell(C) > 0$, the graph G' contains at least two terminal chains with weight prime to ℓ. Therefore, the greatest common divisor s of the multiplicities of G' is prime to ℓ. Let G'' denote the arithmetical tree obtained from G' by dividing all its multiplicities by s. By induction, we know that

$$W_\ell(G'') \text{ is equal to } T_\ell(G''), \text{ and that } e''_G \text{ is equal to } \varepsilon''_G\,.$$

It is easy to check that

$$W_\ell(G) = W_\ell(G'') \cup \{\mathrm{ord}_\ell(s_1), \ldots, \mathrm{ord}_\ell(s_{d-2})\}\,.$$

One also checks that, since $\mathrm{ord}_\ell(s_{d-1}) = \mathrm{ord}_\ell(s_d)$,

$$T_\ell(G) = \big(T_\ell(G'')\setminus\{\mathrm{ord}_\ell(s_d)\}\big) \cup \{\mathrm{ord}_\ell(s_1), \ldots, \mathrm{ord}_\ell(s_{d-1})\}\,.$$

This shows that $W_\ell(G) = T_\ell(G)$. We leave it to the reader to check that $e_G = \varepsilon_G$. □

3. Bounds for Φ

Theorem 3.1. *Let G be a simple arithmetical graph. Then the rational function*

$$f_G(x) := \prod_{i=1}^{n} [(x^{r_i} - 1)/(x-1)]^{(d_i-2)}$$

is a polynomial.

Proof. When G has no nodes, $f_G(x) = 1$. We first prove this theorem for simple trees with exactly one node. Then by induction on the number of nodes, we show that our statement is true for all trees. Finally, by induction on the first Betti number $\beta(G)$ of G, we prove our theorem in general.

Assume that G has exactly one node, of multiplicity r and degree d; denote by $s_1, \ldots, s_d$ the terminal multiplicities of G (or, equivalently, the weights of the chains of G). To prove our theorem, we need to show that the polynomial

$$\prod_{i=1}^{d-2} \frac{x^r - 1}{x^{s_i} - 1}$$

is divisible by $(x^{s_{d-1}} - 1)(x^{s_d} - 1)(x - 1)^{-2}$. That this polynomial is thus divisible is trivial if $s_{d-1} = s_d = 1$. Let $\{\ell_1, \ldots, \ell_k\}$ denote the set of primes occurring in the factorization of $s_{d-1} s_d$.

Recall that

$$(x^r - 1) = \prod_{u|r} \varphi_u(x),$$

where $\varphi_u(x)$ is the minimal polynomial of a primitive u^{th} root of unity. Let

$$g_{l_i}(x) := \prod_{\substack{\ell_i | u | s_{d-1} \\ \ell_1, \ldots, \ell_{i-1} \nmid u}} \varphi_u(x),$$

$$g'_{l_i}(x) := \prod_{\substack{\ell_i | u | s_d \\ \ell_1, \ldots, \ell_{i-1} \nmid u}} \varphi_u(x).$$

Then

$$(x^{s_{d-1}} - 1) = (x - 1) \cdot g_{\ell_1}(x) \cdot \ldots \cdot g_{\ell_k}(x),$$

and

$$(x^{s_d} - 1) = (x - 1) \cdot g'_{\ell_1}(x) \cdot \ldots \cdot g'_{\ell_k}(x).$$

By construction, the polynomial $g_{\ell_i}(x)$ is prime to $g'_{\ell_j}(x)$ if $\ell_i \neq \ell_j$. Note now that, if $\ell_{i_1}, \ldots, \ell_{i_m}$ and $\ell_{j_1}, \ldots, \ell_{j_p}$ are distinct primes not dividing one of the multiplicities $s_1, \ldots, s_{d-2}$, say, not dividing s_1, then

$$(x^r - 1)/(x^{s_1} - 1) \text{ is divisible by } g_{\ell_{i_1}}(x) \cdot \ldots \cdot g_{\ell_{i_m}}(x) \cdot g'_{\ell_{j_1}}(x) \cdot \ldots \cdot g'_{\ell_{j_p}}(x).$$

Recall that for each prime ℓ dividing r, at most $d-2$ terminal multiplicities are divisible by ℓ (Lemma 2.6.b)). Therefore, for each prime ℓ_i dividing s_{d-1}, there exists a multiplicity $s_{\ell_i} \in \{s_1, \ldots, s_{d-2}\}$ such that

$$\ell_i \nmid s_{\ell_i}.$$

Similarly, for each prime ℓ_i dividing s_d, there exists a multiplicity $s'_{\ell_i} \in \{s_1, \ldots, s_{d-2}\}$ such that

$$\ell_i \nmid s'_{\ell_i}.$$

If ℓ_i divides both s_{d-1} and s_d, we may and will pick

$$s_{\ell_i} \neq s'_{\ell_i}.$$

Therefore, for any $i = 1, \ldots, d-2$,

$$\frac{x^r - 1}{x^{s_i} - 1} \quad \text{is divisible by} \quad \prod_{s_{\ell_j} = s_i} g_{\ell_j}(x) \cdot \prod_{s'_{\ell_j} = s_i} g'_{\ell_j}(x).$$

This proves our statement when G is a tree with a unique node.

