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Models of Curves and Finite Covers

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Abstract. Let K be a discrete valuation field with ring of integers $\mathcal{O}_K$. Let $f: X \to Y$ be a finite morphism of curves over K. In this article, we study some possible relationships between the models over $\mathcal{O}_K$ of X and of Y. Three such relationships are listed below.

Consider a Galois cover $f: X \to Y$ of degree prime to the characteristic of the residue field, with branch locus B. We show that if Y has semi-stable reduction over K, then X achieves semi-stable reduction over some explicit tame extension of $K(B)$. When K is strictly henselian, we determine the minimal extension L/K with the property that X_L has semi-stable reduction.

Let $f: X \to Y$ be a finite morphism, with $g(Y) \geqslant 2$. We show that if X has a stable model $\mathcal{X}$ over $\mathcal{O}_K$, then Y has a stable model $\mathcal{Y}$ over $\mathcal{O}_K$, and the morphism f extends to a morphism $\mathcal{X} \to \mathcal{Y}$.

Finally, given any finite morphism $f: X \to Y$, is it possible to choose suitable regular models $\mathcal{X}$ and $\mathcal{Y}$ of X and Y over $\mathcal{O}_K$ such that f extends to a finite morphism $\mathcal{X} \to \mathcal{Y}$? As was shown by Abhyankar, the answer is negative in general. We present counterexamples in rather general situations, with f a cyclic cover of any order $\geqslant 4$. On the other hand, we prove, without any hypotheses on the residual characteristic, that this extension problem has a positive solution when f is cyclic of order 2 or 3.

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Let $\mathcal{O}_K$ be a Dedekind domain with field of fractions K. Let $f: X \to Y$ be a finite morphism of projective, smooth, and geometrically connected curves over Spec(K). In this paper, we study some possible relationships between the models of X and of Y. In the first part of the paper, we look at semi-stable and stable models, while in the second part we investigate regular models.

Let us describe the content of this paper. Definitions and standard facts about models are reviewed in the first section. Let $B \subset Y$ denote the branch locus of f, and let $K(B)$ be the compositum, in an algebraic closure of K, of the residue fields of points of B. In the second section, we consider a Galois cover $X \to Y$ of degree prime to p and show that, if Y has semi-stable reduction over K, then X achieves semi-stable reduction over an explicit tame extension of the field $K(B)$ (Theorem 2.3). When K is strictly Henselian, there exist extensions L_X/K and

L_Y/K minimal with the property that X_{L_X} and Y_{L_Y} have semi-stable reduction. In the third section, we assume that K is strictly Henselian and strengthen the result obtained in the second section to show that for a Galois cover of degree prime to p, the p-part of L_X/K is equal to the compositum of the p-part of L_Y/K and the p-part of $K(B)/K$ (Corollary 3.2). We later completely describe L_X in terms of $L_Y K(B)$ and some vertical ramification data (Theorem 3.9).

Let $f: X \to Y$ be a finite morphism, with $g(Y) \geqslant 2$. In the fourth section, we show that if X has a stable model $\mathcal{X}/\mathcal{O}_K$, then Y has a stable model $\mathcal{Y}/\mathcal{O}_K$, and f extends to a (not necessarily finite) morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ (Proposition 4.4). As a corollary, we give a new proof a theorem of Lange which states that if X has good reduction, then Y has good reduction.

Given any finite morphism $f: X \to Y$ as above, it is interesting in some situations to be able to compare the regular models of X and Y over $\mathcal{O}_K$. In particular, it is natural to wonder whether it is possible to choose suitable regular models $\mathcal{X}/\mathcal{O}_K$ and $\mathcal{Y}/\mathcal{O}_K$ of X and Y such that the morphism f extends to a finite morphism $\varphi: \mathcal{X} \to \mathcal{Y}$. Unfortunately, this is not always possible, as was shown by Abhyankar in [Ab2]. In the sixth section of this paper, we give a local obstruction to this extension problem, and then present counterexamples in rather general situations, with f a cyclic cover of any order $\geqslant 4$. On the other hand, we prove in the last section that when f is cyclic of order 2 or 3, then suitable regular models of X and Y can be chosen such that f extends to a finite morphism between these regular models. The difficulties in the proof of this statement are due to the fact that we do not make any assumption on the residue characteristics. Three preparatory lemmas for Sections 6 and 7 are stated for the convenience of the reader in a separate section, Section 5. The reader may refer to these lemmas on models that dominate regular models as needed while reading the results of Sections 6 and 7.

The properties of models we are concerned with in this article are most often local on $\mathrm{Spec}(\mathcal{O}_K)$. Thus, most of time and unless stated otherwise, $\mathcal{O}_K$ will be a discrete valuation ring. Then v denotes the normalized valuation of K, t a uniformizing element, $k = \mathcal{O}_K/(t)$ the residue field, and $p = \mathrm{char}(k) \geqslant 0$. The special fiber $\mathcal{Y} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(k)$ of a model $\mathcal{Y}/\mathcal{O}_K$ will be denoted by $\mathcal{Y}_s$. The generic fiber $\mathcal{Y} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(K)$ will be denoted by $\mathcal{Y}_K$. For any finite extension L/K, we denote by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. It is a Dedekind domain with finitely many maximal ideals ([Z-S], V.9, Theorem 21). With the exception of section 4, all covers $X \to Y$ considered are Galois. In the last three sections, the residue field k will be assumed to be perfect or algebraically closed.

1. Basic Facts on Models

Let us begin with a short review of facts and notation pertaining to models. In this paper we call a *curve over K* a projective, smooth, and geometrically connected curve over K. A *model* $\mathcal{X}$ of X is an integral normal scheme $\mathcal{X}$, projective and flat over $\mathrm{Spec}(\mathcal{O}_K)$, such that the generic fiber $\mathcal{X}_K$ is isomorphic over $\mathrm{Spec}(K)$ with the given curve X/K.

LEMMA 1.1. *Let X/K be a curve, and let $\mathcal{X}/\mathcal{O}_K$ be an integral scheme such that $\mathcal{X}_K \simeq X$. If $\mathcal{X}_s$ is reduced, then $\mathcal{X}$ is normal.*

Proof. The statement is local and we are reduced to the case of an open $\mathrm{Spec}(A)$, with $A/(t)$ reduced and $A[1/t]$ integrally closed. Let $\alpha \in \mathrm{Frac}(A)$ be an element integral over A. Then $\alpha \in A[1/t]$, and thus it can be written as a/t^i, with $a \in A$ and $a \notin (t)$ if $i > 0$. Let $(a/t^i)^n + \cdots + a_0 = 0$ be an integral relation for α over A. If $i > 0$, then a^n is congruent to zero modulo (t) and, hence, $a \in (t)$. Contradiction. Thus $i \leqslant 0$ and $\alpha \in A$.

A finite surjective morphism of schemes is called a *cover*. A cover $S \to T$ of integral normal schemes is called a *Galois cover with group G* if the extension of function fields $K(T) \to K(S)$ is Galois with group G, and if T is isomorphic to the quotient S/G. Let $f: X \to Y$ be a cover of curves over $\mathrm{Spec}(K)$ and let $\mathcal{Y}$ be a model of Y over $\mathcal{O}_K$. Denote by $N(\mathcal{Y}, K(X))$ the normalization of the scheme $\mathcal{Y}$ in $K(X)$, and by $\varphi: N(\mathcal{Y}, K(X)) \to \mathcal{Y}$ the canonical morphism. By construction, φ is a finite morphism (thus $N(\mathcal{Y}, K(X))$ is of finite type over $\mathrm{Spec}(\mathcal{O}_K)$) if either f is a separable morphism, or $\mathcal{O}_K$ is an excellent ring. When we consider $N(\mathcal{Y}, K(X))$, we shall always assume that either of these hypotheses hold. When no confusion may result, we may denote the model $N(\mathcal{Y}, K(X))$ of X/K simply by $\mathcal{X}$.

Recall that the morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ is unramified at $x \in \mathcal{X}$ if and only if $(\Omega_{\mathcal{X}/\mathcal{Y}})_x = 0$ ([A-K], 3.3). Hence the set $\mathcal{R}$ of ramified points of φ is a closed set of $\mathcal{X}$, called the *ramification locus*. The map φ being a finite morphism, the image of $\mathcal{R}$ in $\mathcal{Y}$ is a closed set $\mathcal{B}$ (endowed with the reduced induced structure) called the *branch locus* of $\mathcal{X}$ over $\mathcal{Y}$. If $f: X \to Y$ is separable, then the morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ is generically unramified and, thus, $\mathcal{B} \neq \mathcal{Y}$. Hence, $\mathcal{B}$ is either empty, or is the union of finitely many components of codimension 1 and of finitely many isolated points. As we shall recall in this paper, the singularities of the normal model $N(\mathcal{Y}, K(X))$ are intimately linked with the branch locus of the given map $f: X \to Y$.

Let D be any irreducible divisor of $\mathcal{Y}$. We say that D is a *horizontal* divisor if D dominates $\mathrm{Spec}(\mathcal{O}_K)$. Otherwise D is contained in the special fiber $\mathcal{Y}_s$ and we say that D is a *vertical* divisor. An irreducible divisor D of $\mathcal{Y}$ is a horizontal divisor in $\mathcal{B}$ if and only if D is the closure in $\mathcal{Y}$ of a point of Y that belongs to the branch locus of the map $f: X \to Y$. An irreducible vertical divisor D is an irreducible component of $\mathcal{Y}_s$ and, as such, has a multiplicity $r_D \geqslant 1$: Let ξ be the generic point of D. Then $\mathcal{O}_{\mathcal{Y},\xi}$ is a regular local ring of dimension 1. Thus it has a (normalized) valuation ν, and we define r_D to be $\nu(t)$, where t is an uniformizing parameter of $\mathcal{O}_K$. The irreducible vertical divisor D is contained in $\mathcal{B}$ if and only if either the map $\varphi^{-1}(D) \to D$ is not separable, or an irreducible component E of $\varphi^{-1}(D)$ has multiplicity in $\mathcal{X}_s$ equal to $r_E = r_D \cdot e_{E/D}$, with $e_{E/D} > 1$. Let η denote the generic point of E in $\mathcal{X}_s$. Then $e_{E/D}$ is the ramification index of η over ξ. An irreducible divisor D is *regular* if it is a regular scheme. An irreducible horizontal divisor D is said to be *smooth* if the map $D \to \mathrm{Spec}(\mathcal{O}_K)$ is unramified.

1.2. REGULAR MODELS

We call a model $\mathcal{Y}/\mathcal{O}_K$ a *regular* model if the scheme $\mathcal{Y}$ is regular. When $\mathcal{Y}$ is a regular model, the finite morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is flat [Mat], 18.H. In this case, the Theorem of the Purity of the Branch Locus [A-K], 6.8, shows that $\mathcal{B}$ is either empty, or equal to $\mathcal{Y}$, or pure of codimension 1 (i.e., is the union of finitely many divisors of $\mathcal{Y}$).

