NÉRON MODELS

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Abstract. Néron models were introduced by André Néron in his seminar at the IHES in 1961. This article gives a brief survey of past and recent results in this very useful theory.

KEYWORDS Néron model, weak Néron model, abelian variety, group scheme, elliptic curve, semi-stable reduction, Jacobian, group of components.

MATHEMATICS SUBJECT CLASSIFICATION: 14K15, 11G10, 14L15

1. Introduction

Néron models were introduced by André Néron (1922-1985) in his seminar at the IHES in 1961. He gave a summary of his results in a 1961 Bourbaki seminar [81], and a longer summary was published in 1962 in Crelle's journal [82]. Néron's own full account of his theory is found in [83]. Unfortunately, this article[1] is not completely written in the modern language of algebraic geometry.

Néron's initial motivation for studying models of abelian varieties was his study of heights (see [84], and [56], chapters 5 and 10-12, for a modern account). In [83], Néron mentions his work on heights[2] by referring to a talk that he gave at the International Congress in Edinburgh in 1958. I have been unable to locate a written version of this talk. A result attributed to Néron in [54], Corollary 3, written by Serge Lang in 1963, might be a result that Néron discussed in 1958. See also page 438 of [55].

There is a vast literature on the theory of Néron models. The first mention of Néron models completely in the language of schemes may be in an article of Jean-Pierre Serre at the International Congress in Stockholm in 1962 ([94], page 195). Michel Raynaud extended Néron's results in [90] and gave a seminar on Néron models at the IHES in 1966-1967. He published [91] in 1970, which remained for many years the most important reference on the subject. Several important results on Néron models, including the semi-stable reduction theorem for abelian varieties, are proved by Alexander Grothendieck in the 1967-69 seminar SGA7 I [42]. The 1970 Springer Lecture Notes [92] by Raynaud includes a wealth of results on groups schemes, including over bases of dimension greater than 1. A nice overview on Néron models is found in the 1986 article [4] by Michael Artin. The 1990 book [14] by Siegfried Bosch, Werner Lüktebohmert, and Michel Raynaud, remains the most detailed source of

Date: February 6, 2017.

[1]On February 14, 1963, Grothendieck writes to Artin: *First I want to ask you if you feel like refereeing Néron's big manuscript on minimal models for abelian varieties (it has over 300 pages). I wrote to Mumford in this matter, who says he will have no time in the next months, do you think you would? Otherwise I will publish it as it is, as it seems difficult to find a referee, and the stuff is doubtlessly to be published, even if it is not completely OK in the details.* [79]

[2]On August 8, 1964, Grothendieck in a letter to Serre, writes: *J'ai essayé d'apprendre la théorie des hauteurs de Néron, et me propose d'essayer de généraliser ses symboles locaux... Les résultats de Néron sont rupinants, et prendront, je crois une grande importance* [95]. (I tried to learn Néron's theory of heights, and I plan to try to generalize his local symbols... Néron's results are terrific, and will become, I think, of great importance.)

information. More recent results are exposited by Lars Halvard Halle and Johannes Nicaise in [43]. This article is a brief survey of this very useful theory.

2. Models

Let R be a noetherian normal integral domain with field of fractions K, and set $S :=$ Spec R. When R is a discrete valuation ring (otherwise known as a local principal ideal domain), we let (π) denote its maximal ideal, and k its residue field, of characteristic $p \geq 0$. Examples of such rings include the power series ring $k[[t]]$, and the ring of p-adic numbers $\mathbb{Z}_p$. When k is algebraically closed, it is a very useful geometric analogy to view Spec R as an open little disk with center at the closed point Spec k. The generic point Spec K of Spec R is then an analogue of an open unit disc minus the origin. The most subtle care is usually needed to handle the case where the residue field k is imperfect, such as when $R = \mathbb{Z}[t]_{(p)}$, with residue field $\mathbb{F}_p(t)$.

Let $X_K \to$ Spec K be any scheme over K. We say that a scheme X endowed with a morphism $f : X \to S$ is a *model*[3] of X_K over S if there is an isomorphism over Spec K between the generic fiber $X \times_S$Spec K of f and the given K-scheme X_K. When the morphism f need not be specified, we will denote the model simply by X/S or X/R.

If $s \in S$, with residue field $k(s)$, then we denote as usual by $X_s := X \times_S$ Spec $k(s)$ the fiber of f over s. Since the field $k(s)$ is likely to be less complicated than the initial field K, so that a fiber X_s might be easier to study than the generic fiber X_K, a possible motivation for studying models could be as follows: Given a well-chosen surjective model $X \to S$ for X_K/K, can we recover interesting information on X_K/K from the study of the fibers $X_s/k(s)$, where s is closed point of S? This is indeed a prominent question in modern arithmetic geometry, highlighted for instance by the Birch and Swinnerton-Dyer Conjecture (see [44], F.4.1.6).

When first encountering the notion of model, it is natural to ask what kind of interesting models does a given scheme X_K/K have. If X_K/K has some nice properties, such as being smooth, or proper, is it always possible to find a model $f : X \to S$ such that f has the same properties? What about if X_K/K is a group scheme: Can we find then a model X/S which is an S-group scheme?

Example 2.1 Consider the simplest case where $X_K :=$ Spec L, with L/K a finite extension, so that X_K/K is proper, and smooth if L/K is separable. We can define in this case a *canonical* model X/S for X_K/K by letting $X :=$ Spec R_L, where R_L is the integral closure of R in L.

We find that the model $f : X \to S$ may not be proper if R is not excellent (since R_L might not be finite over R), and even when X_K/K is also smooth, one does not expect the morphism $f : X \to S$ to be smooth everywhere, as primes might ramify in the extension L/K, or if the ring R is not regular, then f might not be flat.

Example 2.2 The multiplicative group scheme $\mathbb{G}_{m,S}/S$ represents the functor which, to an R-algebra B, associates the group $(B^*, \cdot)$. We can take $\mathbb{G}_{m,S} =$ Spec $R[x, y]/(xy - 1)$. The unit section is $\epsilon : S \to \mathbb{G}_{m,S}$, with $\epsilon^* :$ Spec $R[x, y]/(xy - 1) \to R$, and $x \mapsto 1$ and $y \mapsto 1$. The inverse map sends $x \mapsto y$ and $y \mapsto x$. The multiplication map sends $x \mapsto x \otimes x$.

We present now a model of $\mathbb{G}_{m,K}/K$ with a special fiber isomorphic to the additive group $\mathbb{G}_{a,k}/k$. To do this, let us change coordinates so that the unit element in $\mathbb{G}_{m,K}(K)$ is $(0, 0)$

[3] The use of the terminology 'model' in this context may predate the invention of schemes. For instance, in 1956, Masayoshi Nagata publishes [80], entitled *A General Theory of Algebraic Geometry Over Dedekind Domains, I: The Notion of Models*. As early as 1939, Oscar Zariski in [111] uses the term 'model of a function field F/K' to refer, as we still do now, to a projective variety over K with function field isomorphic to the given field F.

in the new coordinates: we have then $\mathbb{G}_{m,S} = \operatorname{Spec} R[x,y]/(x+y+xy)$. For any $b \in R \setminus \{0\}$, we can construct a new model G_b/S of $\mathbb{G}_{m,K}/K$ as follows:

$$G_b := \operatorname{Spec} R[x,y]/(x+y+bxy).$$

When R is a discrete valuation ring and $b \in (\pi)$, the special fiber of G_b is isomorphic to the additive group $\mathbb{G}_{a,k}/k$. There is a natural morphism of S-group schemes $G_b \to \mathbb{G}_{m,S}$, which corresponds to the morphism of rings

$$R[x,y]/(x+y+bxy) \longleftarrow R[X,Y]/(X+Y+XY)$$

with $X \mapsto bx$ and $Y \mapsto by$. The morphism $G_b \to \mathbb{G}_{m,S}$ is an isomorphism when restricted to the generic fiber, and is an isomorphism when $b \in R^*$. When $b \in (\pi)$, the special fiber of G_b is sent to the neutral element of the special fiber of $\mathbb{G}_{m,S}$.

Example 2.3 Let X_K/K be smooth and proper. We say that X_K/K has *(everywhere) good reduction over* S if there exists a smooth and proper model X/S of X_K over S. In general, given a proper model X/S for X_K/K, the generic smoothness theorem implies that there exists a dense open set $U \subseteq S$ such that the morphism $X_U := X \times_S U \to U$ is smooth and proper. When the generic point η of S is open in S, such as when R is a discrete valuation ring, then the above theorem is satisfied with $U = \{\eta\}$ and does not provide any new information on the model X/S. But when the base has infinitely many points and is of dimension 1, such as when $S = \operatorname{Spec} \mathbb{Z}$ or $S = \operatorname{Spec} k[t]$, then the above theorem implies that all but finitely many fibers $X_s/k(s)$ are smooth and proper.

Given any finite non-trivial extension $L/\mathbb{Q}$, the integral closure $\mathcal{O}_L$ of $\mathbb{Z}$ is such that the morphism $X := \operatorname{Spec} \mathcal{O}_L \to S := \operatorname{Spec} \mathbb{Z}$ is smooth over a dense open set U of $\operatorname{Spec} \mathbb{Z}$ maximal with this property, and a famous theorem of Hermann Minkowski asserts that this maximal U is never equal to $\operatorname{Spec} \mathbb{Z}$. A beautiful generalization to abelian varieties $A/\mathbb{Q}$, conjectured for Jacobians by Igor Shafarevich in 1962 ([101], section 4), and proven independently by Victor Abrashkin [1] and Jean-Marc Fontaine [36], shows that there does not exist any abelian variety of positive dimension $A/\mathbb{Q}$ with everywhere good reduction over $S = \operatorname{Spec} \mathbb{Z}$.

Fix an integer $g > 1$ and a non-empty open set $U \subset \operatorname{Spec} \mathbb{Z}$. Shafarevich also conjectured in 1962 ([101], section 4) that there exist only a finite number of isomorphism classes of curves $X/\mathbb{Q}$ of genus g possessing everywhere good reduction over U (he proved the analogous statement in the case $g = 1$, see e.g., [99], IX.6.1). This conjecture was proved by Gerd Faltings in 1983, and we refer to the survey [74], page 240, for further information on this topic.

Example 2.4 Suppose that R is a discrete valuation ring. When X_K/K is projective, we can obtain a flat projective model X/S of X_K/K by choosing a projective embedding $X_K \to \mathbb{P}^n_K$ and letting X be the schematic closure of the image of X_K in $\mathbb{P}^n_S$ ([14], 2.5). It is easy to build examples, such as $X = \operatorname{Proj} R[x,y,z]/(z^2 - \pi xy)$, where the scheme X is not regular even though the generic fiber X_K/K is smooth, and where the special fiber X_k is not reduced.