Assume now that our theorem is true for trees with $k-1$ nodes and consider a tree G with k nodes. Pick an edge e on a connecting chain; break this connecting chain at e and complete (as in 2.4) to get two graphs G_1 and G_2 whose number of nodes is strictly less than k. Let s denote the weight of the chosen connecting chain. Let s_1 and s_2 denote respectively the greatest common divisor of the multiplicities of G_1 and G_2. Clearly, both s_1 and s_2 divide s. We get two arithmetical trees G'_i, $i = 1, 2$, by dividing all the multiplicities of G_i by s_i, $i = 1, 2$.

Since G is a tree, the following equality holds:

$$\sum_{i=1}^{n} (d_i - 2) = -2.$$

It is then easy to check that:

$$f_G(x) = [f_{G'_1}(x^{s_1}) \cdot (x-1)^2 (x^{s_1} - 1)^{-2}] \cdot [f_{G'_2}(x^{s_2}) \cdot (x-1)^2 (x^{s_2} - 1)^{-2}] \cdot (x^s - 1)^2 (x-1)^{-2}.$$

By induction, $f_{G'_1}(x)$ and $f_{G'_2}(x)$ are polynomials. Since G is an arithmetical graph, $\gcd(s_1, s_2) = 1$. Therefore

$$(x^s - 1)^2 (x-1)^2 (x^{s_1} - 1)^{-2} (x^{s_2} - 1)^{-2}$$

is a polynomial. This concludes the proof of our theorem when G is a tree.

Let G be an arithmetical graph with $\beta(G) > 0$. Pick an edge e on a connecting chain of G such that $G \backslash \{e\}$ is connected. The new arithmetical graph G' obtained by breaking this connecting chain at e and completing (as in 2.4) is such that $\beta(G') < \beta(G)$. Let s denote the weight of the chosen connecting chain. One checks easily that

$$f_G(x) = f_{G'}(x) \cdot (x^s - 1)^2 (x-1)^{-2}.$$

By induction, $f_{G'}(x)$ is a polynomial and, hence, $f_G(x)$ is a polynomial. □

Corollary 3.2 ([Lor 1], 4.6 and 4.7). *Let G be a simple arithmetical graph. Then the rational number*

$$\prod_{i=1}^{n} r_i^{d_i - 2} = f_G(1)$$

is an integer and

$$\deg f_G(x) = \sum_{i=1}^{n} (r_i - 1)(d_i - 2)$$

is a nonnegative integer. □

Corollary 3.3. *Let G be a simple arithmetical graph. The rational function*

$$f_{G,\ell}(x) := \prod_{i=1}^{n} [(x^{\ell^{\mathrm{ord}_\ell(r_i)}} - 1)/(x - 1)]^{d_i - 2}$$

is a polynomial. Moreover, $f_G(x)$ is divisible by $(\prod_{\ell\ \mathrm{prime}} f_{G,\ell}(x))$ and

$$f_{G,\ell}(1) = \ell^{\mathrm{ord}_\ell(f_G(1))}.$$

Proof. Recall the factorization over $\mathbb{Z}$ of $(x^r - 1)$ into cyclotomic polynomials:

$$(x^r - 1) = \prod_{u|r} \varphi_u(x),$$

where $\varphi_u(x)$ is the minimal polynomial of a primitive u^{th} root of unity. This factorization allows us to write

$$f_G(x) = (\prod_{\ell\ \mathrm{prime}} f_{G,\ell}(x)) \cdot p(x)/q(x),$$

where $p(x)$ and $q(x)$ are integral polynomials whose irreducible factors are cyclotomic polynomials of the form $\varphi_u(x)$, with u a composite integer. Moreover, this factorization also shows that $f_G(x)$ is a polynomial if and only if $f_{G,\ell}(x)$ is a polynomial for all prime ℓ and $q(x)$ divides $p(x)$. Hence, Theorem 3.1 implies that $f_{G,\ell}(x)$ is a polynomial for all primes ℓ and that

$$f_G(x) = (\prod_{\ell\ \mathrm{prime}} f_{G,\ell}(x)) \cdot g(x),$$

with $g(x)$ a polynomial. Recall that

$$\varphi_u(1) = 1 \quad \text{if } u \text{ is a composite integer},$$
$$\varphi_u(1) = \ell \quad \text{if } u = \ell^{\mathrm{ord}_\ell(u)}.$$