Let $y \in \mathcal{Y}$ be a closed point. Let $D_1, \ldots, D_r$ be irreducible divisors of $\mathcal{Y}$ containing y. Denote by u_i a local equation of D_i at y. We say that the divisors $D_1, \ldots, D_r$ have *normal crossings* at y (or meet transversally at y) if $u_1, \ldots, u_r$ can be completed to a system of parameters of $\mathcal{O}_{\mathcal{Y},y}$. This definition implies that y is a regular point of D_i. Note however that the residue field of y may not be separable over k.

The following lemmas are well known. We recall them here for the convenience of the reader.

LEMMA 1.3. *Let $\mathcal{Y}/\mathcal{O}_K$ be a model of a curve Y/K. Let $Q \in Y$ be a point rational over an extension L/K unramified over K. If $y \in \overline{\{Q\}} \cap \mathcal{Y}_s$ is regular in $\mathcal{Y}$, then y is regular in $\mathcal{Y}_s$.*

Proof. The valuative criterion for properness shows that for each maximal ideal $\mathfrak{p}$ of $\mathcal{O}_L$, there exists a map $\mathrm{Spec}(\mathcal{O}_{L,\mathfrak{p}}) \to \mathcal{Y}$ that extends the map $\mathrm{Spec}(L) \to \mathcal{Y}$ corresponding to Q. These maps glue together to give a map $\mathrm{Spec}(\mathcal{O}_L) \to \mathcal{Y}$. Since $\mathcal{O}_L$ is unramified over $\mathcal{O}_K$, the extension L/K is separable. Thus $\mathcal{O}_L$ is a finite $\mathcal{O}_K$-module, and the image of $\mathrm{Spec}(\mathcal{O}_L)$ in $\mathcal{Y}$ is closed. Hence, the horizontal divisor $\overline{\{Q\}}$ is the image of the map $\mathrm{Spec}(\mathcal{O}_L) \to \mathcal{Y}$ over $\mathrm{Spec}(\mathcal{O}_K)$. Let $y \in \overline{\{Q\}} \cap \mathcal{Y}_s$, image of a point $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_L)$. The kernel I of the associated ring homomorphism $\mathcal{O}_{\mathcal{Y},y} \to \mathcal{O}_{L,\mathfrak{p}}$ is a prime of height 1. Since $\mathcal{O}_{\mathcal{Y},y}$ is regular and, hence, locally factorial, the prime I is principal, say $I = (w)$. Note now that $\mathcal{O}_{\mathcal{Y},y}/(w, t)$ is isomorphic to $\mathcal{O}_{L,\mathfrak{p}}/(t)$. Since $t\mathcal{O}_{L,\mathfrak{p}}$ is equal to the maximal ideal of $\mathcal{O}_{L,\mathfrak{p}}$ by hypothesis, we find that (w, t) is maximal. Thus w is a local parameter on $\mathcal{Y}_s$ at y.

LEMMA 1.4. *Let $\mathcal{Y}/\mathcal{O}_K$ be a regular model of a curve Y/K. Let C and D be irreducible components in $\mathcal{Y}_s$, of multiplicities r_C and r_D, respectively. Let $y \in \mathcal{Y}_s$, and let $\mathcal{Y}'$ denote the model of Y obtained by blowing up $\mathcal{Y}$ at y. Let $E \subset \mathcal{Y}'$ denote the exceptional divisor.*

(a) *If y is a regular point of C that does not belong to any other component of $\mathcal{Y}_s$, then the multiplicity of E in $\mathcal{Y}'_s$ equals r_C.*

(b) *If $y \in C \cap D$ and does not belong to any other components of $\mathcal{Y}_s$, and if C and D intersect transversally at y, then the multiplicity of E in $\mathcal{Y}'_s$ is $r_C + r_D$.*

Proof. Omitted.

1.5. SEMI-STABLE MODLES

Let $\mathcal{O}_K$ be any Dedekind domain. In this section we call a *curve over $\mathcal{O}_K$* a normal (not necessarily connected) scheme $\mathcal{Y}$ over $\mathcal{O}_K$, flat, of finite type and of relative dimension 1, with smooth generic fiber. We say that $\mathcal{Y}$ is *semi-stable* (over $\mathcal{O}_K$) if every geometric fiber of $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_K)$ is reduced and has at most ordinary double points as singularities ([BLR], 9.2/6). An $\mathcal{O}_K$-scheme finite étale over a semi-stable curve is semi-stable. Let $\mathcal{O}_L$ be any Dedekind domain containing $\mathcal{O}_K$. If $\mathcal{Y}$ is semi-stable over $\mathcal{O}_K$, then $\mathcal{Y}_{\mathcal{O}_L}$ is semi-stable over $\mathcal{O}_L$. Indeed, we only need to check that $\mathcal{Y}_{\mathcal{O}_L}$ is normal, and this fact follows from $\mathcal{Y}_L$ being normal and the closed fibers of $\mathcal{Y}_{\mathcal{O}_L}$ being reduced (1.1). Conversely, if $\mathcal{Y}_{\mathcal{O}_L}$ is semi-stable and if $\mathcal{O}_L$ is finite over $\mathcal{O}_K$, then $\mathcal{Y}$ is obviously semi-stable over $\mathcal{O}_K$. Let us recall the following proposition due to Raynaud.

PROPOSITION 1.6. *Let $\mathcal{O}_K$ be a Dedekind domain. Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a cover of curves over $\mathcal{O}_K$. Assume that $\mathcal{X}$ is semi-stable over $\mathcal{O}_K$. Then $\mathcal{Y}$ is also semi-stable over $\mathcal{O}_K$. Moreover, if $\mathcal{X}$ is smooth at a point x, then $\mathcal{Y}$ is smooth at $\varphi(x)$.*

Proof. The statement is local on $\mathrm{Spec}(\mathcal{O}_K)$, so we may assume that $\mathcal{O}_K$ is local. Since the special fiber of $\mathcal{X}$ is reduced, all irreducible components of $\mathcal{Y}_s$ have multiplicity 1. Thus $\mathcal{Y}_s$ is also reduced. Let $\widehat{\mathcal{O}}_K$ be the completion of $\mathcal{O}_K$. Then the special fiber of $\mathcal{Y}_{\widehat{\mathcal{O}}_K} \to \mathrm{Spec}(\widehat{\mathcal{O}}_K)$ is also reduced (since it is isomorphic to $\mathcal{Y}_s$). Since in addition the generic fiber of $\mathcal{Y}_{\widehat{\mathcal{O}}_K}$ is normal, then $\mathcal{Y}_{\widehat{\mathcal{O}}_K}$ is normal (1.1.). Since $\mathcal{X}_{\widehat{\mathcal{O}}_K}$ is semi-stable, we are reduced to the case where $\mathcal{O}_K$ is complete. Under this additional hypothesis, the proposition is proven in [Ray], Appendice. The last statement concerning smoothness can be found in Raynaud's proof at the top of page 195.

Remark 1.7. We will use Proposition 1.6 in the next section in the case where $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is a Galois cover of degree prime to p. Proposition 1.6 can be proved under this additional hypothesis in the following simpler manner. Let G denote the Galois group of φ. For any ideal $\mathcal{I}$ of $\mathcal{O}_\mathcal{X}$ on which G acts, we have $\mathrm{H}^1(G, \mathcal{I}) = \{1\}$ since $|G|$ is invertible in $\mathcal{O}_\mathcal{X}$ ([Ser], VIII, Section 2, Corollary 1). Thus $(\mathcal{O}_\mathcal{X}/\mathcal{I})^G = \mathcal{O}_\mathcal{X}^G/\mathcal{I}^G$. Taking $\mathcal{I} = t\mathcal{O}_\mathcal{X}$, we find that $\mathcal{Y}_s = \mathcal{X}_s/G$. Since the quotient commutes with formal completion, it is easy to see that $\mathcal{Y}_s$ has at most ordinary double points, and that the image in $\mathcal{Y}_s$ of a smooth point $x \in \mathcal{X}_s$ is smooth.

A model $\mathcal{Y}/\mathcal{O}_K$ of a curve Y/K is said to be *semi-stable* if $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_K)$ is semi-stable. Note that since Y is geometrically connected, so is each closed fiber $\mathcal{Y}_s$, $s \in \mathrm{Spec}(\mathcal{O}_K)$. When such a model exists, we shall say that Y/K has *semi-stable reduction*. Given a possibly singular semi-stable model $\mathcal{Y}/\mathcal{O}_K$ of Y/K, the minimal desingularization $\mathcal{Z}$ of $\mathcal{Y}$ is a regular semi-stable model, and the exceptional locus of $\mathcal{Z} \to \mathcal{Y}$ consists of chains of smooth rational curves with self-intersection -2. Note that this definition of semi-stability implies that the residue field of each singular point of $\mathcal{Y}_s$, $s \in \mathrm{Spec}(\mathcal{O}_K)$, is a separable extension of $k(s)$.

A model $\mathcal{Y}/\mathcal{O}_K$ of a curve Y/K is said to be *semi-stable in a neighborhood of a point* $y \in \mathcal{Y}_s$ if there exists a dense open set $\mathcal{U}$ of $\mathcal{Y}$ containing y and such that $\mathcal{U} \to \mathrm{Spec}(\mathcal{O}_K)$ is semi-stable.

The semi-stable reduction theorem for curves ([D-M], Corollary 2.7) states that, given any curve Y/K, there exists a finite separable extension L/K with the property that Y_L has a semi-stable model $\mathcal{Y}'$ over $\mathcal{O}_L$. More generally, given any model $\mathcal{Y}/\mathcal{O}_K$ of Y/K, there exist such an extension L/K and a semi-stable model $\mathcal{Y}'/\mathcal{O}_L$ with $\mathcal{Y}'$ dominating $\mathcal{Y}_{\mathcal{O}_L}$. This last statement can be proved using rigid analytic methods (see for instance [B-L], Theorem 5.5, and step 3 in the proof of Lemma 7.3, page 377).

1.8. GOOD MODELS

We shall say that a regular model $\mathcal{Y}/\mathcal{O}_K$ of Y/K is *good* if the irreducible components of $\mathcal{Y}_s$ are smooth, if each singular point of $\mathcal{Y}_s$ belongs to exactly two irreducible components of $\mathcal{Y}_s$, and if these components intersect transversally. Given any model $\mathcal{Z}$ of Y, we may, using the embedded resolution of singularities, obtain a good model $\mathcal{Y}$ of Y which dominates $\mathcal{Z}$. The blow-up of a good model at a closed point is again a good model. Note that a regular semi-stable model is not necessarily a good model.