If the scheme X_K/K is proper, then the existence of a proper model X/S follows from Nagata's Compactification Theorem [25], which states that if $f : Y \to S$ is separated and of finite type, then there exists a proper S-scheme $\overline{Y}$ and an open immersion $j : Y \to \overline{Y}$ over S. Let now R be any Dedekind domain. Let X_K/K be a separated scheme of finite type. Then there exists a model $f : X \to S$ of X_K/K which is separated of finite type and faithfully flat ([63], 2.5). Moreover, if X_K/K is also proper (respectively, projective), then so is f.

A given scheme X_K/K has many different models over S, which raises the following natural question. Is it possible, given X_K/K, to find an interesting class of models Y/S of X_K/K such that in that class, there is a *terminal object* Y_0/S, that is, there is an object Y_0/S in the class such that given any model Y/S in the class, there exists a morphism of S-schemes $g : Y \to Y_0$ which induces the identity on the generic fiber. As we indicate below, this question has a positive answer in several important cases.

Remark 2.5 Let us assume that X_K/K has a smooth and proper model X/S of X_K over S. When such a model exists, we are certainly happy to work with it! But how unique is such a model when it exists? Even before schemes were invented in 1956 by Grothendieck[4], one could sometimes recognize some projective objects who might have everywhere good reduction: those defined by a system of equations with coefficients in R such that for every maximal ideal M of R, the reduced system of equations still defines a smooth projective variety over R/M. This condition is not sufficient to insure good reduction in general, so in the case of curves, one would also add that the (geometric) genus of each special fiber is equal to the genus of the generic fiber. Thus, equipped with this definition, one could ask already in 1956 the following question when R is a discrete valuation ring: given two smooth proper models X/S and X'/S of a given smooth proper scheme X_K/K, what can we say about the special fibers X_k/k and X'_k/k. Are they always isomorphic for instance?

A positive answer to this question in the case of elliptic curves was given by Max Deuring in 1955, and soon thereafter, in 1957, by Wei-Liang Chow and Serge Lang for smooth proper curves and for abelian varieties [23]. In 1963, the general case is treated by Teruhisa Matsusaka and David Mumford (see [72], Theorem 1, and also [70], Theorem 5.4). In particular, *suppose given two smooth proper models X/S and Y/S of their irreducible generic fibers X_K/K and Y_K/K, and a K-isomorphism $h : X_K \to Y_K$. Assume that Y_k is not ruled (that is, Y_k is not birational over k to the product of $\mathbb{P}^1_k$ by another scheme). Then X_k and Y_k are birational.* In the case of curves of genus $g > 0$ and of abelian varieties, we conclude that the smooth special fibers are in fact isomorphic. As we shall see in 3.1 and 3.2, the smooth and proper models turn out to be unique terminal objects in these two cases. The situation is already much more delicate for surfaces and we refer to [37] for an introduction in equicharacteristic 0.

We also mention here an example in [70], 5.2, page 261, of a K3 surface with no smooth and proper model over a discrete valuation ring, but such that it has a smooth and proper algebraic space model, indicating that in certain contexts one might profit from working in a category larger than the category of schemes.

Let us return to 2.2 where models of the multiplicative group $\mathbb{G}_{m,K}/K$ are introduced. The following 1979 result of William Waterhouse and Boris Weisfeiler (in [109], use 1.4 with the proof of 2.5) answers in this case the question about the existence of terminal objects posed in 2.4.

Theorem 2.6. *Let R be a discrete valuation ring with maximal ideal (π). Let G/S be a smooth, separated S-group scheme of finite type, with connected fibers, and which is a model of $\mathbb{G}_{m,K}/K$. Then G/S is isomorphic to a group scheme of the form G_{π^n}/S for some $n \geq 0$. In particular, $\mathbb{G}_{m,S}/S$ is a terminal object in the class of such group schemes.*

Group schemes G/S might not be separated (see, e.g., [40], Exposé $\mathrm{VI_B}$, 5.6.4), but if $G \to S$ is a group scheme over S, flat, locally of finite presentation, with connected fibers,

[4] On February 16, 1956, Grothendieck in a letter to Serre mentions 'arithmetic varieties obtained by gluing the spectrum of noetherian commutative rings' [95]. These are schemes, except for the name. In a letter to Serre dated October 17, 1958, Grothendieck uses the name 'scheme' and mentions that Jean Dieudonné is writing the first four chapters of what will become the EGA's [39].

then $G \to S$ is necessarily separated and of finite presentation ([40], Exposé $\mathrm{VI_B}$, Corollary 5.5). This latter result is attributed to Raynaud by Grothendieck in a letter to Mumford on January 23, 1965 [79].

Example 2.7 The additive group $\mathbb{G}_{a,K}/K$ represents the functor which associates to a K-algebra B the additive group $(B,+)$. Let R be a discrete valuation ring with maximal ideal (π). Choosing an identification $\mathbb{G}_{a,K} = \operatorname{Spec} K[x]$, we find that $\mathbb{G}_{a,S} = \operatorname{Spec} R[x]$ is a smooth model of $\mathbb{G}_{a,K}/K$ over S, with $\mathbb{G}_{a,S}(S) = (R,+)$.

It turns out that for any $n \in \mathbb{Z}$, given the subgroup $H_n := (\pi^{-n}R,+)$ of the additive group $(K,+)$, there exists a smooth model G_n/S of $\mathbb{G}_{a,K}/K$ such that $G_n(S) = H_n \subset G_n(K) = K$. Indeed, simply consider $G_n := \operatorname{Spec} R[\pi^n x]$. The natural inclusion $R[\pi^n x] \to R[\pi^{n-1}x]$ induces a morphism of S-group schemes $G_{n-1} \to G_n$. This morphism induces the inclusions of groups $G_{n-1}(S) \subsetneq G_n(S) \subsetneq \mathbb{G}_{a,K}(K)$. Thus, if there existed a terminal object G/S in the class of smooth models of finite type of $\mathbb{G}_{a,K}/K$, we would have $G(S) = G(K)$. This is not possible because in this example, $G(K) = K$ is unbounded (see Definition 4.1), while $G(S)$ is always bounded when G/S is of finite type, as we explain in 4.2.

Let R be a discrete valuation ring. Given a proper model X/S of its generic fiber, and given a closed subscheme Y_k of the special fiber X_k, we can perform the blowing-up of $X' \to X$ of Y_k on X. By construction, X'/S is a new proper S-model of X_K. The preimage of Y_k is a codimension one closed subscheme of X', and thus has the same dimension as X_k. In particular, X'_k in general is not irreducible and, hence, is not smooth, even when X/S is smooth.

Definition 2.8 Let R be a discrete valuation ring. Let X/S be an S-scheme of finite type. Let Y_k be a closed subscheme of its special fiber, and let $\mathcal{I}$ denote the sheaf of ideal of $\mathcal{O}_X$ defining Y_k. Let $X' \to X$ be the blowing-up of Y_k on X, and let $u : X'_\pi \to X$ denote its restriction to the open subscheme of X' where $\mathcal{I}\mathcal{O}_{X'}$ is generated by π. The morphism $u : X'_\pi \to X$ is called *the dilatation of* Y_k *on* X ([14], 3.2, page 64).

When X/S is a group scheme and Y_k is a subgroup scheme of X_k, then X'_π/S is a group scheme, and $u : X'_\pi \to X$ is a homomorphism of S-group schemes ([14], 3.2/2 d)). When X/S is smooth and Y_k/k is a smooth k-subscheme of X_k, then X'_π/S is smooth ([14], 3.2/3, page 64).

The examples of morphisms of group schemes exhibited in 2.2 and 2.7 are dilatations of the origin. Additional examples of dilatations are found in the next example.

Example 2.9 Many of us might have started looking at models when working with elliptic curves. In this setting, one can begin a pretty good theory using only Weierstrass equations. Suppose that an elliptic curve X_K/K is given in the projective plane over K by an equation

$$y^2z + a_1xyz + a_3yz^2 - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3) = 0,$$

with $a_i \in R$ and discriminant $\Delta \neq 0$. This same equation defines a closed subscheme X/S inside the projective plane $\mathbb{P}^2_S$. Thus, X/S is a proper flat model of X_K/K.

Consider now the largest open subscheme $E \subset X$ such that E/S is smooth. It turns out that the scheme E/S is in fact a smooth S-group scheme with connected fibers, with a group scheme structure which extends the given group structure on E_K. Without giving details, let us say that in each fiber, three points on a line add up to zero in the group structure, in the same way as they do in the generic fiber.

When R is a discrete valuation ring with valuation v, the following notion of minimality is very useful. Consider all possible models over R of X_K/K obtained using only changes of variables that do not change the form of the Weierstrass equation. For each such model, we can consider the valuation $v(\Delta)$ of the discriminant of the defining Weierstrass equation.

Since this valuation is a non-negative integer, the models where $v(\Delta)$ is minimal are of special interest, and might have 'terminal properties' in this class of objects.

Let us look at one particular example. Assume that the characteristic of K is not 2 or 3. Then the affine equation $y^2 = x^3 + \pi^6$, when homogenized, defines an elliptic curve over K, and also defines a model X/S of its generic fiber. We let E/S denote its associated smooth S-group scheme model of its generic fiber. In this case, E is obtained from X by removing the closed point corresponding to the singular point of the special fiber of X/S. The special fiber of E is isomorphic to the additive group over k. The discriminant of the given Weierstrass equation is $\Delta = -2^4 3^3 \pi^{12}$. Using the change of variables $y = \pi^3 Y$ and $x = \pi^2 X$, we obtain a new Weierstrass equation $Y^2 = X^3 + 1$, with discriminant $\Delta_0 = -2^4 3^3$. When the characteristic of the residue field k is not 2 or 3, then the model E_0/S associated with this new equation is regular, and its special fiber consists of an elliptic curve. The model is thus smooth, and is an S-group scheme. In this example, the model E_0/S with minimal valuation of the discriminant is 'nicer' than the original model E/S.