Hence,

$$f_{G,\ell}(1) = \ell^{\mathrm{ord}_\ell(f_G(1))}. \quad \square$$

Corollary 3.4 ([Lor 1], 4.7). *Let G be a simple arithmetical graph. Then*

$$\operatorname{ord}_\ell\left(\prod r_i^{d_i-2}\right)\cdot(\ell-1)\leqq \deg\left(f_{G,\ell}(x)\right),$$

and hence

$$\sum_{\ell\text{ prime}} \operatorname{ord}_\ell\left(\prod r_i^{d_i-2}\right)\cdot(\ell-1)\leqq \sum (r_i-1)(d_i-2).$$

Proof. The polynomial $f_{G,\ell}(x)$ factors into a product of irreducible cyclotomic polynomials of the form $\varphi_a(x)$, with $a=\ell^{\operatorname{ord}_\ell(a)}$. It is well known that for such a polynomial,

$$\varphi_a(1)=\ell.$$

Since by 3.3,

$$f_{G,\ell}(1)=\ell^{\operatorname{ord}_\ell(f_G(1))},$$

we conclude that $f_{G,\ell}(x)$ factors into a product of $\operatorname{ord}_\ell(f_G(1))$ irreducible polynomials. Since the degree of $\varphi_a(x)$, with $a=\ell^{\operatorname{ord}_\ell(a)}$, is at least equal to $\ell-1$, we find that

$$\operatorname{ord}_\ell\left(f_G(1)\right)\cdot(\ell-1)\leqq \deg\left(f_{G,\ell}(x)\right).$$

Taking the sums of these inequalities for all prime ℓ proves our corollary. □

Corollary 3.5. *Let G be a simple arithmetical tree and assume that Condition* C_ℓ *holds for G. Write*

$$\Phi_\ell=\prod_{i=1}^{s(\ell)}\mathbb{Z}/\ell^{a_i}\mathbb{Z},\quad a_i\in\mathbb{N}.$$

Then

$$\sum_{i=1}^{s(\ell)}(\ell^{a_i}-1)\leqq \deg\left(f_{G,\ell}(x)\right).$$

Proof. We use Theorem 2.1 to write

$$\Phi_\ell=\prod_{C_i\text{ node}}\left(\prod_{j=1}^{d_i-2}\mathbb{Z}/\ell^{\operatorname{ord}_\ell(r_i/w_{i,j})}\mathbb{Z}\right),$$

where the weights $w_{i,j}$, $j=1,\dots,d_i$, of the chains of G containing C_i have been ordered in such a way that

$$\operatorname{ord}_\ell(w_{i,1})\geqq\dots\geqq\operatorname{ord}_\ell(w_{i,d_i}).$$

Then

$$\sum_{\text{nodes }C_i}\sum_{j=1}^{d_i-2}\left(\ell^{\operatorname{ord}_\ell(r_i/w_{i,j})}-1\right)\leqq\sum_{\text{nodes }C_i}\sum_{j=1}^{d_i-2}\left(\ell^{\operatorname{ord}_\ell(r_i)}-\ell^{\operatorname{ord}_\ell(w_{i,j})}\right)$$

$$=\sum_{\text{nodes }C_i}\left(\ell^{\operatorname{ord}_\ell(r_i)}-1\right)(d_i-2)-\sum_{\text{nodes }C_i}\sum_{j=1}^{d_i-2}\left(\ell^{\operatorname{ord}_\ell(w_{i,j})}-1\right).$$

It follows from Proposition 2.9 that

$$\sum_{\text{nodes } C_i}\left(\sum_{j=1}^{d_i-2} \ell^{\operatorname{ord}_\ell(w_{i,j})}-1\right)=\sum_{i=1}^{\delta}(\ell^{\operatorname{ord}_\ell(t_i)}-1).$$

If follows from the definition of the integers $t_1, \ldots, t_\delta$ that

$$\sum_{i=1}^{\delta}(\ell^{\operatorname{ord}_\ell(t_i)}-1)=\sum_{\text{terminal vertices }(C_j, r_j)}(\ell^{\operatorname{ord}_\ell(r_j)}-1).$$