LEMMA 1.9. *Let $\mathcal{Y}/\mathcal{O}_K$ be any regular model of Y/K. Let $\mathcal{B}$ be any divisor on $\mathcal{Y}$ with smooth horizontal components. Then it is possible to perform a sequence of blow-ups along closed points, starting with $\mathcal{Y}$, to obtain a new regular model $\mathcal{Y}'/\mathcal{O}_K$ of Y/K such that the preimage $\mathcal{B}'$ of $\mathcal{B}$ in $\mathcal{Y}'$ is a divisor with normal crossings, and such that the horizontal part of $\mathcal{B}'$ is the disjoint union of irreducible components. Moreover, if $\mathcal{Y}$ is semi-stable, then so is $\mathcal{Y}'$.*

Proof. Let C and D be two irreducible divisors on the regular surface $\mathcal{Y}$. If C and D intersect at a closed point y of $\mathcal{Y}$, then the intersection multiplicity $(C \cdot D)_y$ of C and D is a nonnegative integer that decreases after a blow-up; that is, if $\widetilde{\mathcal{Y}}$ is the blow-up of $\mathcal{Y}$ at y with exceptional fiber E, and $\tilde{C}$ and $\tilde{D}$ are the strict transforms of C and D, then $\sum_{z \in E} (\tilde{C} \cdot \tilde{D})_z < (C \cdot D)_y$ (see [Sha], page 100). Moreover, E intersects $\tilde{C}$ and $\tilde{D}$ with normal crossings. Thus a sequence of blow-ups will produce the desired model $\mathcal{Y}'$.

Assume now that $\mathcal{Y}$ is a regular semi-stable model. Since a horizontal component D of $\mathcal{B}$ is smooth (and thus étale) by hypothesis, each point y of $D \cap \mathcal{Y}_s$ is a regular point of $\mathcal{Y}_s$ (Lemma 1.3) with residue field $k(y)$ separable over k. Hence, y belongs to a unique irreducible component of multiplicity 1 of $\mathcal{Y}_s$, and the exceptional fiber E of the blow-up $\widetilde{\mathcal{Y}} \to \mathcal{Y}$ has multiplicity 1 (Lemma 1.4) and is geometrically reduced. Thus $\widetilde{\mathcal{Y}}$ is regular and semi-stable. Since the strict transform $\tilde{D}$ of D is again smooth, the next blow-up will also be semi-stable. Hence $\mathcal{Y}'$ is regular and semi-stable.

2. Towards Semi-Stable Reduction

Let X/K be any curve. The semi-stable reduction theorem states the existence of an extension L/K such that X_L has semi-stable reduction, but this theorem does not provide information on how to determine explicitly such an extension L/K and a semi-stable model for X_L. Let $\mathcal{X}$ be a good model of X over $\mathcal{O}_K$ (see 1.8). T. Saito [Sai] has given an effective method for determining, in terms of the graph of $\mathcal{X}_s$, whether there exists a tamely ramified extension L/K such that X_L has semi-stable reduction. When such is the case, the normalization of $\mathcal{X}_{\mathcal{O}_L}$ has only Hirzebruch–Jung singularities, and such singularities can be explicitly resolved given the special fiber $\mathcal{X}_s$ of $\mathcal{X}/\mathcal{O}_K$ (see [Vie], pages 299–302, [Lip1], pages 206–212, [Pin], pages 12–15, or [BPV], pages 80–85). Moreover, it is also known in this case that X will achieve semi-stable reduction over a totally ramified extension L/K of order equal to the least common multiple of the multiplicities of the components of $\mathcal{X}_s$. Thus in the tamely ramified case, much is known concerning L/K and the semi-stable model of X_L over $\mathcal{O}_L$.

The situation is quite different when X achieves semi-stable reduction only after a wildly ramified extension L/K. In this case, it is usually not possible, given the combinatorial description of $\mathcal{X}_s$, to make any guess regarding which extensions L/K will lead to semi-stable reduction for X. The importance of Theorem 2.3 below lies in the fact that in many interesting cases, such as the case of a tame Galois cover of the projective line (compare with [Bro], Section 4, and [Kau], Section 4, for cyclic covers of $\mathbb{P}^1$ and hyperelliptic curves, respectively), it gives an explicit description of an extension L/K that leads to semi-stable reduction and a semi-stable model of X_L over $\mathcal{O}_L$.

Recall that if a group G acts on a scheme X and $x \in X$, then the *inertia group* I_x at x is the set of automorphisms $\sigma \in G$ such that $\sigma(x) = x$ and σ induces the identity on the residue field of x. Assume that the quotient X/G exists. Let x' be the image of x in X/I_x; then $X/I_x \to X/G$ is étale at x'. Recall also that a model $\mathcal{X}/\mathcal{O}_K$ is flat. Thus a closed point $x \in \mathcal{X}$ is smooth if and only if x is smooth on $\mathcal{X}_s$. Moreover, when x is smooth, then $\mathcal{O}_{\mathcal{X},x}$ is regular since $\mathcal{O}_K$ is regular.

LEMMA 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be models of curves, and let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a Galois cover of degree prime to p. Assume that $\mathcal{Y}$ is semi-stable and $\mathcal{X}_s$ is reduced. Let $\mathcal{R}$ and $\mathcal{B}$ be the ramification and branch loci of φ, respectively. Let $x \in \mathcal{X}_s$. Then the following properties hold.*

(a) *If $x \notin \mathcal{R}$, then φ is étale at x.*
(b) *If $x \in \mathcal{R}_s$, and $\varphi(x)$ is smooth in both $\mathcal{Y}_s$ and $\mathcal{B}$, then x is smooth in $\mathcal{X}_s$. Moreover, the inertia group I_x is cyclic.*

(c) *If x is an isolated point of $\mathcal{R}$, then x is an ordinary double point of $\mathcal{X}_s$.*

Proof. (a) Since $\mathcal{Y}$ is normal and, hence, geometrically unibranch ([EGA], 0.23.2.1), Part (a) follows from [EGA], IV.18.10.1.

To prove Parts (b) and (c), let us show first that $\mathcal{X}_s$ is geometrically reduced. Since $\mathcal{X} \to \mathcal{Y}$ has degree prime to p and since the curve $\mathcal{Y}_s$ is geometrically reduced, the residue fields of the components of $\mathcal{X}_s$ are separable extensions of k. Thus, since $\mathcal{X}_s$ is reduced, it is also geometrically reduced ([EGA], IV.4.6.1). Therefore, for any extension $\mathcal{O}_L/\mathcal{O}_K$, $\mathcal{X}_{\mathcal{O}_L}$ has reduced special fibers and is normal (1.1.). It is sufficient to prove the lemma after a base change M/K. So we can assume that X has semi-stable reduction over $\mathcal{O}_K$ with a regular semi-stable model having smooth components, that k is algebraically closed, and that the generic points of the horizontal components of $\mathcal{R}$ are all rational over K.

Let $\psi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the minimal desingularization of $\mathcal{X}$ ([Lip2], 27.3). Since $\mathcal{X}_s$ is reduced and X has semi-stable reduction, $\widetilde{\mathcal{X}}$ is semi-stable. The group I_x acts on $\widetilde{\mathcal{X}}$; let $\mathcal{Z} := \mathcal{X}/I_x$, and denote by z the image of x in $\mathcal{Z}$. Then $\mathcal{Z} \to \mathcal{Y}$ is étale in a neighborhood of z. Denote by $\lambda\colon \widetilde{\mathcal{X}}/I_x \to \mathcal{Z}$ the canonical birational morphism. Since $\mathcal{Z}$ is semi-stable in a neighborhood of z and $\widetilde{\mathcal{X}}/I_x$ is semi-stable, the components of $\lambda^{-1}(z)$ are smooth projective lines over k. Note that $\widetilde{\mathcal{X}}/I_x$ is regular at any point of $\lambda^{-1}(z)$ smooth in $(\widetilde{\mathcal{X}}/I_x)_s$. Thus the map $\widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}/I_x$ is flat over these points.

(b) Assume that x is not smooth in $\mathcal{X}_s$. Since the points of R are rational over K, ψ is not an isomorphism (see 1.3). Since $\varphi(x)$, and thus z, is smooth, there is a component Δ of $\lambda^{-1}(z)$ which meets the other components of $(\widetilde{\mathcal{X}}/I_x)_s$ in one point z' only (5.2(a)). Let Γ be a component of $\psi^{-1}(x)$ lying over Δ. Since $\Delta \setminus \{z'\}$ is contained in the regular locus of $\widetilde{\mathcal{X}}/I_x$, we can use the theorem of the purity of the branch locus to argue that $\Gamma \to \Delta$ is ramified only over z' and at the specialization of a horizontal component of $\mathcal{R}$ in $\widetilde{\mathcal{X}}_s$. (Note that there may be no such specialization on Γ.) Our hypothesis that $\varphi(x)$ is a smooth point on $\mathcal{B}$ shows that there is a unique (rational) point in the branch locus of $X \to X/I_x$ which specializes to z. Thus the morphism $\Gamma \to \Delta$ is ramified over at most two points of Δ. The Riemann–Hurwitz formula shows that any tame cover $C \to \mathbb{P}^1_k$, étale outside of two points, is a cover totally ramified in both points, and C is a projective line. Thus Γ meets the other components of $\widetilde{\mathcal{X}}_s$ in a single point only. Since $\widetilde{\mathcal{X}}$ is semi-stable, Γ is an exceptional divisor. This contradicts the minimality of $\widetilde{\mathcal{X}} \to \mathcal{X}$, and thus x is smooth in $\mathcal{X}_s$.

Since $|I_x|$ is invertible in $\mathcal{O}_{\mathcal{X},x}$, we have $\mathrm{H}^1(\langle\sigma\rangle, \mathcal{O}_{\mathcal{X},x}) = \{0\}$ for all $\sigma \in I_x$ ([Ser], VIII, Section 2, Corollary 1). The reader will check that the canonical homomorphism $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{X},x}) \to \mathrm{Aut}_k(\mathcal{O}_{\mathcal{X}_s,x})$ is injective when restricted to I_x. Since $\mathcal{O}_{\mathcal{X}_s,x}$ is a discrete valuation ring whose residue characteristic is prime to $|I_x|$, I_x is cyclic.