There is a morphism of S-group schemes $E \to E_0$ obtained as a composition

$$E = E_3 \to E_2 \to E_1 \to E_0,$$

where each intermediate morphism corresponds to blowing up the origin in the group law of the special fiber, and removing the strict transform of the old special fiber. Such a morphism is a dilatation. Let us explain below how this composition is constructed. Let $x_i := x/\pi^i$ and $z_i := z/\pi^i$. We start by blowing up the ideal (x, z) in the affine chart given by $z - (x^3 + z^3)$. Since we remove the old special fiber, we get an S-morphism $E_1 \to E_0$ given in an affine chart by

$$R[x, z]/(z - (x^3 + z^3)) \longrightarrow R[x_1, z_1]/(z_1 - \pi^2(x_1^3 + z_1^3)), \text{with } x \mapsto \pi x_1, z \mapsto \pi z_1.$$

In this chart for E_1, we find that the origin is given by the ideal (π, z_1). Blowing it up and removing the old special fiber gives

$$R[x_1, z_1]/(z_1 - \pi^2(x_1^3 + z_1^3)) \longrightarrow R[x_1, z_2]/(z_2 - (\pi x_1^3 + \pi^4 z_2^3),$$

and a final blow up of (π, z_2) gives

$$R[x_1, z_2]/(z_2 - (\pi x_1^3 + \pi^4 z_2^3) \longrightarrow R[x_1, z_3]/(z_3 - (x_1^3 + \pi^6 z_3^3).$$

We recognize that $z_3 - (x_1^3 + \pi^6 z_3^3)$, when homogenized, defines the plane curve given by the equation $y^2 z_3 - (x_1^3 + \pi^6 z_3^3)$, which provides the model E/S once the singular point of the special fiber is removed.

3. (Strong) Néron Models

The question in 2.4 regarding the existence of a terminal object in a class of models has a positive answer in two important cases that we now discuss.

Theorem 3.1. *Let R be a Dedekind domain. Let X_K/K be a smooth, proper, geometrically connected, curve of genus $g > 0$. Consider the class of all models $Y \to S$ of X_K/K such that Y is a regular scheme and such that the morphism $Y \to S$ is proper and flat. Then this class of models contains a unique terminal object, called the minimal regular model of the curve X_K/K.*

Néron proved this theorem in 1961 when X/K is an elliptic curve and R is a discrete valuation ring with perfect residue field. Steven Lichtenbaum, in his 1964 Harvard thesis, and independently Shafarevich, proved the existence of a minimal model given the existence of a regular model. The existence of a desingularization of a scheme of dimension 2 was proved in various settings by Zariski (1939), Sheeram Abhyankar (1956), and Joseph Lipman (1969). Accounts of these results can be found in [5], [22], and [59], Chapter 9.

Theorem 3.2. *Let R be a Dedekind domain. Let A_K/K be an abelian variety. Consider the set of all smooth group schemes $Y \to S$ of finite type which are models of A_K/K. Then this class of models contains a unique terminal object A/S, called the Néron model of the abelian variety A_K/K.*

This theorem, and the stronger statement 3.8, were proved by Néron in 1961 when R has perfect residue fields. In both situations 3.1 and 3.2, we can say a lot more:

Theorem 3.3. *Let R be a Dedekind domain. Let X_K/K be a smooth, proper, geometrically connected, curve of genus $g > 0$. Let Y/S and Z/S be two regular models of X_K/K, proper and flat over S. Assume that there exists a morphism of S-schemes $g : Y \to Z$. Then g is the composition of a sequence of elementary S-morphisms $Y = Z_n \to Z_{n-1} \to \cdots \to Z_1 \to Z_0 = Z$, where each Z_i/S is a regular proper flat model of X_K/K, and each morphism $Z_i \to Z_{i-1}$ is the blow-up of a closed point in a closed fiber of Z_{i-1}/S.*

Theorem 3.4. *Let S be the spectrum of a henselian discrete valuation ring R. Let $g : F \to G$ be an S-morphism of smooth S-group schemes of finite type. Suppose that $g_K : F_K \to G_K$ is an isomorphism. If g is separated, then g is the composition of a sequence of elementary S-morphisms $F = G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 = G$, where each G_i/S is a model of G_K/K and each morphism $G_i \to G_{i-1}$ is a morphism of S-group schemes obtained as the dilatation of a smooth center in G_{i-1}/S.*

A proof of Theorem 3.3 can be found in [59], 9.2.2, while a proof of Theorem 3.4 is found in [61], 2.3. While we have stated the above results in a way that emphasizes the similarities in the two cases, Néron was in fact able to prove that the Néron model of an abelian variety satisfied an even stronger property, the Néron mapping property:

Definition 3.5 Let X_K/K be a scheme, smooth, separated, and of finite type over K. We say that a scheme X/S, smooth, separated, and of finite type over S, is a (strong) *Néron model* of X_K/K if, given any smooth scheme Y/S and any K-morphism $g_K : Y_K \to X_K$, there exists a unique S-morphism $g : Y \to X$ whose pull-back under $\operatorname{Spec} K \to S$ is the given morphism g_K; In other words, such that the natural map

$$\mathrm{Hom}_{S\text{-schemes}}(Y, X) \longrightarrow \mathrm{Hom}_{K\text{-schemes}}(Y_K, X_K)$$

is bijective. We call this universal property *Néron's mapping property.*

The (strong) Néron model, if it exists, is then unique up to a unique isomorphism. Néron was interested in models of abelian varieties for arithmetic reasons, as he was interested in understanding heights of points. In arithmetic, not only are we interested in models X/S of X_K/K, but we would like to have a reduction map from K-rational points on X_K to $k(s)$-rational points on X_s, for $s \in S$. We always have the following natural maps of sets when X/S is separated:

$$X_s(k(s)) \longleftarrow X(S) \hookrightarrow X_K(K)$$

Thus a natural reduction map $red : X_K(K) \to X_s(k(s))$ can be defined if the natural inclusion $X(S) \hookrightarrow X_K(K)$ is a bijection. If the model X/S is proper, then by the valuative criterion of properness, the inclusion $X(S) \hookrightarrow X_K(K)$ is always a bijection. The next lemma follows immediately from the universal property of the Néron model. When R is a discrete valuation ring and L/K is a finite separable extension, a local ring R' with field of fractions L defines a smooth morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ (in fact, etale morphism) if and only if the maximal ideal M_R of R generates the maximal ideal $M_{R'}$ of R' and the residue field extension $R'/M_{R'}$ is separable over R/M_R.

Lemma 3.6. *Assume that $X_K \to \operatorname{Spec} K$ is smooth, separated, of finite type, and has a Néron model X/S over S. Then the natural map $X(S) \hookrightarrow X(K)$ is a bijection. Moreover, for any etale morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ as above, the map $X(R') \hookrightarrow X(L)$ is a bijection.*

Instead of considering all etale local algebras over R as we do in the above lemma, one can use limit arguments and consider instead the strict henselization[5] R^{sh} of R, with field of fractions K^{sh}. If X/S is the Néron model of its generic fiber X_K/K, then the natural injection $X(R^{\mathrm{sh}}) \hookrightarrow X(K^{\mathrm{sh}})$ is a bijection. In fact, we have a useful converse for smooth group schemes ([14], 7.1, Theorem 1):

Theorem 3.7. *Let R be a discrete valuation ring. Let G/S be a smooth separated group scheme of finite type. If the natural map $G(R^{\mathrm{sh}}) \hookrightarrow G(K^{\mathrm{sh}})$ is a bijection, then G/S is the Néron model of its generic fiber.*

In particular, if A/S is an abelian scheme, then it is the Néron model of its generic fiber.

Theorem 3.8. *Let R be an excellent discrete valuation ring with field of fractions K. Let K^{sh} denote the field of fractions of the strict henselization R^{sh} of the ring R. Let G_K/K be a smooth commutative algebraic group such that $G_{K^{\mathrm{sh}}}/K^{\mathrm{sh}}$ does not contain any subgroup scheme isomorphic either to the multiplicative group $\mathbb{G}_{m,K^{\mathrm{sh}}}/K^{\mathrm{sh}}$ or to the additive group $\mathbb{G}_{a,K^{\mathrm{sh}}}/K^{\mathrm{sh}}$. Then G_K/K admits a smooth Néron model G/S of finite type over S.*

This theorem was proved by Néron in 1961 for abelian varieties when the residue field is perfect, and by Raynaud in general ([14], 10.2, Theorem 1). Extensions of this result to the case where R is any excellent Dedekind domain are discussed in [14], 10.3, and two related open conjectures when K has positive characteristic are discussed on pages 310 and 314.

Let $R \subset R'$ be a local extension of discrete valuation rings with fields of fractions K and K'. Assume that R' has ramification index 1 over R. This is the case for instance if R' is the completion of R. Let G_K/K be a smooth group scheme of finite type. Then G_K/K admits a Néron model over R if and only if $G_{K'}/K'$ admits a Néron model over R' ([14], 7.2/1).

3.9 Let G/S be a smooth separated group scheme of finite type. Then there exists an open subgroupscheme G^0/S of G/S such that for each $s \in S$, the fiber $(G^0)_s \to \operatorname{Spec} k(s)$ is the connected component of the identity $(G_s)^0$ of the fiber $G_s \to \operatorname{Spec} k(s)$ (see [40], $\mathrm{VI_B}$, 3.10, page 344).

Example 3.10 For the terminal model $\mathbb{G}_{m,S}/S$ of $\mathbb{G}_{m,K}/K$ introduced in 2.2, we find that $\mathbb{G}_{m,S}(S) \subseteq \mathbb{G}_{m,K}(K)$ is simply the inclusion $R^* \subseteq K^*$. Hence, this model does not provide a reduction map $red : \mathbb{G}_{m,K}(K) \to \mathbb{G}_{m,k}(k)$, and $\mathbb{G}_{m,S}/S$ is not the Néron model of $\mathbb{G}_{m,K}/K$.

Let R be a discrete valuation ring. There exists a smooth separated model G/S of $\mathbb{G}_{m,K}/K$, locally of finite type over S, which satisfies the Néron mapping property ([14], Example 5, page 291). The associated S-morphism of group schemes $\mathbb{G}_{m,S} \to G$ is an open immersion which identifies $\mathbb{G}_{m,S}$ with the subgroup G^0/S. The special fiber is an extension of the constant group scheme $\mathbb{Z}$ by $\mathbb{G}_{m,k}$.