Since, by definition,

$$\deg f_{G,\ell}(x)=\sum_{\text{nodes } C_i}(\ell^{\operatorname{ord}_\ell(r_i)}-1)(d_i-2)-\sum_{\text{terminal vertices }(C_j, r_j)}(\ell^{\operatorname{ord}_\ell(r_j)}-1),$$

our corollary is proved. □

Corollary 3.6. *Let X/K be a smooth proper geometrically irreducible curve. Let A/K be the jacobian of X/K and assume that it has toric rank equal to zero. Let $\mathscr{L}$ denote the set of primes $\ell \neq p$ such that A/K has potentially good ℓ-reduction. Write the ℓ-part of the group of components of A/K as $\Phi_\ell = \prod_{i=1}^{s(\ell)} \mathbb{Z}/\ell^{a_i}\mathbb{Z}$. Then*

$$\sum_{\ell\in\mathscr{L}}\left(\sum_{i=1}^{s(\ell)}(\ell^{a_i}-1)\right)+\sum_{\ell\notin\mathscr{L}}\operatorname{ord}_\ell(|\Phi|)(\ell-1)\leqq 2u_K.$$

Proof. Choose a good regular model $\mathscr{X}/\mathcal{O}_K$ of X/K. Under the hypothesis that $t_K=0$, we showed in [Lor2], 1.5, that

$$|\Phi|=\prod r_i^{d_i-2}=f_G(1).$$

Therefore, it follows from 2.2, 3.4 and 3.5, that

$$\sum_{\ell\in\mathscr{L}}\left(\sum_{i=1}^{s(\ell)}(\ell^{a_i}-1)\right)+\sum_{\ell\notin\mathscr{L}}\operatorname{ord}_\ell(|\Phi|)(\ell-1)\leqq\sum_{\ell}\deg\left(f_{G,\ell}(x)\right).$$

We showed in [Lor2], 2.3, that

$$\deg\left(f_G(x)\right)\leqq 2u_K,$$

which proves our corollary. □

4. Existence of jacobians with given group of components

It is natural to wonder how sharp our bound for Φ is. The following proposition provides a partial answer to this question.

Proposition 4.1. *Let Φ be any finite abelian group of odd order. Write* $\Phi = \prod_{i=1}^{v} \Phi_{l_i}$, *with*

$$\Phi_{\ell_i} = \prod_{j=1}^{s(\ell_i)} \mathbb{Z}/\ell_i^{a_{ij}}\mathbb{Z}.$$

There exists a complete discrete valuation field K of equicharacteristic zero and a curve X/K such that:

(i) *The jacobian A/K of X/K has toric rank equal to zero and the group of components of its Néron model is isomorphic to Φ.*

(ii) *Let u_K denote the unipotent rank of A/K. Then*

$$2u_K = \sum_{i=1}^{v} \sum_{j=1}^{s(\ell_i)} (\ell_i^{a_{ij}} - 1).$$

(iii) *X/K has a good model whose graph satisfies Condition C_ℓ, for all prime ℓ.*

(iv) *The abelian variety A/K has potentially good reduction.*

Proof. Let X/K be a curve and let $\mathscr{X}/\mathcal{O}_K$ be a regular model of X. We denote by $(G(\mathscr{X}), M(\mathscr{X}), R(\mathscr{X}))$ the triple consisting of the graph, the intersection matrix, and the vector of multiplicities associated to the special fiber of $\mathscr{X}$. We let

$$T(\mathscr{X}) := (g(C_1), \ldots, g(C_n))$$

denote the vector whose components are the genus of the irreducible components of $\mathscr{X}_k$.

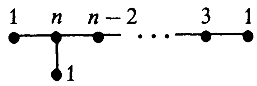
Winters' Existence Theorem [Win] states that, given a simple arithmetical graph (G, M, R) and a vector T of nonnegative integers, there exists a complete field K with a discrete valuation, and a curve $X(G)/K$ with a good regular model $\mathscr{X}/\mathcal{O}_K$ such that

$$(G(\mathscr{X}), M(\mathscr{X}), R(\mathscr{X}), T(\mathscr{X})) = (G, M, R, T).$$

Our aim is to construct an arithmetical tree $G = G(\Phi)$ depending on Φ, in such a way that the properties (i) – (iv) hold for the jacobian of the associated curve $X(G)$.