(c) If x is regular on $\mathcal{X}$, then x is an ordinary double point since $\widetilde{\mathcal{X}}$ is semi-stable. If x is not regular, then let Γ be a component of $\psi^{-1}(x)$, and let Δ be its image in $\widetilde{\mathcal{X}}/I_x$. As in (b), we find that $\Gamma \to \Delta$ is ramified only over the intersection points of Δ (since x is isolated in $\mathcal{R}$, no point of the ramification locus of $X \to X/I_x$ specializes to a point of Γ). Using the same argument as in (b), we conclude that Δ must intersect $(\widetilde{\mathcal{X}}/I_x)_s$ in at least two distinct points. We now claim that Δ can meet the components of $\lambda^{-1}(z)$ in at most two points. Indeed, let $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{X}}/I_x$ denote the minimal desingularization of $\widetilde{\mathcal{X}}/I_x$. Let η be the composition $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{X}}/I_x \to \mathcal{Z}$. Since $\mathcal{Z}$ is semi-stable in a neighborhood of z, the point z is a rational singularity, and we may apply Lemma 5.2.(a) to show that the dual graph of $\eta^{-1}(z)$ is a tree. Since the singularities of $\widetilde{\mathcal{X}}/I_x$ are resolved by chains of projective lines, we conclude that the curve $\lambda^{-1}(z)$ is tree-like. If $\lambda^{-1}(z)$ contains a curve that meets the rest of $\lambda^{-1}(z)$ in three or more points, then there exists at least three curves in $\lambda^{-1}(z)$ that each meet the rest of the $\lambda^{-1}(z)$ in exactly one point. We obtain a contradiction as follows: each of these three curves must meet at least two components of $(\widetilde{\mathcal{X}}/I_x)_s$, but since $\mathcal{Z}$ is semi-stable in a neighborhood of z, there are at most two irreducible components of $(\widetilde{\mathcal{X}}/I_x)_s$ that intersect $\lambda^{-1}(z)$.

Since a component Γ of $\psi^{-1}(x)$ is smooth by construction, we conclude as in (b) that Γ is a projective line with self-intersection -2. The only configurations of n smooth projective lines of self-intersection -2 are a cycle of n curves, or a chain of n curves meeting the rest of the fiber at the first and last curves on the chain. Since in our case the configuration is not equal to the whole special fiber, it cannot be a cycle, and thus x is an ordinary double point.

Remark 2.2. Part (b) of the above lemma is also an easy consequence of facts on tame fundamental groups of regular schemes ([G-M], Theorem 2.3.2). The Galois hypothesis can sometimes be removed in the statement of the lemma. Part (b) of Lemma 2.1. is true for tamely ramified covers (see [Ful], 3.3 and 3.4). Saïdi ([Said2], Théorème 3.2) proved Part (c) for covers $\mathcal{X} \to \mathcal{Y}$ étale in a neighborhood of x but possibly not at x. He uses a rather sophisticated ‘local Hurwitz formula’ due to Kato, and to Matignon–Youssefi.

Let X/K be a scheme of finite type over K. For any finite subset of closed points T of X, let $K(T)$ denote the compositum in an algebraic closure of K of the residue fields of the points of T. The field $K(T)$ is the smallest extension of K over which all points of T are rational.

THEOREM 2.3. *Let $f: X \to Y$ be a Galois cover of curves over K, of degree prime to p, with branch locus B. Let $M := K(B)$ and fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_M$ lying over (t). Assume that Y_M has semi-stable reduction. Let $\mathcal{Y}/\mathcal{O}_{M,\mathfrak{p}}$ be a semi-stable model of Y_M such that the points of $B_M \subset Y_M(M)$ specialize to distinct smooth points of $\mathcal{Y}_s$. Denote by $\Delta_1, \dots, \Delta_d$ the components of $\mathcal{Y}_s$ which are in the vertical branch locus of $N(\mathcal{Y}, K(X_M)) \to \mathcal{Y}$, with ramification indices $e_1, \dots, e_d$. Let $\mathcal{O}_L/\mathcal{O}_{M,\mathfrak{p}}$ be a totally ramified extension of degree* $\mathrm{lcm}(e_1, \dots, e_d)$. *Then $N(\mathcal{Y}_{\mathcal{O}_L}, K(X_L))$ is a semi-stable model of X_L over $\mathcal{O}_L$.*

Moreover, the preimage under $N(\mathcal{Y}_{\mathcal{O}_L}, K(X_L)) \to \mathcal{Y}_{\mathcal{O}_L}$ *of a smooth point consists of smooth points.*

Proof. The existence of the desired (even regular) model $\mathcal{Y}/\mathcal{O}_M$ is established in Lemma 1.9. By Abhyankar's Lemma ([SGA1], Exposé X, 3.6), $N(\mathcal{Y}_{\mathcal{O}_L}, K(X_L)) \to \mathcal{Y}_{\mathcal{O}_L}$ has no vertical ramification. Thus $N(\mathcal{Y}_{\mathcal{O}_L}, K(X_L))_s$ is reduced. Moreover, the hypothesis on the specialization of B_M implies that the branch locus of $N(\mathcal{Y}_{\mathcal{O}_L}, K(X_L)) \to \mathcal{Y}_{\mathcal{O}_L}$ consists of disjoint horizontal smooth components (over $\mathcal{O}_L$) and isolated points. Thus we can apply Lemma 2.1 to conclude.

Remark 2.4. In Theorem 3.9 below, we study the smallest extension of $K(B)$ needed to achieve semi-stable reduction. Under the hypothesis that the Galois cover $f\colon X \to Y$ is of degree prime to p, Theorem 2.3 shows that only a tame extension of $K(B)$ is necessary. T. Saito pointed out to us that this fact can also be obtained by computing vanishing cycles on X. The following example shows that when the degree of f is divisible by p, then $K(B)$ may equal K while any extension L/K such that X_L has semi-stable reduction is a wild extension.

Consider the Fermat quotient $X/\mathbb{Q}$, given by the equation $v^p = u(1-u)$. Let $f\colon X \to \mathbb{P}^1_{\mathbb{Q}}$ be the projection onto the u-axis. Then the branch locus of f consists in the three points 0, 1, and ∞, all three rational over $\mathbb{Q}$. It is shown in [McC] that if $2^p - 2$ is not divisible by p^2, then the curve X does not admit semi-stable reduction over any extension of $\mathbb{Q}$ which is tamely ramified at p.

Note that in this example, the special fiber of the natural model $\mathcal{Y}$ of $\mathbb{P}^1_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$ (associated to $\mathrm{Spec}(\mathbb{Z}_p[u])$) is in the branch locus of the normalization map $\varphi\colon N(\mathcal{Y}, K(X_{\mathbb{Q}_p})) \to \mathcal{Y}$. Note also that the regular point $y = (p, 1/2)$ on $\mathcal{Y}_s$ is a smooth point of the branch locus of φ, but that its preimage $\varphi^{-1}(y)$ is a singular point on the special fiber of $N(\mathcal{Y}, K(X_{\mathbb{Q}_p}))$.

Remark 2.5. Let K be of characteristic 2, and consider the curve X/K given by $v^2 + uv + u^3 + t = 0$. The projection to the u-axis $X \to \mathbb{P}^1_K$ is an Artin–Schreier cover. The branch locus of this cover is smooth at the point (t, u), but the special fiber of the affine chart $\mathrm{Spec}(\mathcal{O}_K[u, v]/(v^2 + uv + u^3 + t))$ is singular at the point (t, u, v). As we have seen in Lemma 2.1(b), this phenomenon cannot happen when the cover is of degree prime to p. More information on Artin–Schreier covers with smooth branch loci in characteristic 2 can be found in [Tak].

3. The Extension to Semi-Stability

Let us assume in this section that $\mathcal{O}_K$ is strictly Henselian. Let X/K be any curve. We denote by L_X/K any extension of K minimal with the property that X has semi-stable reduction over L_X. Let $f\colon X \to Y$ be a Galois cover over $\mathrm{Spec}(K)$ of degree prime to p, with branch locus $B \subset Y$. Recall that $K(B)$ denotes the compositum, in an algebraic closure of K, of the residue fields of the points of B.

Our aim in this section is to describe an extension L_X in terms of an extension L_Y, the extension $K(B)$, and some vertical ramification data.

In general, such an extension L_X/K is not unique. However, when either $g(X) \geqslant 2$ or $X(K) \neq \emptyset$, then there is a unique Galois extension L_X/K minimal with the property that X has semi-stable reduction over L_X (see [Des], 5.7–5.15). The same is true for any $g \geqslant 0$ if one knows that X has semi-stable reduction over a tamely ramified extension of K. Note that when k is algebraically closed and $g(X) = 0$, then $X(K) \neq \emptyset$ ([Ser], X.7), and thus $L_X = K$. In any case, it is always possible to find a separable extension L/K such that X_L/L has semi-stable reduction. Indeed, since the curve X/K is geometrically reduced, it has a point defined over a separable extension M/K. Thus the results recalled above imply the existence of a Galois extension L/M such that X_L/L has semi-stable reduction. Note however Example 4.8 in [K-U], where a curve of genus 1 achieves semi-stable reduction over a purely inseparable extension of K. Recall (1.6) that in general, given $f: X \to Y$ and any extension L_X/K, we can find an extension L_Y/K contained in L_X.

THEOREM 3.1. *Let K be strictly henselian. Let $f: X \to Y$ be a Galois cover of curves over* $\mathrm{Spec}(K)$*, with Galois group G of order prime to p, and ramification locus $R \subset X$. Then the following properties are true.*

(a) *The extension $K(R)/K$ is Galois.*
(b) *Let L_X/K be any finite extension of K such that X has semi-stable reduction over L_X. Then* $[L_X K(R) : L_X] \leqslant \gcd(2, |G|)$.
(c) *If $g(X) > 0$ and X has potentially good reduction, then $K(R) \subseteq L_X$.*

Theorem 3.1 has the following interesting application. Let us call *p-part of a Galois extension L/K* the following subextension L^H of L/K. Under our hypotheses on K, the inertia group I of L/K is equal to the full Galois group of L/K. Thus the p-Sylow subgroup P of I is normal with cyclic quotient of order m. Hence, I contains a unique cyclic normal subgroup H of order m, and L^H is the unique maximal extension of K in L whose degree over K is a power of p. When M/K is any finite extension, we call *p-part of M* the intersection of M with the p-part of the Galois closure of M/K. This extension is the unique maximal extension of K in M whose degree over K is a power of p.

Let ℓ be any prime different from p. We call *ℓ-part of a Galois extension L/K* the unique subextension of L of degree $\ell^{\mathrm{ord}_\ell(m)}$ over K. When M/K is any finite extension, we call *ℓ-part of M* the intersection of M with the ℓ-part of the Galois closure of M/K.