Example 3.11 Assume that the characteristic of K is not 2. Let $L := K(\sqrt{d})$ be a quadratic extension. Consider the norm torus T_K/K given by the affine equation $x^2 - dy^2 = 1$. The

[5]Recall that one way to construct R^{sh} is as follows. Consider a separable closure K_s of K, and choose a maximal ideal M_s in the integral closure R_s of R in K_s such that $M_s \cap R = (\pi)$. Let $\mathcal{G}$ denote the Galois group of K_s/K. Let $I := I_{M_s/(\pi)}$ denote the inertia subgroup at M_s, that is, the set of $\sigma \in \mathcal{G}$ such that $\sigma(M_s) = M_s$ and such that the induced morphism on residue fields $R_s/M_s \to R_s/M_s$ is the identity. Then the localization of the fixed subring R_s^I at the maximal ideal $M_s \cap R_s^I$ is the strict henselization R^{sh} of R ([14], 2.3, Proposition 11). The field of fractions of R^{sh} is K_s^I. The residue field of R^{sh} is a separable closure of k.

multiplication map $T(K)\times T(K)\to T(K)$ is given by $(x,y)\cdot(x',y') := (xx'+dyy', xy'+x'y)$, with neutral element $(1,0)$ and inverse $(x,y)\to(x,-y)$. It is easy to check that T_L/L is L-isomorphic to $\mathbb{G}_{m,L}$.

Consider the model T/S of T_K/K given as $T := \operatorname{Spec} R[x,y]/(x^2-dy^2-1)$. This is a separated group scheme of finite type over S, and when the characteristic of k is not 2, it is smooth. In the specific case where $d=\pi$ and T/S is smooth, it is not hard to check that $T(R^{\mathrm{sh}})\hookrightarrow T(K^{\mathrm{sh}})$ is a bijection, so that T/S is the Néron model of T_K. We find that in this case the special fiber $T_k := \operatorname{Spec} k[x,y]/(x^2-1)$ is the extension of the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{G}_{a,k}$. Further explicit examples of Néron models of norm tori of higher dimension are found in [60].

In Example 3.11, the Néron model of T_K/K is an *affine* group scheme T/S. It is a general theorem of Sivaramakrishna Anantharaman (in his thesis under Raynaud) that if R is a Dedekind domain and G/S is a flat separated group scheme of finite type with generic fiber G_K/K affine, then G itself is affine ([2], 2.3.1, p. 30). This result does not extend to bases of dimension larger than 1, as Raynaud has given an example over $R=\mathbb{Q}[X,Y]$ of a smooth model G/S with connected fibers of $\mathbb{G}_{m,K}\times\mathbb{G}_{m,K}$ that is not affine ([92], VII.3, page 115).

Example 3.12 In the case of a Weierstrass equation with coefficients in R, we defined in 2.9 a proper flat model X/S, and a smooth S-group scheme E/S with $E\subseteq X$, both models of their generic fibers $X_K=E_K$. By construction, we always have a reduction map $X_K(K)\to X_s(k(s))$ since X/S is proper. But we may fail to have a well-defined reduction map $X_K(K)\to E_s(k(s))$ when E/S is not proper.

For instance, assume now that the characteristic of K is not 2 or 3, and consider the equation $y^2=x^3+\pi^2$ defining X/S, with the S-rational point $(0,\pi)$. Reducing modulo a maximal ideal that contains π, we obtain a special fiber $y^2=x^3$, and the point $(0,\pi)$ reduces to the singular point of the special fiber. (This in particular shows that the singular point of the special fiber is also singular as a point of X). Thus $(0,\pi)\in X_K(K)$ cannot be reduced in $E_k(k)$ in a natural way. In particular, E/S is not the Néron model of E_K/K, as follows from Lemma 3.6.

The universal property of the Néron model $\mathcal{E}/S$ of E_K/K implies the existence of a morphism of S-group schemes $E\to\mathcal{E}$. When R is a discrete valuation ring with maximal ideal (π), the Weierstrass equation defining E/K has discriminant $\Delta=-2^4 3^3\pi^4$ and in particular has minimal valuation when R has residue characteristic different from 2 and 3. It follows (see 6.5) that the morphism $E\to\mathcal{E}$ is an open immersion in this case.

Example 3.13 Let R be a henselian discrete local ring, and let R^{sh} denote its strict henselization, with field of fractions K^{sh}. Let X_K/K be smooth, separated, and of finite type. When $X_K(K^{\mathrm{sh}})=\emptyset$, then X_K viewed as an S-scheme as $X_K\to\operatorname{Spec} K\to S$, is a Néron model of X_K.

Example 3.14 Néron and Raynaud gave necessary and sufficient conditions for the existence of the Néron model of a K-torsor under a smooth group scheme ([14], 6.5, Corollary 4). In [62], 4.13, Qing Liu and Jilong Tong show that if X_K/K is a smooth proper scheme which admits a proper model X/S over a Dedekind scheme S with X regular and such that no geometric fiber $X_{\overline{s}}$, $s\in S$, contains a rational curve, then the smooth locus X^{sm}/S of X/S is the (strong) Néron model of X_K/K. Adrian Vasiu in [106], 4.4, provides classes of projective varieties over certain fields K which have projective (strong) Néron models and which often do not admit finite maps into abelian varieties over K. See also [107], section 6.

Example 3.15 To give only one indication of the subtleties that need to be considered when k is not perfect, let us note that if k is perfect, then any smooth connected group scheme

G_k/k such that $G_{\overline{k}}/\overline{k}$ is $\overline{k}$-isomorphic to $\mathbb{G}_{a,\overline{k}}$ is already k-isomorphic to $\mathbb{G}_{a,k}$. In other words, there are no twisted forms of $\mathbb{G}_{a,k}$ ([108], 17.7, Corollary).

This is no longer the case when k is imperfect. Consider for instance the closed subgroup scheme G_k/k of $\mathbb{G}_{a,k} \times \mathbb{G}_{a,k}$ given by the equation $x - x^p + ty^p = 0$. When t is not a p-th power in k, the group G_k/k is smooth connected of dimension 1, but it is not k-isomorphic to $\mathbb{G}_{a,k}$, as it can easily be checked that there exists no k-isomorphism $k[x,y]/(x - x^p + ty^p) \to k[u]$. But over the purely inseparable extension $F := k(\sqrt[p]{t})$, the group G_F/F is F-isomorphic to $\mathbb{G}_{a,F}$.

Assume now that K is the field of fractions of a discrete valuation ring R with maximal ideal (π) and residue field k. When K is of characteristic $p > 0$, the group scheme H/S defined by $H := \operatorname{Spec} R[x,y]/(x - x^p + \pi y^p)$ is smooth, and one can check directly that it is the (strong) Néron model of its generic fiber by checking that $H(R^{\mathrm{sh}}) = H_K(K^{\mathrm{sh}})$. The special fiber H_k/k is defined by $\operatorname{Spec} k[x,y]/(x - x^p)$ and is thus an extension of the constant group scheme $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{G}_{a,k}$.

4. (Weak) Néron models

Néron's main theorem on the existence of models is Theorem 4.3 below. Let us start with the following definition (see [14], 1.1/2).

Definition 4.1 Let $R \subset R'$ be a local extension of discrete valuation rings with field of fractions K and K'. The valuations on K and K' give rise to absolute values, that we choose so that $|\ \ |_{K'}$ extends $|\ \ |_K$. This compatible absolute value will be denoted by $|\ \ | : K' \to \mathbb{R}$. Let X_K/K be a scheme of finite type. When $x \in X_K(K')$, and $g \in \mathcal{O}_{X_K}(U)$ for some neighborhood U of x, we may view $g(x)$ as an element of K', so that $|g(x)|$ is defined. Let $E \subset X_K(K')$. The function g is *bounded* on E if E is in the domain of g and the set $\{|g(x)|, x \in E\}$ is bounded. A subset of $\mathbb{A}^n_K(K')$ is *bounded* if each coordinate function is bounded on the subset.

If X_K is affine, we say that E is *bounded in* X_K is there exists a closed K-immersion $X_K \hookrightarrow \mathbb{A}^n_K$ for some n mapping E onto a bounded subset of $\mathbb{A}^n_K(K')$. In general, E is bounded on X_K is there exists a covering of X_K by finitely many affine open subsets U_i of X_K, $i = 1, \dots, d$, and a decomposition $E = \cup_{i=1}^d E_i$ with $E_i \subset U_i(K')$ and such that, for each i, the set E_i is bounded in the affine U_i. The definition is independent of the closed immersion and of the affine cover, and both parts are compatible (see [14], page 9).

Example 4.2 Let $S := \operatorname{Spec} R$ with R as above. Let X_K/K be a K-scheme separated of finite type, and let X/S be a separated model of finite type. Let $E := X(S)$, considered as a subset of $X_K(K)$. Then E is bounded in X_K. Indeed, cover X with finitely many affine open sets U_i. Let $\sigma : S \to X$ be a section. There exists an index i such that the image of σ is contained in U_i, with U_i of the form $U_i = \operatorname{Spec} R[x_1, \dots, x_n]/(g_1, \dots, g_m)$. The section σ corresponds to a point of the form $(r_1, \dots, r_n) \in R^n$. We have a natural embedding $U_{i,K} \to X_K$. Any affine covering of the image of $U_{i,K}$ allows us to express the coordinate y_j in that covering as a function in $K[x_1, \dots, x_n]/(g_1, \dots, g_m)$, and such function is bounded when evaluated on R-points $(r_1, \dots, r_n)$.

We may now state Néron's main result on models ([14], 3.5, Theorem 2).

Theorem 4.3. *Let R be a discrete valuation ring. Let X_K/K be a smooth separated K-scheme of finite type. If $X_K(K^{\mathrm{sh}})$ is bounded, then there exists a smooth separated model X/S of finite type of X_K/K over S such that the natural inclusion $X(R^{\mathrm{sh}}) \subseteq X_K(K^{\mathrm{sh}})$ is a bijection.*

A smooth separated model X/S of X_K/K over S as in Theorem 4.3 is called a *weak Néron model* of X_K/K. In the case of group schemes, the above theorem can be strengthened ([14], 4.3-4.4, and also 6.5, Corollary 4):

Theorem 4.4. *Let R be a discrete valuation ring. Let G_K/K be a smooth K-group scheme of finite type. Then G_K/K admits a (strong) Néron model G/S if and only if $G_K(K^{\mathrm{sh}})$ is bounded in G_K.*

Suppose that X_K/K is a smooth proper K-scheme with a proper model X/S. Recall that we say that $Y \to X$ is a *desingularization* of X if Y is a regular scheme and $Y \to X$ is a proper surjective birational morphism inducing an isomorphism over the regular locus of X. It is a very difficult theorem of Hironaka that such desingularization of X exists when R is of equicharacteristic 0. But when such Y exists, then the smooth locus $Y^{sm} \to S$ of $Y \to S$ is a Néron model of X_K/K ([14], page 61).