Definition/Construction 4.2. Let G_1 and G_2 be two arithmetical graphs. Let $C_i \in G_i$, $i = 1, 2$, denote a terminal vertex of multiplicity s. The *joint* of G_1 and G_2 at C_1 and C_2 is a new arithmetical graph G obtained as follows: as a topological space, it is the disjoint union of G_1 and G_2 with the points C_1 and C_2 identified in a single point C_0. The vertex C_0 in G has multiplicity s. We give to the other vertices in G a multiplicity equal to their multiplicity in G_1 or G_2. It is easy to verify that G is an arithmetical graph. In fact, the "self-intersection" of C_0 in G is equal to the sum of the "self-intersections" of C_1 in G_1 and C_2 in G_2.

4.3. For any *odd* integer n, we denote by $G(n)$ the following graph:



4.4. The tree $G(\Phi)$ needed to prove our proposition is obtained by "joining" the graphs in the set

$$\{G(\ell_i^{a_{ij}}) \mid i = 1, \ldots, v; \quad 1 \leqq j \leqq s(\ell_i)\}.$$

More precisely, for each ℓ_i, construct a graph $G(\ell_i, s(\ell_i))$ obtained as follows: let

$$G(\ell_i, 1) := G(\ell_i^{a_{i1}}),$$

and let

$$G(\ell_i, j) := \text{a joint of } G(\ell_i, j-1) \text{ with } G(\ell_i^{a_{ij}}).$$

Then let $G := G(\Phi)$ be the graph $G[v]$ obtained as follows: let

$$G[1] := G(\ell_1, s(\ell_1)),$$

and let

$$G[i] := \text{a joint of } G[i-1] \text{ with } G(\ell_i, s(\ell_i)).$$

Let m denote the number of vertices of $G(\Phi)$. Let M and R denote the intersection matrix and vector of multiplicities associated to $G(\Phi)$. Let T be a null vector with m components. Recall that the genus g of the curve $X(G)/K$ associated to the type (G, M, R, T) using Winters' Theorem can be expressed as follows:

$$2g(X(G)) = 2\sum_{i=1}^{m} r_i g(C_i) + \sum_{i=1}^{m} (r_i - 1)(d_i - 2) + 2\beta(G),$$

(see for instance [Lor2], 2.1). Using this formula, one easily checks that:

$$2g(X(G)) := \sum_{i=1}^{v} \sum_{j=1}^{s(\ell_i)} (\ell_i^{a_{ij}} - 1).$$

The fact that G is a tree and that $T = (0, \ldots, 0)$ insures that the genus g is equal to the unipotent rank of the jacobian A/K of $X(G)$ (see 1.3). An easy application of Theorem 2.1 shows that the group of components of A is isomorphic to Φ. It is obvious by construction that (iii) is true. Part (iv), that is, the fact that A has potentially good reduction, follows from 1.9 and 1.11; we leave the details of the proof of (iv) to the reader. □

Corollary 4.5. *Let g be any positive integer. Let $\Phi = \prod_{\ell \neq 2} \Phi_\ell$ be any finite abelian group of odd order. Write $\Phi_\ell = \prod_{i=1}^{s(\ell)} \mathbb{Z}/\ell^{a_i}\mathbb{Z}$. Assume that*

$$\sum_{\ell \neq 2} \left(\sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1) \right) \leqq 2g.$$

Then there exists a field K and an abelian variety A/K of dimension g with purely additive reduction, potentially good reduction, and such that

$$\Phi(A) \cong \Phi .$$

Proof. Let K be any complete field with a discrete valuation of equicharacteristic zero. There always exists an elliptic curve E/K with additive reduction, potentially good reduction and whose group of components is trivial. Given the group Φ, our previous theorem shows the existence of such a field K and of an abelian variety B/K such that

$$2\dim(B) = \sum_{\ell \neq 2} \left(\sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1) \right).$$

Let

$$2f := 2g - \sum_{\ell \neq 2} \left(\sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1) \right).$$

Let A/K denote the product of B with f copies of an elliptic curve E/K as above. □

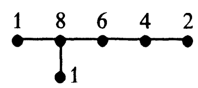
Remark 4.6. When $p \neq 2$, we found no curve of genus 2 whose jacobian has purely additive reduction, potentially good reduction, and whose group of components is isomorphic to

$$\mathbb{Z}/4\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

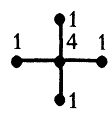
There are curves of genus 3 whose jacobian has potentially good reduction and group of components isomorphic to

$$\mathbb{Z}/4\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

A good model of such a curve may have a graph of the form:



or



This follows immediately from 2.1. We do not know whether any of the groups

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \quad \text{and} \quad \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$$

can be realized as the group of components of a jacobian of dimension 3 having purely additive reduction and potentially good reduction.

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