COROLLARY 3.2. *Let $f: X \to Y$ be a Galois cover over* $\mathrm{Spec}(K)$*, with Galois group G of order prime to p. Let $B \subset Y$ denote the branch locus of f. Let q be any prime that does not divide the degree of f (q may be equal to p). Then there exist extensions L_X and L_Y such that $L_Y \subseteq L_X$ and such that the q-part of L_X is equal to the q-part of $L_Y K(B)$.*

Proof. Let $\mathcal{L}_X$ denote the set of extensions L_X/K minimal with the property that X achieves semi-stable reduction over L_X. Let $\mathcal{L}_Y$ denote the same set relative to Y. For any $L_Y^1 \in \mathcal{L}_Y$, Theorem 2.3 shows that there is an extension $L/L_Y^1K(B)$ of degree prime to q such that X_L has semi-stable reduction. Hence there exists $L_X^1 \in \mathcal{L}_X$ such that $L_X^1 \subseteq L$. So the q-part of L_X^1 is contained in the q-part of $L_Y^1K(B)$. Conversely, any $L_X^1 \in \mathcal{L}_X$ contains a $L_Y^2 \in \mathcal{L}_Y$ (1.6.). Theorem 3.1 implies that $L_X^1K(B)$ and L_X^1 have the same q-part (note that $K(B) \subseteq K(R)$ and, by hypothesis, $q \neq 2$ if $|G|$ is even). Thus the q-part of L_X^1 contains that of $L_Y^2K(B)$. We have the inclusions:

$$q\text{-part of } L_Y^2K(B) \subseteq q\text{-part of } L_X^1 \subseteq q\text{-part of } L_Y^1K(B).$$

This procedure can be continued in an obvious way to obtain a decreasing chain of extensions of K. Since this chain becomes stationary after finitely many steps, we can find $L_X \supseteq L_Y$ such that L_X and $L_YK(B)$ have the same q-part.

The following general lemma proves Part (a) of Theorem 3.1.

LEMMA 3.3. *Let K be any field of characteristic $p \geqslant 0$. Let $f\colon X \to Y$ be a finite separable morphism of smooth (but not necessarily proper) curves over K. Assume that f is tamely ramified (that is, for any $x \in X$, the ramification index $e_{x/f(x)}$ is prime to p and the residue field extension $K(x)/K(f(x))$ is separable). Let $R \subset X$ be the ramification locus of f. Then $K(R)/K$ is a Galois extension.*

Proof. Since f is defined over K, $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ leaves $K(R)$ stable. Thus it remains only to prove that $K(R)$ is separable over K. To prove this fact, we may assume that K is separably closed, and show that $K(R) = K$. Let $x \in R$ and $y := f(x)$. Let $L := K(y) = K(x)$, and $d := [L : K]$. We will prove that $d = 1$.

The base change $Y_L \to Y$ is a homeomorphism. Denote again by y the preimage in Y_L of y. Then $\mathcal{O}_{Y_L,y} = \mathcal{O}_{Y,y} \otimes_K L$. Since $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_L,y}$ has trivial residue field extension, its ramification index is d. Consider the extension $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$. It has ramification index $e_x > 1$ and the tameness assumption implies that the residue extension is trivial. Thus there exists an Eisenstein polynomial $P(T) \in \widehat{\mathcal{O}}_{Y,y}[T]$ of degree e_x such that $\widehat{\mathcal{O}}_{X,x} = \widehat{\mathcal{O}}_{Y,y}[T]/(P)$. Tensoring by L, one gets

$$\widehat{\mathcal{O}}_{X_L,x} = \widehat{\mathcal{O}}_{X,x} \otimes_K L = \widehat{\mathcal{O}}_{Y_L,y}[T]/(P).$$

Let $\mathfrak{m}$ be the maximal ideal of $\widehat{\mathcal{O}}_{Y_L,y}$. If $d > 1$, then $P(T) \in (T, \mathfrak{m})^2$. This contradicts the regularity of $\widehat{\mathcal{O}}_{X_L,x}$. So $d = 1$ and $y \in Y(K)$.

EXAMPLE 3.4. The following example shows that the statement of Lemma 3.3 does not hold if the degree of f is divisible by p. Let $K = k(a, b)$ denote the field of rational functions in two variables. Consider the plane projective curve X/K defined by the affine equation $v^p + (u^p - b)v + au = 0$. The reader will easily check that this curve is smooth over K. Let $f\colon X \to \mathbb{P}^1_K$ denote the projection onto the u-axis. The point y corresponding to the ideal $(u^p - b)$ in $K[u]$ is in the branch locus of f and its residue field is inseparable over K.

To prove Parts (b) and (c) of Theorem 3.1, we may assume that $K = L_X$. To check that Part (b) holds when $g(X) = 0$, note that a Galois morphism $f: X \to Y$ of degree d prime to p between curves of genus zero is ramified in at most three points, with ramification indices (d, d), $(2, 2, d/2)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$. Except in the case where $d = 4$ and the ramification indices are $(2, 2, 2)$, at most two of the indices are equal, and thus a point in the branch locus is defined over K or over a quadratic extension of K. The same statement holds for the case $(2, 2, 2)$ since in this case the morphism can be factored into the composition of two cyclic covers of degree 2, and the ramification of each cover can be analyzed separately. The case (d, d) with d odd is the only case where $|G|$ can be odd. Since $X \to Y$ is branched at two points, it is cyclic. Then it is easy to see that the extension $K(X)/K(Y)$ can always be given by an equation of the form $v^d = u$, and thus the branch locus is always defined over K.

When $g(X) \geqslant 1$, Parts (b) and (c) are proved by applying the following proposition to the minimal regular model (which is semi-stable since $K = L_X$) of X over $\mathcal{O}_K$.

PROPOSITION 3.5. *Let K be strictly Henselian. Let $f: X \to Y$ be a Galois cover of curves over* $\mathrm{Spec}(K)$, *with Galois group G of order prime to p. Assume that X has a semi-stable model $\mathcal{X}$ over $\mathcal{O}_K$ such that G acts on $\mathcal{X}$. Let $\varphi: \mathcal{X} \to \mathcal{Y} := \mathcal{X}/G$ be the quotient map. Let $\overline{\{R\}}$ be the Zariski closure in $\mathcal{X}$ of the ramification locus R. Let $x \in \overline{\{R\}} \cap \mathcal{X}_s$ and let $\mathcal{C}$ be the connected component of $\overline{\{R\}}$ that contains x.*

- (a) *If $\mathcal{X}_s$ is smooth at x, then $\mathcal{C} \simeq$* $\mathrm{Spec}(\mathcal{O}_K)$.
- (b) *If x is a double point of $\mathcal{X}_s$, then $\varphi(\mathcal{C})$ has degree* 2 *over $\mathcal{O}_K$, and $|G|$ is even. Moreover, $\mathcal{C}_K \subset X(K)$ if and only if $\varphi(\mathcal{C}_K) \subset Y(K)$.*
- (c) *Assume that x is a double point of $\mathcal{X}_s$ and that $\mathcal{C}_K \subset X(K)$. Then there exists a semi-stable model $\mathcal{Y}_1$ of Y which dominates $\mathcal{Y}$ such that the points of $\varphi(\mathcal{C}_K)$ specialize to two distinct smooth points of $(\mathcal{Y}_1)_s$ and such that $N(\mathcal{Y}_1, K(X))$ is semi-stable.*

Proof. We will assume in the proofs of (a) and (b) that K is complete. Indeed, assume that the proposition holds in this case. Denote by $\hat{K}$ the completion of K. Then $\hat{K}(R)$ is tamely ramified over $\hat{K}$, and thus $K(R)$ is tamely ramified over K. Therefore, since K is strictly Henselian, $K(R)$ and $\hat{K}$ are linearly disjoint over K. Thus Parts (a) and (b) are true over K.

(a) Since $\mathcal{X}_s \to \mathcal{Y}_s$ is tamely ramified at x, we can apply Lemma 3.3. to conclude that x is rational over k. Then the completion of the local ring $\mathcal{O}_{\mathcal{X},x}$ is isomorphic to $\mathcal{O}_K[[v]]$. Let σ be a generator of the cyclic inertia group I_x (2.1. (b)). Since the order of σ is not divisible by the residual characteristic, there exists a change of variables $v \mapsto w$ such that $\sigma^*: \mathcal{O}_K[[w]] \to \mathcal{O}_K[[w]]$ is of the form $\sigma^*(w) = \xi_n w$, with $\xi_n \in \mathcal{O}_K$ an nth root of unity (see, e.g., [Lor], 1.3). Then the ramification of the inclusion $\mathcal{O}_K[[w]]^{\langle\sigma^*\rangle} \subseteq \mathcal{O}_K[[w]]$ occurs at the ideal (w). Thus $\mathcal{C}$ is a section of $\mathcal{X}$ over $\mathcal{O}_K$.

(b) Let us now consider the case where x is not smooth. Then x is an ordinary double point of $\mathcal{X}_s$ and, as such, is rational over k.

LEMMA 3.6. *Let K be a discrete valuation field. Let $\mathcal{X}/\mathcal{O}_K$ be a semi-stable (not necessarily regular) model of a curve X/K. Let x be a singular point of $\mathcal{X}_s$, rational over k. Let G be a finite group of automorphisms of $\mathcal{X}$. Assume that the inertia group I_x is not trivial and of order prime to p. Assume also that the elements of I_x either do not permute the irreducible components of $\mathcal{X}_s$ passing through x or, if $\mathcal{X}_s$ is irreducible at x, do not permute the tangent directions at x. Then x is isolated in the ramification locus of $\mathcal{X} \to \mathcal{X}/G$.*

Proof. Since $I_x \neq \{\text{id}\}$, x is ramified. Assume that x is not isolated in the ramification locus. Then there exists a closed point $P \in \mathcal{X}_K$ which specializes to x and such that the inertia group I_P is not trivial. After extending K if necessary and replacing $\mathcal{X}$ by its base change, we can assume that P is rational over K. This implies that x is singular in $\mathcal{X}$.

Let $\psi: \widetilde{\mathcal{X}} \to \mathcal{X}$ denote the minimal desingularization of x. Then $\widetilde{\mathcal{X}}$ is semi-stable, and P specializes to a smooth point $\tilde{x}$ of $\widetilde{\mathcal{X}}_s$. Let $\Gamma \subseteq \psi^{-1}(\tilde{x})$ be the irreducible component of $\widetilde{\mathcal{X}}_s$ passing through $\tilde{x}$. Since $I_P \subseteq I_x$, the hypothesis on the action of I_x implies that I_P acts on Γ, fixing the two intersection points of Γ with the other components of $\widetilde{\mathcal{X}}_s$. Furthermore, since Γ has multiplicity 1, the morphism $\Gamma \to \Gamma/I_P$ has degree I_P and, thus, I_P is a subgroup of $\text{Aut}(\Gamma)$. The morphism $\Gamma \to \Gamma/I_P$ is ramified in at least three points (the intersection points and $\widetilde{x}$) and $\Gamma \simeq \mathbb{P}^1_k$. Using the Riemann–Hurwitz formula and the fact that I_P is cyclic, the reader will check that such a morphism cannot exist.