Remark 4.5 A smooth proper K-scheme X_K/K has more than one weak Néron model, and even the most obvious of such models, such as the smooth proper model $\mathbb{P}^n_S/S$ of the projective space $\mathbb{P}^n_K/K$, might not satisfy Néron's universal mapping property 3.5. Indeed, there are K-automorphisms $\varphi_K : \mathbb{P}^n_K \to \mathbb{P}^n_K$ which do not extend to an S-morphism $\mathbb{P}^n_S \to \mathbb{P}^n_S$. For instance, let R be a discrete valuation ring with maximal ideal (π), and consider $\mathbb{P}^1_K = \mathrm{Proj}(K[x,y])$, with the automorphism φ_K induced by $x \mapsto x$ and $y \mapsto \pi y$. Then φ_K extends to a morphism $\varphi : \mathbb{P}^1_S \setminus \{P\} \to \mathbb{P}^1_S$, where P is the point in $\mathbb{P}^1_S = \mathrm{Proj}(R[x,y])$ corresponding to the homogeneous ideal (x,π). The morphism φ cannot be extended to a morphism $\varphi' : \mathbb{P}^1_S \to \mathbb{P}^1_S$ because such morphism would be proper, with closed image, while the image of this morphism is the open generic fiber union with the closed point corresponding to the homogeneous ideal (x,π).

Remark 4.6 Let X_K/K be a smooth separated K-scheme of finite type, and let $\varphi_K : X_K \to X_K$ be a finite K-morphism. In arithmetic dynamics, a *weak Néron model for the pair* $(X_K/K, \varphi_K)$ is a pair $(X/S, \varphi)$, where X/S is a weak Néron model of X_K/K and $\varphi : X \to X$ is a finite morphism which extends φ_K. The existence of such weak Néron model pair has important applications regarding the properties of the canonical height attached to the pair $(X_K/K, \varphi_K)$ (see [18], and [46]).

Remark 4.7 Let G_K/K be a connected reductive group scheme. When K is the field of fractions of a discrete valuation ring R, the study of G_K/K involves group schemes G/S which are models of G_K/K over S. Much of this theory is found in [17], where what we call model is called (in French) 'un prolongement', that is, an extension (see 1.2.4). We remark below that the models G/S of a connected reductive group G_K/K introduced in the theory of algebraic groups are rarely Néron models of their generic fiber.

When K is the field of fractions of a henselian discrete valuation ring, the Theorem of Borel-Tits-Rousseau (see [89]) states that $G_K(K)$ is bounded if and only if G_K/K is anisotropic (i.e., does not contain a split multiplicative group). When K is any discrete valuation field, we can apply this theorem to K^{sh}, and obtain from 4.4 that G_K/K has a Néron model over S if and only of $G_{K^{\mathrm{sh}}}/K^{\mathrm{sh}}$ is anisotropic. It turns out that there are few anisotropic groups over a strictly henselian field. Indeed, recall that the maximal unramified extension K^{sh} of K is C_1 if R is complete with a perfect residue field ([53], Theorem 12). When the field K is C_1, a theorem of Steinberg shows that the group G_K/K contains a Borel subgroup defined over K ([93], III, 2.3, Theorem 1', and [10], 8.6, top of page 484). Over any field, if the group has a Borel subgroup defined over K, then it contains a non-central split K-torus ([9], 20.6 (ii)), and thus is not anisotropic.

Recall that when a field is C_1, then its Brauer group $\mathrm{Br}(K)$ is trivial. When there is a non-trivial element in $\mathrm{Br}(K)$, corresponding to a central division algebra D/K, then one finds a K-anisotropic group as follows. For any field K and D a central division over K of rank n^2 with $n > 1$, the K^{sep}/K-form of SL_n given by the K-group $SL_1(D)$ of units of reduced norm 1 in D (i.e., representing the functor assigning to any K-algebra A the kernel of the polynomial map $\text{Nrd} : (D \otimes_K A)^* \to A^*$ given by the reduced norm) is K-anisotropic.

Let R be a discrete valuation ring. Let X_K/K be a smooth separated K-scheme of finite type such that $X_K(K^{\text{sh}})$ is bounded. As we mentioned above, such an object can have more than one weak Néron model X/S. It is natural to look for invariants attached to a weak Néron model which do not depend on the choice of a weak Néron model for X_K/K. This is achieved by François Loeser and Julien Sebag in the next theorem (see [64], and also [85] and [35]) as follows.

Definition 4.8 Let F be any field. We denote by $K_0(\text{Var}_F)$ the *Grothendieck ring of varieties* over F. As an abelian group, $K_0(\text{Var}_F)$ is generated by the isomorphism classes $[X]$ of separated F-schemes of finite type X, modulo the following relations: if X is a separated F-scheme of finite type and Y is a closed subscheme of X, then $[X] = [Y] + [X \setminus Y]$, where $X \setminus Y$ is endowed with the induced structure of open subscheme. Note that this relation shows that $[\emptyset]$ is the neutral element for the addition in $K_0(\text{Var}_F)$, and that $[X] = [X_{\text{red}}]$, where X_{red}/F denotes the reduced scheme structure on X/F.

The group $K_0(\text{Var}_F)$ is endowed with the unique ring structure such that $[X]\cdot[X'] = [X \times_F X']$ for all F-schemes X and X' of finite type. The identity element for the multiplication is the class $[\operatorname{Spec} F]$, that we shall denote by 1. We denote by $K_0^{\text{mod}}(\text{Var}_F)$ the quotient of $K_0(\text{Var}_F)$ by the ideal generated by the elements of the form $[X] - [Y]$, where X and Y are separated F-schemes of finite type such that there exists a finite, surjective, purely inseparable F-morphism $Y \to X$.

Returning the case of a discrete valuation ring R with residue field k, we set $K_0^R(\text{Var}_k) := K_0(\text{Var}_k)$ if R has equal characteristic, and $K_0^R(\text{Var}_k) := K_0^{\text{mod}}(\text{Var}_k)$ if R has mixed characteristic. Let $\mathbb{L}_k := [\mathbb{A}^1_k]$. There exists a unique ring homomorphism $\chi : K_0^R(\text{Var}_k)/(\mathbb{L}_k - 1) \to \mathbb{Z}$ which sends the class $[X_k]$ of a separated k-scheme of finite type X_k to the Euler characteristic with proper support $\chi(X_k)$.

Theorem 4.9. *Assume that R has perfect residue field k. Let X_K/K be a separated smooth scheme of finite type with a weak Néron model X/S. Then the class $[X_k]$ in $K_0^R(\text{Var}_k)/(\mathbb{L}_k - 1)$ is independent of the choice of the weak Néron model.*

The proof of this theorem involves rigid analytic geometry, and it would take us too far afield to attempt to introduce this subject in this survey. But rigid analytic geometry is an important tool in the study of Néron models and their groups of components, as we shall indicate in section 7 when we briefly discuss the rigid analytic uniformization of an abelian variety. Néron models in the rigid analytic context are discussed in [15] and [48].

5. The Néron model of an abelian variety

Let R be a discrete valuation ring with residue field k and characteristic $p \geq 0$. Let A_K/K be an abelian variety of dimension $g > 0$, and denote by A/S its (strong) Néron model. Let A_k/k denote the special fiber of A/S. It is a smooth commutative k-group scheme, and as such, it contains a largest connected smooth commutative subgroup scheme A_k^0/k, *the connected component of the identity*, and the quotient $\Phi_A := A_k/A_k^0$ is a finite étale k-group scheme called *the group of components*. By definition, the following sequence is an exact sequence of group schemes:

$$(0) \longrightarrow A_k^0 \longrightarrow A_k \longrightarrow \Phi_A \longrightarrow (0).$$

Assume now that k is perfect. Claude Chevalley proved (see [24]) that every connected smooth commutative k-group scheme is an extension of an abelian variety B/k by a smooth[6] connected linear k-group scheme L/k, so that we have an exact sequence

$$(0) \longrightarrow L \longrightarrow A_k^0 \longrightarrow B \longrightarrow (0).$$

Every smooth connected commutative linear k-group scheme is the product of a unipotent group U/k by a torus T/k when k is perfect (see [14], page 243-244).

We call $u_K := \dim(U)$ the *unipotent rank* of A_K, $t_K := \dim(T)$ the *toric rank* of A_K, and $a_K := \dim(B)$ the *abelian rank* of A_K. By construction,

$$g = u_K + t_K + a_K.$$

When $\mathrm{char}(k) > 0$ and $u > 1$, there exist smooth commutative unipotent group schemes U/k of dimension u which are not isomorphic to $\mathbb{G}_{a,k}^u$. Results on the unipotent part of the special fiber of a Néron model are found for instance in [32] (5.4/2), [52], and [86].

Example 5.1 Let A_K/K be an elliptic curve, so that $g = 1$. It follows in this case that exactly one of the integers u_K, t_K, and a_K is positive, and we say that the elliptic curve has *additive reduction* if $u_K = 1$, *multiplicative reduction* if $t_K = 1$, and *good reduction* if $a_K = 1$. The elliptic curve is said to have *split* multiplicative reduction if the torus T/k is split over k, that is, if it is k-isomorphic to $\mathbb{G}_{m,k}$ (see 3.11 for an example of a non-split torus).

In general, an abelian variety of dimension g is said to have *purely additive reduction* if $u_K = g$, *purely multiplicative reduction* if $t_K = g$, and *good reduction* if $a_K = g$. Obviously, these are not the only possible types of reduction for abelian varieties when $g > 1$, as is easily seen by taking products of elliptic curves of various types of reduction.

Let $(H, +)$ be any group and let n be a positive integer. Denote by $[n] : H \to H$ the multiplication-by-n map. When H is commutative, this map is a group homomorphism, and we denote by $H[n]$ its kernel. Let now G/S be a group scheme, and denote again by $[n] : G \to G$ the S-morphism which, given T/S, induces the multiplication-by-n on the group $G(T)$. When G/S is commutative, we denote by $G[n]/S$ the kernel of $[n]$.

Theorem 5.2. *Let G be any smooth commutative S-group scheme of finite type. Let n be a positive integer invertible in R. Then*

(a) *The S-group homomorphism $[n] : G \to G$ is étale.*
(b) *Let R be a discrete valuation ring with field of fractions K and residue field k. Then the reduction map $G[n](S) \to G_k[n](k)$ is injective.*

Part (a) is proved when R is a field in [40] VII_A, 8.4, Proposition, page 472, and [41], XV, 1.3, page 352. It is found in general in [14], 7.3/2. Part (b) is proved in [14], 7.3/3. Note that in (b), if G/S is the Néron model of its generic fiber, then $G[n]/S$ is the Néron model of its generic fiber. Indeed, (a) shows that the group scheme $G[n]/S$ is étale, with $G[n]$ closed in G, and it inherits the Néron mapping property from the one of G/S.

Keep the hypotheses of (b). The kernel of the reduction map $red : G(K) \longrightarrow G_k(k)$ is often denoted by $G^1(K)$. Part (b) shows that $G^1(K)$ does not contain any point of order n when n is invertible in R. This theorem can also be obtained as a standard application of the structure of the multiplication-by-n on a formal group, such as in [44], C.2.

Example 5.3 Let n be an integer coprime to a prime p. Choose a primitive n-th root of unity $\zeta_n \in \overline{\mathbb{Q}}$ and choose a prime ideal $\mathfrak{p}$ of the ring $\mathbb{Z}[\zeta_n]$ that contains p. Let $R := \mathbb{Z}[\zeta_n]_{\mathfrak{p}}$ and $k := R/\mathfrak{p}$. Then the kernel of $[n] : \mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$ is given by $\mathrm{Spec}\, R[x]/(x^n - 1)$, and it

[6]When k is not assumed to be perfect, every connected smooth commutative k-group scheme is an extension of an abelian variety by a possibly non-smooth affine k-group scheme L/k. See [14], page 243, and [24]. See also [105] for recent work on a possible substitute to Chevalley's theorem when k is imperfect.

is a standard fact about cyclotomic fields that the reduction homomorphism $\mathbb{G}_{m,R}[n](R) \to \mathbb{G}_{m,k}[n](k)$ is injective: since the polynomial $x^{n-1} + \cdots + x + 1 = \prod_{a=1}^{n-1}(x - \zeta_n^a)$ evaluated at 1 is not zero modulo $\mathfrak{p}$, the class of each element $1 - \zeta_n^a$ is not zero in $R/\mathfrak{p}$.

Let us return to our study of the special fiber of the Néron model A/R of an abelian variety A_K/K. Assume that n is invertible in R. Let $\overline{k}$ denote an algebraic closure of k. Then

$$U(\overline{k})[n] = (0), \quad T(\overline{k})[n] \cong (\mathbb{Z}/n\mathbb{Z})^{t_K}, \quad \text{and} \quad B(\overline{k})[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2a_K}$$

(see, e.g., [78], Application 3 on page 62). If n is prime with $n > 2g+1$, then $\Phi_A(\overline{k})[n]$ can be generated by t_K elements. This latter statement follows from [68], Theorem 2.15.

To introduce the next major result, let us consider the following natural question. It may be that a smooth proper K-scheme X_K might not have good reduction over S. It is natural to wonder whether it could be possible to 'improve the reduction type' if we increased the field of coefficients of our equations. Indeed, more possibilities for the coefficients means more possible changes of variables, and thus could lead to better integral equations for models, equations which would not be available when working only over K. In other words, let K'/K be a finite extension, and denote by R' the integral closure of R in K', with $S' = \operatorname{Spec} R'$. Is it possible to find an extension K'/K such that X'_K has a model X'/S' over S' which is 'better' than the model already available over K?

It is easy to produce examples of such improvements in the reduction type of the model after base change. For instance, the elliptic curve $y^2 = x^3 + \pi$ has additive reduction over R when $p \neq 2, 3$, but when $K' := K(\sqrt[6]{\pi})$, we can make the change of variables $X \mapsto x/\sqrt[6]{\pi}^2$ and $Y \mapsto y/\sqrt[6]{\pi}^3$ and obtain a model with equation $Y^2 = X^3 + 1$, giving good reduction modulo a maximal ideal containing $\sqrt[6]{\pi}$. The phenomenon observed in this example is in fact quite general for abelian varieties:

Theorem 5.4. *Let R be a Dedekind domain with field of fractions K. Let A_K/K be an abelian variety with Néron model A/R. Then there exists a finite Galois extension L/K such that the Néron model of A_L/L over* $\operatorname{Spec} R_L$ *has no unipotent subgroup in its closed fibers, where R_L is the integral closure of R in L. When R is strictly henselian, then there exists a unique Galois extension L/K, minimal with the property that the Néron model of A_L/L over* $\operatorname{Spec} R_L$ *has no unipotent subgroup in its special fiber.*

Theorem 5.4 is called the Semi-Stable Reduction Theorem. It was stated as a question by Serre in a letter to Andrew Ogg dated August 8, 1964, and also at the end of a letter to Grothendieck on August 13, 1964 [95]. Néron's view on the question is given in the endnote 176.1 in [95]. Grothendieck, in a 12-page letter to Serre dated October 3 and 5, 1964 ([95], page 208), refers to the question as 'your [Serre's] conjecture' and sketches ideas for a proof. The theorem is proved by Grothendieck in [42], Exposé IX, 3.6 (see [30], 5.15, for the uniqueness). The theorem was also proved by Mumford in characteristic different from 2 ([28], Introduction, and [79], 38, letter, 1967 undated).

We say that A_K/K has *semi-stable reduction over* R if $g = a_K + t_K$. Let $R \subset R'$ be any local extension of discrete valuation rings with fields of fractions K and K'. Let A_K/K be an abelian variety with semi-stable reduction over R. Let A/R denote its Néron model and let A'/R' denote the Néron model of $A_{K'}/K'$ over R'. The terminology *semi-stable* is justified by the fact that the natural R'-morphism $A \times_R R' \to A'$ induces an isomorphism $(A \times_R R')^0 \to (A')^0$ on the identity components ([14], 7.4/4).

Suppose that R is henselian. We say that the abelian variety A_K/K of dimension g has *potentially good reduction* (resp., *potentially purely multiplicative reduction*) if there exists a finite extension F/K such that A_F/F has good reduction over the integral closure R_F of R in F (resp., has toric rank equal to g).

Raynaud gave the following explicit choice of an extension that leads to semi-stable reduction (see [42], Exposé IX, 4.7, and [97] for refinements). Let K_s denote a separable closure of the field K. Let ℓ be prime and invertible in R. Let K_ℓ/K be a finite separable extension such that $A(K_\ell)[\ell] = A(K_s)[\ell] = A(\overline{K})[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$. Let R_ℓ denote the integral closure of R in K_ℓ.

Theorem 5.5. *Let R be a discrete valuation ring. Let A_K/K be an abelian variety of positive dimension. Let $\ell \geq 3$ be prime, and invertible in R. Then the Néron model of A_{K_ℓ}/K_ℓ over* $\operatorname{Spec} R_\ell$ *has no unipotent subgroup in its special fibers.*

Let $\mathrm{Gal}(K_s/K)$ denote the Galois group of K_s/K. Let ℓ be any prime number. Consider the *Tate module* $T_\ell(A_K) := \varprojlim A_K[\ell^n](K_s)$, where the inverse limit is taken over the system of multiplication-by-ℓ maps $A_K[\ell^n](K_s) \to A_K[\ell^{n-1}](K_s)$. This module is naturally equipped with an action of $\mathrm{Gal}(K_s/K)$, and this representation $\rho_\ell : \mathrm{Gal}(K_s/K) \to \mathrm{Aut}(T_\ell(A_K))$ is the main *ℓ-adic representation* attached to the abelian variety A_K/K. When ℓ is coprime to $\mathrm{char}(K)$, the Tate module is a free $\mathbb{Z}_\ell$-module of rank $2g$.

Assume now that K is the field of fractions of a discrete valuation ring R with valuation v. Let $\overline{v}$ denote an extension of v to K_s and denote by $I(\overline{v})$ and $D(\overline{v})$ the associated inertia and decomposition subgroups of $\mathrm{Gal}(K_s/K)$. Grothendieck gave a (co)homological criterion for semi-stability: he showed in [42], IX, 3.5, that the abelian variety A_K/K has semi-stable reduction over R if and only if there exists a prime ℓ invertible in R and a submodule T_0 of the Tate module $T_\ell(A_K)$ stable under the action of the inertia group $I(\overline{v})$ such that $I(\overline{v})$ acts trivially on both T_0 and $T_\ell(A_K)/T_0$. In particular, every element of $I(\overline{v})$ acts unipotently on $T_\ell(A_K)$. He further showed that given a continuous linear representation such as ρ_ℓ (with a mild hypothesis C_ℓ on k), there exists an open subgroup H of $I(\overline{v})$ such that $\rho_\ell(s)$ is unipotent for all $s \in H$ ([96], page 515, or [42], I, 1.1). Using the semi-stable reduction theorem, Grothendieck was able to answer positively, for abelian varieties, questions of Serre and John Tate in [96], Appendix, Problem 2, regarding the characteristic polynomial of the image under ρ_ℓ of a Frobenius element $F_v \in D(\overline{v})$ when k is finite (see [42], IX, 4.3). We refer to the survey [47] for more information on these questions.

6. The Néron model of a Jacobian

Let R be a discrete valuation ring with field of fractions K and residue field k. Let X_K/K be a proper smooth geometrically connected K-scheme. Associated with X_K/K is the Picard scheme $\mathrm{Pic}_{X_K/K}$, whose tangent space at the identity is $H^1(X_K, \mathcal{O}_{X_K})$ ([49], 5.11). This group scheme may not be reduced when K is of characteristic $p > 0$. Its connected component of the identity $\mathrm{Pic}^0_{X_K/K}$ is a proper scheme ([14], 8.4/3) and is smooth if and only if it has dimension $\dim_K(H^1(X_K, \mathcal{O}_{X_K}))$. Assume now that $\mathrm{Pic}^0_{X_K/K}/K$ is smooth. Associated to the *Picard variety* $\mathrm{Pic}^0_{X_K/K}$ is its dual abelian variety (that is, the connected component of the identity of the Picard scheme of $\mathrm{Pic}^0_{X_K/K}$), called the *Albanese variety* of X_K/K.

When X_K/K is a proper smooth geometrically connected curve over K of genus $g > 0$, the Picard variety is called the *Jacobian* J_K/K of X_K/K, has dimension g, and is principally polarized (so that in particular, it is isomorphic over K to its dual abelian variety). Since X_K/K has a minimal regular model X/S, and the Jacobian J_K/K has a (strong) Néron model J/S, it is natural to wonder what are the relationships between these two canonical objects. The main reference for this type of questions is the article [91] by Raynaud, and chapters 8 and 9 in [14]. An account of results in the case where X_K/K is an elliptic curve is found in [59], 10.2, or [100], Chapter IV. When X_K/K has genus 1 with no K-rational point, the models of X_K and of its Jacobian are compared in [61]. When X_K/K has dimension greater than one, the relationship between a semi-factorial model of X_K/K and properties of the Néron model of the Picard variety of X_K/K are explored in articles of Cédric Pépin ([87] and [88]).