Let us return to the proof of (b). Since x in this case is not isolated in the ramification locus of $\mathcal{X} \to \mathcal{Y}$, Lemma 3.6. implies that I_x must either permute the components of $\mathcal{X}_s$ passing through x or permute the tangent directions at x. Hence I_x is obviously an extension $1 \to J_x \to I_x \to \mathbb{Z}/2\mathbb{Z} \to 0$. Since $\mathcal{X}/I_x \to \mathcal{Y}$ is étale in a neighborhood of the image of x in $\mathcal{X}/I_x$, and $\mathcal{X} \to \mathcal{X}/J_x$ is étale outside of x in a neighborhood of x (3.6, 2.1(a)), we may restrict our attention to the double cover $\mathcal{X}/J_x \to \mathcal{X}/I_x$. We claim that the image of x in $\mathcal{X}/I_x$ is a smooth point. This statement is local and follows from our next lemma.

LEMMA 3.7. *Let K be a discrete valuation field. Let $\mathcal{X}/\mathcal{O}_K$ be a good regular model or a semi-stable model of a curve X/K. Let x be a singular point of $(\mathcal{X}_s)_{\text{red}}$, rational over k. Let σ be an involution of $\mathcal{X}$ which fixes x and which either permutes the components of $\mathcal{X}_s$ containing x or, if $\mathcal{X}_s$ is irreducible at x, permutes the tangent directions at x. Then the quotient $\mathcal{Y} := \mathcal{X}/\langle\sigma\rangle$ is regular at the image y of x. Moreover, if $\mathcal{X}_s$ is reduced, then $\mathcal{Y}$ is smooth at y.*

Proof. Let A be the formal completion $\widehat{\mathcal{O}}_{\mathcal{X},x}$. By hypothesis, the maximal ideal of A is generated by t, u_1, u_2, with a relation $(t^n) = (u_1^r u_2^r)$, $n, r \in \mathbb{N}$. We have

$\widehat{\mathcal{O}}_{\mathcal{Y},y} = A^{\langle\sigma\rangle}$. Set $u := u_1 + \sigma(u_1)$, and $v := u_1\sigma(u_1)$. Consider the continuous homomorphism $\phi: \widehat{\mathcal{O}}_K[[U, V]] \to A$ defined by $\phi(U) = u$ and $\phi(V) = v$. Let $C = \mathrm{Im}(\phi) \subseteq A^{\langle\sigma\rangle}$. One easily checks that $\{1, u_1\}$ is a basis for A over C, and thus A is integral over C. The maximal ideal of C is generated by t, u, and v. Since σ permutes the components, $t^n v^{-r}$ is an unit in C; and since either n or r is 1, we find that C is regular. It follows that $C = A^{\langle\sigma\rangle}$. Moreover, if $\mathcal{X}_s$ is reduced, then $r = 1$. So $C/tC = k[[u]]$ is formally smooth over k. Thus Lemma 3.7. is proved.

Let us now study the ramification and branch loci of the quotient map $\mathcal{X}/J_x \to \mathcal{X}/I_x$. Denote by x_1 and y_1 the images of x in $\mathcal{X}/J_x$ and $\mathcal{X}/I_x$. Let $C := \widehat{\mathcal{O}}_{\mathcal{X}/I_x,y_1}$. Lemma 3.7. shows that $C = \mathcal{O}_K[[u]]$ and that since $p \neq 2$, the ring $A := \widehat{\mathcal{O}}_{\mathcal{X}/J_x,x_1}$ is generated over C by an element w satisfying a quadratic relation $w^2 - \alpha(u) = 0$. Moreover, modulo (t), this relation must give an ordinary double point, so that Weierstrass' Preparation Theorem implies that $\alpha(u) = a(u)m(u)$, with $a(u)$ a distinguished polynomial of degree 2, and $m(u)$ a unit. Since $p \neq 2$, A contains the square root of $m(u)$. So we can assume that $m(u) = 1$. Thus A is generated over C by $\{1, w\}$, with $w^2 = a(u)$. Thus the ramification locus $\mathcal{R}$ is defined by (w) in a neighborhood of x. This achieves the proof of Part (b) of the proposition since $A/(w) = C/(a(u))$ and the latter has degree two over $\mathcal{O}_K$.

Let us prove Part (c) of Proposition 3.5. Consider the minimal desingularization $\mathcal{X}_1 \to \mathcal{X}$ of $\mathcal{X}$. Then G acts on $\mathcal{X}_1$, and we denote by $\mathcal{Y}_1$ the quotient. Clearly, $\mathcal{Y}_1$ is semi-stable and dominates $\mathcal{Y}$. Since $\mathcal{X}_1$ is regular, the points of $\mathcal{C}_K$ specialize to two distinct smooth points x_1 and x_2 of $(\mathcal{X}_1)_s$ (use Part a)). Proposition 1.6. implies that the points in the image of $\mathcal{C}_K$ also specialize to smooth points of $(\mathcal{Y}_1)_s$. We claim that these points are distinct. Indeed, the Going-Down Theorem 4.2 implies that if the two points in the image of $\mathcal{C}_K$ pass through the same point y of $(\mathcal{Y}_1)_s$, then there are two points of the ramification locus of f that specialize to x_1 (use the fact that the morphism f is Galois), which is a contradiction.

Remark 3.8. It is natural to wonder, in the case where the group G of $X \to Y$ has order divisible by p, whether the p-part of the extension L_X/K can also be described in terms of the p-part of L_Y/K and of some explicit data coming from the geometry of the cover $f: X \to Y := X/G$. We showed in Remark 2.4 that, contrary to the tame case, this 'explicit data' cannot be the field of rationality $K(B)$ of the branch locus of f.

The example presented in 2.4 is a wild cover $f: X \to \mathbb{P}^1_K$, where X is given by the affine equation $v^p = u(1-u)$, and f is the projection to the u-axis. The reader will note that a twist of X has good reduction over a tame extension of K. Indeed, the curve X is isomorphic over K to the curve given by $1/4 - u^2 = v^p$. Thus the twist $1/4 - u^2 = v^p/4$ is isomorphic to $1 - z^2 = v^p$. Theorem 2.3. can be applied to the hyperelliptic curve $z^2 = 1 - v^p$ to show that this curve achieves good reduction after a tame extension of K (the reader may also proves this fact using this explicit equation). Thus in this example the p-part of the extension L_X/K is in some sense 'explained' by the fact that a twist of X has good reduction over a tame extension of K.

The following is an example of an elliptic curve E/K over the maximal unramified extension K of $\mathbb{Q}_2$ such that all its points of order 2 are K-rational and such that the extension L_E/K has order 24. In particular, none of the twists of E/K has good reduction over K. To find examples of such a curve, one can use the tables in the corollary to Theorem 3 of [Kra]. The curve $v^2 = u(u+1)(u+4)$ is such a curve with discriminant $2^8 3^2$ and conductor 24. Note that $[L_E : K]$ is divisible by 3, even though 3 is prime to p and the points of the branch locus are all K-rational. As we saw in Corollary 3.2, this cannot happen when p does not divide the degree of the morphism $f: X \to Y$. Note also that the minimal extension of K such that one of the twists of E has good reduction is explicitly computable ([Ive]).

Let us consider a final example of a p-cover. Let K be the maximal unramified extension of $\mathbb{Q}_p(\zeta_p)$, with uniformizer $t := 1 - \zeta_p$. Let $m > 1$ be an integer prime to p. Consider the curve X/K given by the affine equation $v^p = u^m + t^{-p}$. The automorphism $v \mapsto \zeta_p v$ allows us to view X as a p-cover of $\mathbb{P}^1$. The branch locus of this cover is only defined over a tamely ramified extension of K, even though X has already good reduction over $\mathcal{O}_K$. Indeed, the change of variable $v = w - 1/t$ shows that X has good reduction over $\mathcal{O}_K$, with reduction of the form $w^p + w = u^m$.

Given a Galois cover $f: X \to Y$ of degree prime to p, Theorem 2.3 exhibits an extension L/K such that X_L has semi-stable reduction. A 'piece' of this extension is described using the vertical ramification indices of a well-chosen morphism of models $N(\mathcal{Y}, K(X)) \to \mathcal{Y}$. In our next theorem, we determine exactly which vertical ramifications need to be 'killed' for X to obtain semi-stable reduction.

THEOREM 3.9. *Let K be strictly Henselian. Let $f: X \to Y$ be a Galois cover of curves over K, of degree prime to p, with branch locus B. Let $M := K(B)$. Assume that Y_M has semi-stable reduction. Let $\mathcal{Y}/\mathcal{O}_M$ be a semi-stable model of Y_M such that the points of $B_M \subset Y_M(M)$ specialize to distinct smooth points of $\mathcal{Y}_s$. Denote by $\Delta_1, \ldots, \Delta_d$ the components of $\mathcal{Y}_s$ which are in the vertical branch locus of $N(\mathcal{Y}, K(X_M)) \to \mathcal{Y}$, with ramification indices $e_1, \ldots, e_d$. Denote by $\overline{\{B_M\}}$ the closure of B_M in $\mathcal{Y}$. Consider the set $\mathcal{F}$ of components Δ of $\mathcal{Y}_s$ such that either $p_a(\Delta) \geqslant 1$ or Δ contains at least three points of $\overline{\{B_M\}} \cup (\mathcal{Y}_s)_{\mathrm{sing}}$. If $\mathcal{F}$ is empty, then X_M has semi-stable reduction over M. If $\mathcal{F}$ is not empty, let L be the totally ramified extension of M of degree $\mathrm{lcm}(e_i \mid \Delta_i \in \mathcal{F})$.*

(a) *Then L contains an extension L_X.*
(b) *Assume that $g(X) \geqslant 2$ or, if $g(X) = 1$, that X has potentially good reduction. Then $[L : L_X] \leqslant \gcd(2, |G|)$. Moreover, L/M is the unique minimal extension of M such that X_L has semi-stable reduction.*

Proof. (a) Let $\mathcal{Y}$ be any semi-stable model of Y_M such that the closure of B in $\mathcal{Y}$ is contained in a smooth open subset of $\mathcal{Y}$ (Lemma 1.9). Consider the set

$\mathcal{F} = \mathcal{F}(\mathcal{Y})$ of components Δ of $\mathcal{Y}_s$ such that either $p_a(\Delta) \geqslant 1$ or Δ contains at least three points of $\overline{\{B\}} \cup (\mathcal{Y}_s)_{\rm sing}$. If $\mathcal{F} = \emptyset$, then either $Y \simeq \mathbb{P}^1_M$ and $|B| \leqslant 2$, or Y is an elliptic curve with multiplicative reduction and $B = \emptyset$. In the first case, $g(X) = 0$, and since the branch locus of f is M-rational, $X(M) \neq \emptyset$. Thus X/M has semi-stable reduction. In this second case, X is also an elliptic curve and is isogenous to Y; thus X has also multiplicative reduction. Therefore, if $\mathcal{F}$ is empty, X has semi-stable reduction over M and (a) holds.

Assume for the remainder of this proof that $\mathcal{F} \neq \emptyset$ (this is true under the assumption of (b)). Let $\mathcal{Y} \to \mathcal{Y}'$ be the contraction of the components of $\mathcal{Y}_s$ which do not belong to $\mathcal{F}$ (see [BLR], Section 6.7, Proposition 4). We leave it to the reader to check that the closure $\overline{\{B\}}'$ of B in $\mathcal{Y}'$ is again contained in a smooth open subset of $\mathcal{Y}'$. This follows from the fact that the exceptional components in the minimal desingularization $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ do not belong to $\mathcal{F}(\tilde{\mathcal{Y}})$. Moreover, the reader will check that:

LEMMA 3.10. *The model $\mathcal{Y}'$ of Y described above is the unique semi-stable model minimal among all semi-stable models $\mathcal{Z}$ of Y for which all points of B specialize to distinct smooth points of $\mathcal{Z}_s$. The construction of $\mathcal{Y}'$ commutes with base change.*

Make the extension $\mathcal{O}_L/\mathcal{O}_M$ to kill the vertical ramification. Then, as in 2.3, Lemma 2.1 can be applied to show that $N(\mathcal{Y}'_{\mathcal{O}_L}, K(X_L))$ is semi-stable. Hence, L contains an extension L_X.

(b) Using Theorem 3.1(b), we see that to prove the stated inequality, it is sufficient to prove that $L \subseteq L_X M$. Let $\mathcal{X} := N(\mathcal{Y}', K(X_M))$. Let us show first that L is the smallest extension of M such that the normalization of $\mathcal{X}_{\mathcal{O}_L}$ has reduced special fiber. Let E/K be such an extension. For any $\Delta_i \in \mathcal{F}$, let ξ denote the generic point of a component of $\mathcal{X}_s$ lying over Δ_i, and let η be a point of the normalization of $\mathcal{X}_{\mathcal{O}_E}$ lying over ξ. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_\xi & \longrightarrow & \mathcal{O}_\eta \\ \uparrow & & \uparrow \\ \mathcal{O}_K & \longrightarrow & \mathcal{O}_E \end{array}$$

of discrete valuation rings. Comparing the ramification indices in this diagram, we find that e_i divides $[E : K]$. Thus $[L : K]$ divides $[E : K]$. Since L/K is totally and tamely ramified and K is strictly henselian, we find that $L \subseteq E$.

Let $F := L_X M$. Let $\mathcal{W}$ be the stable (resp. smooth if $g(X) = 1$) model of X_F (see the beginning of Section 4). By uniqueness of $\mathcal{W}$, G acts on $\mathcal{W}$. Applying Proposition 3.5 to $\mathcal{W} \to \mathcal{W}/G$, we see that there exists a dominant morphism $\mathcal{Y}_1 \to \mathcal{W}/G$ such that $\mathcal{Y}_1$ and $N(\mathcal{Y}_1, K(X_F))$ are both semi-stable, and the Zariski closure of B_F in $\mathcal{Y}_1$ is smooth and contained in the smooth locus of $\mathcal{Y}_1$ (when $|G|$ is odd, pick $\mathcal{Y}_1 := \mathcal{W}/G$). Thus $\mathcal{Y}_1$ dominates $\mathcal{Y}'_{\mathcal{O}_F}$ and, hence, $N(\mathcal{Y}_1, K(X_F))$

dominates $N(\mathcal{Y}'_{\mathcal{O}_F}, K(X_F))$. In particular, $N(\mathcal{Y}'_{\mathcal{O}_F}, K(X_F))_s$ is reduced. Thus $L \subseteq F$ and, hence, $L = F$.

Suppose that N/M is a subextension of L such that X_N has semi-stable reduction. Then N contains an extension L'_X/K, and the discussion above shows that $L'_X M = L$. Since $L'_X M \subseteq N$, we find that $N = L$.

Remark 3.11. Keep the notation and hypotheses as in 3.9(b). Since $L = F$, we find that we have a dominant morphism $\mathcal{Y}_1 \to \mathcal{Y}'_{\mathcal{O}_L}$. Assume that $|G|$ is odd. Then $\mathcal{Y}_1 = \mathcal{W}/G$ by construction, and since $N(\mathcal{Y}'_{\mathcal{O}_L}, K(X_L))$ is semi-stable (Theorem 2.3) and dominated by $\mathcal{W}$, it follows that $N(\mathcal{Y}'_{\mathcal{O}_L}, K(X_L)) = \mathcal{W}$. In particular, if $g(X) \geqslant 2$, then $N(\mathcal{Y}'_{\mathcal{O}_L}, K(X_L))$ is the stable model of X_L. When $|G|$ is even, the situation is more complicated, but it should also be possible to find a semi-stable model $\mathcal{Y}''/\mathcal{O}_M$ of Y_M such that $N(\mathcal{Y}''_{\mathcal{O}_L}, K(X_L)) = \mathcal{W}$ (i.e., stable if $g(X) \geqslant 2$ or smooth if $g(X) = 1$).

4. Extending Covers to Stable Models

Let X be a (proper, smooth and geometrically connected) curve over K. Assume that $g(X) \geqslant 2$. A semi-stable model (see 1.5) $\mathcal{X}$ is said to be *stable* if any irreducible component of the geometric special fiber $\mathcal{X}_{\bar{s}}$ isomorphic to $\mathbb{P}^1$ meets the other components of $\mathcal{X}_{\bar{s}}$ in at least three points ([D-M], Definition 1.1). A stable model $\mathcal{X}$ together with the isomorphism $\mathcal{X}_K \cong X$ is unique ([D-M], Lemma 1.12). Let $\mathcal{X}_0$ be the minimal regular model of X over $\mathcal{O}_K$. Then X admits a stable model if and only if $\mathcal{X}_0$ is semi-stable. In fact, given any regular semi-stable model $\mathcal{X}$ of X, the stable model of X is obtained by contracting all the smooth rational curves of self-intersection -2 in $\mathcal{X}_s$ ([D-M], Section 1). Let K^{sh} be the strict henselization of K. Since the minimal regular model commutes with étale extensions of $\mathcal{O}_K$ (see for instance [Li1], Section 8, Lemme 11), X has a stable model over $\mathcal{O}_K$ if and only if $X_{K^{\mathrm{sh}}}$ has a stable model over $\mathcal{O}_{K^{\mathrm{sh}}}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be stable curves over $\mathcal{O}_K$ with smooth generic fibers X and Y, respectively. Let $f: X \to Y$ be a finite morphism. In this section, we investigate whether it is possible to extend f to a morphism from $\mathcal{X}$ to $\mathcal{Y}$. It is easy to see that, in general, f cannot be extended to a finite morphism, unless $\mathcal{X}$ is assumed to be smooth (see Corollary 4.10 below). However, we prove in Proposition 4.4 that the answer to the above question is positive if one does not require the extended morphism to be finite.

To begin, let us recall some well known facts on the birational geometry of normal surfaces. Let X be a curve over K. Let $\mathcal{O}$ be a valuation ring of $K(X)$ dominating $\mathcal{O}_K$. For any model $\mathcal{W}$ of X over $\mathcal{O}_K$, let $\mathrm{Spec}(\mathcal{O}) \to \mathcal{W}$ be the birational morphism given by the valuative criterion of properness. The image of the closed point of $\mathrm{Spec}(\mathcal{O})$ in $\mathcal{W}_s$ is called *the center of $\mathcal{O}$ in $\mathcal{W}$*. Note that for any generic point ξ of $\mathcal{W}_s$, $\mathcal{O}_{\mathcal{W},\xi}$ is a discrete valuation ring of $K(X)$ as $\mathcal{W}$ is normal. The following lemma is well known and useful.

LEMMA 4.11. *Let X be as above. Let $\mathcal{X}$ and $\mathcal{W}$ be two models of X over $\mathcal{O}_K$. Assume that for any generic point ξ of $\mathcal{W}_s$, the center of $\mathcal{O}_{\mathcal{W},\xi}$ in $\mathcal{X}$ is a generic point of $\mathcal{X}_s$. Then the birational map $\mathcal{X} \to \mathcal{W}$ induced by the identity map $\mathcal{X}_K \simeq \mathcal{W}_K$ is a morphism.*

Proof. Let $\Gamma \subseteq \mathcal{X} \times_{\mathcal{O}_K} \mathcal{W}$ be the graph of the birational map $T\colon \mathcal{X} \to \mathcal{W}$. Suppose that there exists a fundamental point $x \in \mathcal{X}$ of T (see [Har], V.5.1); x is closed. Let $p_1\colon \Gamma \to \mathcal{X}$ be the first projection. Then Zariski's Main Theorem (see for instance [Har], V.5.2) implies that $T(x) := p_2(p_1^{-1}(x))$ is connected and of dimension $\geqslant 1$. So there is a generic point ξ of $\mathcal{W}_s$ which lies in $T(x)$. Since T^{-1} is defined at ξ, we have $T^{-1}(\xi) = x$. This contradicts the assumption that $T^{-1}(\xi)$ is a generic point of $\mathcal{X}_s$.

4.2. Another useful tool in this section is the going-down property ([Mat], 5.E.v, p. 34). Let B be an integral domain, let A be a normal subring of B over which B is integral. Let $\mathfrak{m}$ be a prime ideal of A, let $\mathfrak{n}$ be a prime ideal of B lying over $\mathfrak{m}$. Then, given any prime ideal $\mathfrak{p}$ contained in $\mathfrak{m}$, there exists a prime ideal $\mathfrak{q}$ of B contained in $\mathfrak{n}$ and lying over $\mathfrak{p}$ (i.e., such that $\mathfrak{q} \cap A = \mathfrak{p}$).

Let E be a (possibly not normal or not reduced) connected curve over an algebraically closed field. For any closed point $x \in E$, denote by $m_{E,x}$ the number of points lying over x in the normalization of E (which is, by definition, the normalization of E_{red}).