Assume now that k is algebraically closed. Fix a smooth proper geometrically connected curve X_K/K of genus $g > 0$, and let X/S be a proper flat model of X_K such that X is regular. Then the special fiber X_k/k is the union of irreducible components C_i/k, $i = 1, \dots, n$, proper over k, of multiplicity r_i in X_k, and of geometric genus $g(C_i)$. The *intersection matrix* M associated with the model X is the symmetric matrix $((C_i \cdot C_j)_{1 \le i,j \le n})$, where $(C_i \cdot C_j)$ is the intersection number of the curves C_i and C_j on the regular scheme X. Let $R \in \mathbb{Z}^n$ denote the transpose of the vector $(r_1, \dots, r_n)$. Let $r := \gcd(r_1, \dots, r_n)$. Then $MR = (0, \dots, 0)^t$, and R/r generates the kernel of the linear map $M : \mathbb{Z}^n \to \mathbb{Z}^n$.

It is always possible to perform a sequence of blowups of closed points on X to obtain a new regular model with the following properties:

(1) Each reduced irreducible component C_i of the special fiber is smooth of (geometric) genus $g(C_i) \ge 0$.

(2) Each intersection number $(C_i \cdot C_j)$ equals either 0 or 1 when $i \ne j$.

We associate to the special fiber X_k/k a *connected graph* G whose vertices are the components C_i, $i = 1, \dots, n$, and we link in G two distinct components C_i and C_j by exactly $(C_i \cdot C_j)$ edges. The main topological invariant of the graph G is its Betti number $\beta(G) := m - n + 1$, where m is the total number of edges of G. Recall that the *degree* of a vertex in a graph is the number of edges attached to the vertex. Letting $d_i := \sum_{j \ne i}(C_i \cdot C_j)$ denote the degree of C_i in G, we find that $2\beta(G) - 2 = \sum_{i=1}^{n} d_i - 2$. The adjunction formula ([59], 9.1.37) implies that:

$$2g = 2\beta(G) + 2\sum_{i=1}^{n} g(C_i) + \sum_{i=1}^{n}(r_i - 1)(d_i - 2 + 2g(C_i)).$$

Theorem 6.1. *Let R be a discrete valuation ring with algebraically closed residue field k. Let X_K/K be a smooth proper geometrically connected curve of genus $g > 0$. Assume that X_K/K has a regular model X/S with the above properties* (1) *and* (2)*, and with $r = 1$. Let J/S denote the Néron model of the Jacobian of X_K. Then*

(a) *The abelian rank of J_K is $a_K := \sum_{i=1}^{n} g(C_i)$.*
(b) *The toric rank of J_K is $t_K := \beta(G)$.*
(c) *The group of components $\Phi_J(k)$ is isomorphic to the torsion subgroup of the group $\mathbb{Z}^n/\mathrm{Im}(M)$. More precisely, let $D = \mathrm{diag}(f_1, \dots, f_{n-1}, 0)$ denote the Smith normal form of M, with $f_i \mid f_{i+1}$ for $i = 1, \dots, n-2$. Then $\Phi_J(k)$ is isomorphic to $\prod_{i=1}^{n-1} \mathbb{Z}/f_i\mathbb{Z}$.*

Parts (a) and (b) follow from a very deep theorem of Raynaud where he first shows that the connected component of the identity $\mathrm{Pic}^0_{X/S}$ of the Picard functor $\mathrm{Pic}_{X/S}$ is represented by a smooth separated scheme over S, and then identifies it with the connected component of the identity of the Néron model of the Jacobian J_K ([14], 9.5, Theorem 4, and see also [61], 7.1, for remarks when k is only separably closed). It follows that to compute the invariants a_K and t_K, it suffices to understand the structure of group scheme of the special fiber of $\mathrm{Pic}^0_{X/S}$, which is nothing but $\mathrm{Pic}^0_{X_k/k}$ ([14], 9.2). Part (c) is another theorem of Raynaud ([14], 9.6, Theorem 1).

Remark 6.2 Given a field K and an abelian variety A_K/K, there exists a curve X_K/K with Jacobian J_K/K endowed with a surjective K-morphism $J_K \to A_K$. This theorem was originally proved when K is algebraically closed by Matsusaka [71]. See [77], 10.1, for a proof when K is infinite, and [38], 2.5, in general. When K is the field of fractions of a discrete valuation ring R, the morphism $J_K \to A_K$ induces a natural homomorphism $J \to A$ between the Néron models of J_K and A_K. One can show that if J has semi-stable reduction, then so does A ([14], 7.4/2). It follows that the proof of the semi-stable reduction theorem for a general abelian variety A_K/K can be reduced to proving it for Jacobians.

The natural homomorphism $J \to A$ induces a group homomorphism $\Phi_J \to \Phi_A$. Unfortunately, this group homomorphism need not be surjective (see, e.g., [14], 7.5/7), and thus in general one cannot expect to be able to infer properties of Φ_A from properties of Φ_J.

Remark 6.3 We remark here that to prove the Semi-Stable Reduction Theorem 5.4 for an abelian variety A_K/K, it suffices to prove the following fact: *For some ℓ invertible in R, the group $\Phi_A(\overline{k})[\ell]$ can be generated by t_K elements.* Indeed, let K_s and $\overline{K}$ denote the separable closure and the algebraic closure of the field K. Let ℓ be a prime invertible in R. The multiplication-by-ℓ morphism $[\ell] : A_K \to A_K$ is finite étale of degree $2g$. Thus, there exists a finite separable extension K_ℓ/K such that

$$A(K)[\ell] \subseteq A(K_\ell)[\ell] = A(K_s)[\ell] = A(\overline{K})[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}.$$

It follows from Theorem 5.2 (b) and the above fact that

$$\dim_{\mathbb{Z}/\ell\mathbb{Z}}(A(K)[\ell]) \leq 2t_K + 2a_K$$

and, more importantly since $g = u_{K_\ell} + t_{K_\ell} + a_{K_\ell}$,

$$\dim_{\mathbb{Z}/\ell\mathbb{Z}}(A(K_\ell)[\ell]) = 2g \leq 2t_{K_\ell} + 2a_{K_\ell}.$$

An immediate consequence is that $u_{K_\ell} = 0$, and thus we have shown the existence of a finite separable extension over which A_K/K acquires semi-stable extension.

Remark 6.4 Artin and Winters used 6.2 and 6.3 to give a proof of the semi-stable reduction theorem in [3], as we now explain. Let X_K/K be a smooth proper geometrically connected curve with a K-rational point and with a regular model X satisfying (1) and (2) above. Let J/S denote the Néron model of the Jacobian of X_K/K. Raynaud's Theorem 6.1 (c) gives a completely combinatorial description of the group of components of a Jacobian in term only of the intersection matrix M (the existence of a K-rational point implies that $r = 1$). Artin and Winters prove the following fact in [3]: As before, write $X_k = \sum_{i=1}^n r_i C_i$. Write as in Theorem 6.1 that $\Phi_J(\overline{k}) = \prod_{i=1}^{n-1} \mathbb{Z}/f_i\mathbb{Z}$ with $f_i \mid f_{i+1}$ for $i = 1, \ldots, n-2$. When $0 \leq \beta(G) < n-1$, define Υ such that

$$\prod_{i=1}^{n-1} \mathbb{Z}/f_i\mathbb{Z} = \Upsilon \times \prod_{i=n-\beta(G)}^{n-1} \mathbb{Z}/f_i\mathbb{Z}.$$

When $\beta(G) \geq n-1$, set $\Upsilon = (0)$. *Then Artin and Winters show that there exists a constant $c = c(g)$ depending on g only such that $|\Upsilon| \leq c$.* Hence, it follows immediately that there exists a prime ℓ invertible in R such that $\Phi_J(\overline{k})[\ell]$ can be generated by t_K elements. The constant c was later made explicit in [69], 1.5, and [68], 4.21. A functorial variant of this result applicable to any abelian variety is suggested in 7.3.

Explicit regular models of curves have been computed in many important cases. When X_K/K is an elliptic curve, Tate's algorithm [104] (when k is perfect) and Szydlo's algorithm [103] (when k is imperfect), take as input a Weierstrass equation for X_K with coefficients in R, and output an equation with discriminant having minimal valuation among all possible R-integral Weierstrass equations for X_K/K. The algorithms also produce an explicit description of the minimal regular model X/S of X_K/K. When k is algebraically closed, the combinatorics of the possible special fibers of X/S is encoded in what is called a *Kodaira type* (see e.g., [100] page 365, or [59], pp. 486-489, when k is perfect). When k is separably closed but imperfect, several new types of reduction can occur in addition to the classical Kodaira types. These new types are described for instance in [61], Appendix A. There are also new reduction types for curves of genus 1 listed in [61], and it would be interesting to show that these new combinatorial types all arise as reduction types of curves of genus 1.

Assume now that k is algebraically closed with $\mathrm{char}(k) \neq 2$. When X_K/K is a curve of genus 2 given by an explicit hyperelliptic equation, Liu's algorithm [58] produces an explicit description of the minimal regular model X/S of X_K/K. Note that in this case, there are over 100 possible different types of reduction. The description of a regular model of the modular curve $X_0(N)/\mathbb{Q}_p$ for $p \geq 5$ dividing N, of $X_1(p)/\mathbb{Q}_p$, and of the Fermat curve $F_p/\mathbb{Q}_p$ and $F_p/\mathbb{Q}_p(\zeta_p)$, can be found in [29], [31], [26], [21], and [75], respectively.

Once the minimal regular model of an elliptic curve is computed explicitly using Tate's algorithm, the following theorem provides a description of the Néron model of the elliptic curve ([14], 1.5, Proposition 1, or [59], 10.2.14):

Theorem 6.5. *Let R be a discrete valuation ring. Let X_K/K be an elliptic curve. Consider a Weierstrass equation for X_K with coefficients in R and whose discriminant has minimal valuation among all such equations. Let X_0/S denote the closed subscheme of $\mathbb{P}^2_S/S$ defined by the Weierstrass equation. Then X_0 has at most one singular point. Let E_0 denote the largest open subscheme of X_0 such that E_0/S is smooth.*

(a) *If X_0 is regular, then E_0/S is the Néron model of X_K over S.*
(b) *If X_0 has a singular point, let $X \to X_0$ denote the desingularization of X_0, obtained for instance using Tate's algorithm when k is perfect. Let E denote the largest open set of X such that E/S is smooth. Then E/S is the Néron model of X_K over S, and the scheme E_0/S can be identified with an open subscheme of E, namely the scheme E^0/S defined in* 3.9.

We can view the projective scheme X/S in the above theorem as a natural regular compactification of the Néron model E/S. In higher dimension, the existence of a semi-factorial projective compactification of a Néron model is proved in [88], 6.4. Compactifications of Néron models of Jacobians of stable curves are considered in [19]. The question regarding the existence of a good compactification of the Néron model on page 318 of [56] is still open.

Example 6.6 The theory of models of curves or of group schemes over bases S of dimension greater than 1 is quite subtle. We refer to [27], [45], or [107], for such questions, and we mention here only an example of Raynaud.

Assume that $S = \mathrm{Spec}\, R$ is normal, so that each point $s \in S$ of codimension 1 is such that the local ring $\mathcal{O}_{S,s}$ is a discrete valuation ring. Let E_K/K be an elliptic curve. For each $\mathrm{Spec}\, \mathcal{O}_{S,s}$ with s of codimension 1, we can construct $E^0(s)/\mathrm{Spec}\, \mathcal{O}_{S,s}$, the connected component of the identity of the Néron model $E(s)/\mathrm{Spec}\, \mathcal{O}_{S,s}$ of E_K/K over $\mathrm{Spec}\, \mathcal{O}_{S,s}$. It is then natural to ask whether there exists a smooth group scheme E^0/S such that for each $s \in S$ of codimension 1, the base change $E^0 \times_S \mathrm{Spec}\, \mathcal{O}_{S,s}$ is isomorphic over $\mathrm{Spec}\, \mathcal{O}_{S,s}$ to the group scheme $E^0(s)/\mathrm{Spec}\, \mathcal{O}_{S,s}$. It turns out that the answer to this question can be negative. Indeed, Raynaud exhibits a normal scheme S of dimension 3 and an elliptic curve E_K/K where no such group scheme E^0/S exists ([92], Remarque XI 1.17 (2), page 174-175. Page 175 is missing in the book [92]. A copy of the correct page 175 is available on the website [65]).

7. The group of components

Let R be a discrete valuation ring. Let A_K be an abelian variety over K with dual A'_K. Denote by A and A' the corresponding (strong) Néron models and by Φ_A and $\Phi_{A'}$ their groups of components. In [42], Exp. VII-IX, Grothendieck used the notion of biextension invented by Mumford to investigate how the duality between A_K and A'_K is reflected on the level of Néron models. In fact, the essence of the relationship between A and A' is captured by a bilinear pairing

$$\langle\ ,\ \rangle \colon \Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

introduced in [42], Exp. IX, 1.2, and which represents the obstruction to extending the Poincaré bundle $\mathcal{P}_K$ on $A_K \times A'_K$ to a biextension of $A \times A'$ by $\mathbb{G}_{\mathrm{m},R}$.

Grothendieck conjectured in [42], Exp. IX, 1.3, that the pairing $\langle\ ,\ \rangle$ is perfect. The conjecture has been established in various cases, notably:

(a) by Grothendieck for ℓ-parts with ℓ prime and invertible in R, as well as in the semi-stable reduction case; see [42], Exp. IX, 11.3 and 11.4. See also [7] and [110].
(b) by Lucile Bégueri [6] for valuation rings R of mixed characteristic with perfect residue fields,
(c) by William McCallum [76] for finite residue fields,
(d) by Siegfried Bosch [11] for abelian varieties with potentially purely multiplicative reduction, again for perfect residue fields.
(e) by Bosch and Lorenzini [13] for the Jacobian of a smooth proper geometrically connected curve X/K having a K-rational point when X_K/K has a regular model X/S such that every irreducible component of the special X_k is geometrically reduced (this condition is automatic when k is algebraically closed, but may also hold in many cases when k is separably closed and imperfect). See also [67] and [87].

That the pairing is perfect in general when k is perfect has been announced by Takashi Suzuki in [102].

Using previous work of Bas Edixhoven [32] on the behavior of component groups under the process of Weil restriction, Alessandra Bertapelle and Bosch [8] gave the first counter-examples to Grothendieck's conjecture when the residue field k of R is not perfect. A counter-example where A_K is a Jacobian is given in [13], and a further counter-example where A_K is an elliptic curve can be found in [67].

Remark 7.1 When the pairing is perfect, we obtain as a consequence that the finite groups of components $\Phi_A(\overline{k})$ and $\Phi_{A'}(\overline{k})$ are isomorphic. There is no known proof of this result that does not rely on the perfectness of the pairing. It would be interesting to know whether it is always the case that $\Phi_A(\overline{k})$ and $\Phi_{A'}(\overline{k})$ are isomorphic. Examples of abelian varieties A_K/K that are not isomorphic to their dual are found for instance in [66], Remark 3.16.

Remark 7.2 The perfectness of Grothendieck's pairing does not imply that the étale group schemes Φ_A/k and $\Phi_{A'}/k$ are k-isomorphic. In particular, it does not imply that the groups $\Phi_A(k)$ and $\Phi_{A'}(k)$ are isomorphic. However, we do not know of any example where $|\Phi_A(k)|$ and $|\Phi_{A'}(k)|$ are not equal. As shown in [66], 4.3, when k is finite, $|\Phi_A(k)| = |\Phi_{A'}(k)|$.

In the particular case where k is finite, the integer $|\Phi_A(k)|$ is called the *Tamagawa number* of A_K/K at (π). When K is a number field with ring of integers $\mathcal{O}_K$, the product of the Tamagawa numbers at all maximal ideals of $\mathcal{O}_K$ is a term appearing in the Birch and Swinnerton-Dyer conjectural formula for the leading term in the L-function of A_K/K (see [44], F.4.1.6). Another term appearing in this formula, the Tate-Shafarevich group, can be given an integral cohomological interpretation using the Néron model, up to some 2-torsion ([73], Appendix). Further information on $\Phi_A(k)$ in general can be found in [13].

Let ℓ be a prime invertible in R. The possible abelian groups that arise as the ℓ-part of a group of component are completely understood thanks to the work of Edixhoven [33]. The p-part of the group $\Phi_A(\overline{k})$ is much less understood and much more difficult to study, as it is not related in general with the p-torsion in $A_K(K)$, contrary to the case of the prime-to-p part of $\Phi_A(\overline{k})$.

The study of the group Φ_A has involved since the very beginning of the theory in [42] the use of rigid geometry, in the form of the rigid analytic uniformization of an abelian variety, further refined in the work of Bosch and Xavier Xarles in [16]. We discuss below just enough of this theory to be able to define a functorial subgroup of $\Phi_A(\overline{k})$ and state the main open question regarding its size. We follow the notation in [16].

Assume that R is a complete discrete valuation ring with algebraically closed residue field k. There exist a semi-abelian K-group scheme E_K and a lattice M_K in E_K of maximal rank such that as rigid K-group, A_K is isomorphic to the quotient E_K/M_K. The semi-abelian variety E_K is the extension of an abelian variety B_K with potentially good reduction by a torus T_K. Each of these objects has both a (usual) Néron model, and a formal Néron model (see [15]), with the property that the group of components of the formal Néron model is isomorphic to the group of components of the Néron model. Thus we obtain a natural map of groups of components $\Phi_E \to \Phi_A$ from the rigid analytic morphism $E_K \to A_K$.

The torus T_K contains a maximal split subtorus T'_K, of dimension t_K. The Néron mapping property induces morphisms of Néron models and homomorphisms of groups of components $\Phi_{T'} \to \Phi_T \to \Phi_E$. Let us define $\Sigma \subseteq \Phi_A(\overline{k})$ as the image of $\Phi_{T'}(\overline{k})$ under the composition $\Phi_{T'} \to \Phi_T \to \Phi_E \to \Phi_A$. (Similarly, Φ_T, $(\Phi_T)_{\mathrm{tors}}$, Φ_E, and $(\Phi_E)_{\mathrm{tors}}$, can be used to define functorial subgroups of Φ_A, and we refer to [16] for a study of these subgroups.)

Since T'_K is a split torus of rank t_K, we find that $\Phi_{T'}(\overline{k}) \cong \mathbb{Z}^{t_K}$. It follows that the subgroup Σ can be generated by t_K elements.

Conjecture 7.3. *The order of the quotient $\Phi_A(\overline{k})/\Sigma$ can be bounded by a constant c depending on u_K only.*

In particular, when $t_K = 0$, then $|\Phi_A(\overline{k})|$ itself would be bounded by a constant c depending on u_K only. As mentioned in 6.4, this statement is true for the Jacobian of a curve X_K/K having a K-rational point (and $t_K = 0$). Joseph Silverman in [98] proved that the prime-to-p part of $|\Phi_A(\overline{k})|$ is bounded by a constant depending only on g when A_K/K has potentially good reduction and suggested that the same result would remain true for the full group $\Phi_A(\overline{k})$ under that assumption. That a subgroup such as Σ exist in general is suggested in [68], 1.8. It follows from [16], 5.9, that the prime-to-p part of $|\Phi_A(\overline{k})/\Sigma|$ is bounded by a constant depending only on u_K.

Let L/K be the finite Galois extension minimal with the property that A_L/L has semi-stable reduction over L. It follows for instance from [69], 3.1, that if q is prime and divides $[L : K]$, then $q \leq 2g + 1$. The extension L/K is not easy to determine in practice when it is *wild*, that is, when p divides $[L : K]$ (see for instance [50], [51], and [57]). It is shown in [60], 1.8, that if $t_K = 0$, then $[L : K]^2$ kills the group $\Phi_A(\overline{k})$, so that when $t_K = 0$, the group $\Phi_A(\overline{k})$ has a non-trivial p-part only when L/K is wild.

This is only a short survey, and much more could be said on the extension L/K and on the natural morphism $A \times_R R_L \to A^{ss}$, where A^{ss}/R_L is the Néron model of A_L/L over the integral closure R_L of R in L. Some of these topics were covered by Johannes Nicaise at the conference in Bordeaux. We refer the reader to the book [43] for a complete exposition, as well as to the recent papers [20] and [34].

Acknowledgements

This text is a written version of a series of lectures delivered at the Sino-French Conference in Algebraic and Arithmetic Geometry in Bordeaux, France, May 23-27, 2016. It is my pleasure to thank Qing Liu for the invitation to speak at the conference. It is also my pleasure to thank Siegfried Bosch, Brian Conrad, Qing Liu, Johannes Nicaise, Michel Raynaud, and the referee, for useful comments.

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