LEMMA 4.3. *Let $\psi\colon \mathcal{W} \to \mathcal{Z}$ be a finite surjective morphism of normal flat $\mathcal{O}_K$-schemes of finite type of dimension 2. Let $z \in \mathcal{Z}_s$ be a closed point and let $w \in \psi^{-1}(z)$. Let Δ be any irreducible component of $\mathcal{Z}_s$ containing z.*

(a) *There is an irreducible component of $\mathcal{W}_s$ containing w which maps surjectively onto Δ.*

(b) *Assume that K is complete and that k is algebraically closed. Let E be the union of the irreducible components of $\psi^{-1}(\Delta)$ passing by w. Then $m_{E,w} \geqslant m_{\Delta,z}$.*

Proof. (a) Apply the going-down property to the case where $A = \mathcal{O}_{\mathcal{Z},z}$, $B = (\psi_*\mathcal{O}_{\mathcal{W}})_z$, $\mathfrak{m}$ is equal to the maximal ideal of A, and $\mathfrak{n}$ is equal to the maximal ideal of B corresponding to w. (b) Using the embedded resolution of curve singularities ([Sha], page 38), there is a birational morphism $\pi\colon \widetilde{\mathcal{Z}} \to \mathcal{Z}$ such that the strict transform $\widetilde{\Delta}$ of Δ by π is smooth. Let $\widetilde{\mathcal{W}}$ be the normalization of $\widetilde{\mathcal{Z}}$ in $K(\mathcal{W})$. Then one has a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{W}} & \xrightarrow{\lambda} & \mathcal{W} \\ {\scriptstyle\widetilde{\psi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{\mathcal{Z}} & \xrightarrow{\pi} & \mathcal{Z} \end{array}$$

Let $\widetilde{E}$ be the strict transform of E by λ. To prove Lemma 4.3, it is enough to show that the map $\widetilde{\psi}: \lambda^{-1}(w) \cap \widetilde{E} \to \pi^{-1}(z) \cap \widetilde{\Delta}$ is surjective. Let $w_1 \in \lambda^{-1}(w) \cap \widetilde{E}$, $z_1 = \widetilde{\psi}(w_1)$ and $z_2 \neq z_1$ be another point of $\pi^{-1}(z) \cap \widetilde{\Delta}$. Since $\pi^{-1}(z)$ is connected, there is a chain of irreducible components $\Delta_1, \ldots, \Delta_r$ of $\pi^{-1}(z)$ such that $z_1 \in \Delta_1$ and $z_2 \in \Delta_r$. Using (a), one gets a connected chain of irreducible components $\Gamma_1, \ldots, \Gamma_r, \Gamma_{r+1}$ of $\widetilde{\mathcal{W}}_s$ such that $w_1 \in \Gamma_1$, $\widetilde{\psi}(\Gamma_i) = \Delta_i$ for all $i \leqslant r$, and $\widetilde{\psi}(\Gamma_{r+1}) = \widetilde{\Delta}$. Let $w_2 \in \Gamma_r \cap \Gamma_{r+1}$ be a point lying over z_2. By construction, $\lambda(\cup_{1\leqslant i\leqslant r}\Gamma_i)$ contains $\lambda(w_1) = w$ and is connected. Moreover, $\lambda(\cup_{1\leqslant i\leqslant r}\Gamma_i) \subseteq \psi^{-1}(z)$, and thus is equal to w. Hence, $w \in \lambda(\Gamma_{r+1})$ and $\Gamma_{r+1} \subseteq \widetilde{E}$. Thus $w_2 \in \lambda^{-1}(w) \cap \widetilde{E}$ with $\widetilde{\psi}(w_2) = z_2$.

The above lemma is also proved in Youssefi's thesis in the context of valued function fields ([Y-M], Remarque, page 120).

PROPOSITION 4.4. *Let $\mathcal{O}_K$ be a discrete valuation ring. Let $\mathcal{X}$ and $\mathcal{Y}$ be stable curves over $\mathcal{O}_K$ with smooth generic fibers X and Y (in particular, $g(Y) \geqslant 2$). Let $f: X \to Y$ be a finite morphism. Then the following properties hold:*

(a) *The morphism f extends to a (not necessarily finite) morphism of $\mathcal{O}_K$-schemes $\varphi: \mathcal{X} \to \mathcal{Y}$.*
(b) *Let $y \in \mathcal{Y}_s$ be a singular point. If $\varphi^{-1}(y)$ is finite, then it is contained in the singular locus of $\mathcal{X}_s$.*

Proof. (a) We need to show that the rational map $\mathcal{X} \to \mathcal{Y}$ is a morphism. Let $\mathcal{O}_L$ be any discrete valuation ring dominating $\mathcal{O}_K$. If $\mathcal{X}_{\mathcal{O}_L} \to \mathcal{Y}_{\mathcal{O}_L}$ is a morphism, then $\mathcal{X}_R \to \mathcal{Y}_R$ is a morphism for some sub-$\mathcal{O}_K$-algebra R of $\mathcal{O}_L$ of finite type. Let $\overline{\Gamma}_f \subset \mathcal{X} \times_{\mathcal{O}_K} \mathcal{Y}$ be the graph of $\mathcal{X} \to \mathcal{Y}$ and let $p: \overline{\Gamma}_f \to \mathcal{X}$ be the first projection. Since $R/\mathcal{O}_K$ is flat and of finite type (thus universally open), the graph of $\mathcal{X}_R \to \mathcal{Y}_R$ is $(\overline{\Gamma}_f)_R$. So p_R is an isomorphism. But $R/\mathcal{O}_K$ is faithfully flat, so p is already an isomorphism. This means that $\mathcal{X} \to \mathcal{Y}$ is defined everywhere. Since the stable model commutes with base change, we are allowed to make any extension $\mathcal{O}_L/\mathcal{O}_K$ in the course of the proof. Hence, we will assume that K is complete and that k is algebraically closed.

Let $\psi: \mathcal{W} \to \mathcal{Y}$ be the normalization of $\mathcal{Y}$ in X. It is a finite morphism since K is complete, hence excellent. For any irreducible component Γ of $\mathcal{W}_s$, a theorem of Epp ([Epp], Theorem 2.0) proves the existence of a discrete valuation ring $\mathcal{O}_L$ finite over $\mathcal{O}_K$ with the following property: let $\mathcal{W}'$ be the normalization of $\mathcal{W}_{\mathcal{O}_L}$, then any irreducible component of $\mathcal{W}'_s$ lying over Γ is of multiplicity 1. Let F be the compositum of all the $\mathcal{O}_L$'s. Since a component of multiplicity 1 remains of multiplicity 1 after any base extension, the normalization of $\mathcal{W}_{\mathcal{O}_F}$ has reduced special fiber. Thus, after a suitable base change if necessary, we may assume that $\mathcal{W}_s$ is reduced.

The statement (a) is equivalent to saying that the birational map $\mathcal{X} \to \mathcal{W}$ is a morphism. Let $\pi: \widetilde{\mathcal{W}} \to \mathcal{W}$ be the minimal desingularization of $\mathcal{W}$. We claim that for any irreducible component Γ of $\mathcal{W}_s$, the strict transform $\widetilde{\Gamma}$ of Γ in $\widetilde{\mathcal{W}}$ cannot be exceptional of the first or second kind (that is, $\widetilde{\Gamma}$ cannot be isomorphic to $\mathbb{P}^1$ with $\widetilde{\Gamma}^2 = -1$ or -2). This claim implies that $\widetilde{\mathcal{W}}$ is the minimal regular model and that $\widetilde{\Gamma}$ is mapped onto a component of $\mathcal{X}_s$. Lemma 4.11 shows then that $\mathcal{X} \to \mathcal{W}$ is a morphism.

It remains to prove the claim. Let Γ be a component of $\mathcal{W}_s$ such that $\widetilde{\Gamma}$ is exceptional of the first or second kind. Let Δ be its image in $\mathcal{Y}$. Let $y \in \Delta$ be a singular point of $\mathcal{Y}_s$. Let $x \in \pi^{-1}(\psi^{-1}(y)) \cap \widetilde{\Gamma}$ and let $w = \pi(x)$. We want to show first that x is singular in $\widetilde{\mathcal{W}}_s$. It is easy to see, using Lemma 4.3, that $m_{\mathcal{W}_s,w} \geqslant m_{\mathcal{Y}_s,y} \geqslant 2$. If π is an isomorphism over w, we are done. Otherwise, $\pi^{-1}(w)$ is a connected one-dimensional curve. Hence there is an irreducible component Γ' of $\widetilde{\mathcal{W}}_s$, contained in $\pi^{-1}(w)$ and containing x. So $\Gamma' \neq \widetilde{\Gamma}$, and $x \in \widetilde{\Gamma} \cap \Gamma'$. Thus x is singular in $\widetilde{\mathcal{W}}_s$.

Since $\widetilde{\Gamma}$ is rational and dominates Δ, the latter is also rational. Since $\mathcal{Y}_s$ is stable, either Δ contains at least three intersection points, or it contains one double point and one intersection point. By hypothesis, $\widetilde{\Gamma}$ is regular. So there are at least three points in $\widetilde{\Gamma}$ lying over singular points of $\mathcal{Y}_s$ contained in Δ. These three points are all singular in $\widetilde{\mathcal{W}}_s$ as we just showed, but this contradicts the hypothesis on $\widetilde{\Gamma}$. Thus the claim is proved.

(b) We may assume that K is complete and k is algebraically closed. Then (b) is a consequence of Lemma 4.3(b) or Proposition 1.6.

T. Saito informed us that Proposition 4.4(a) is proved in [Moc], Lemma 8.3, in the case where the morphism $f: X \to Y$ is étale.

Remark 4.5. Note that Proposition 4.4(a) is not true if one replaces ‘stable curves’ by ‘minimal regular models’. Indeed, consider a Galois cover $f: X \to Y$ with Galois group G. Let $\mathcal{X}/\mathcal{O}_K$ be the minimal regular model of X and let $x \in \mathcal{X}_s$ be an intersection point of two irreducible components Γ_1 and Γ_2 of $\mathcal{X}_s$. The group G acts on $\mathcal{X}$. It may well happen that the inertia group I_x is nontrivial and leaves fixed each Γ_i. If $\mathcal{X}$ is semi-stable, then the image y of the point x in the quotient $\mathcal{X}/G$ is a singular point (see, for instance, the beginning of the proof of 6.2). Thus we may expect that the minimal regular model $\mathcal{Y}$ of Y has ‘too many components’ coming from the desingularization of y for the morphism f to extend to a morphism from $\mathcal{X}$ to $\mathcal{Y}$.

Remark 4.6. If $X \to Y$ is separable, then Proposition 4.4(a) can be proved in a somewhat simpler manner as follows. As explained before, we are allowed to extend the base field K. Let us do it in such a way that: (1) the Galois closure $\hat{X}$ of $X \to Y$ is geometrically connected and smooth; (2) the residue field k is algebraically closed; and (3) $\hat{X}$ admits a stable model $\hat{\mathcal{X}}$ over $\mathcal{O}_K$. Let G denote the group $\mathrm{Gal}(K(\hat{X})/K(Y))$. Let H be the subgroup $\mathrm{Gal}(K(\hat{X})/K(X))$ of G. Then $\mathcal{X}_1 := \hat{\mathcal{X}}/H$ and $\mathcal{Y}_1 := \hat{\mathcal{X}}/G$ are semi-stable (Proposition 1.6), and f extends to $\varphi_1: \mathcal{X}_1 \to \mathcal{Y}_1$ by Galois property. Let Γ be any component of $(\mathcal{X}_1)_s$ mapped to a closed point of the stable model $\mathcal{X}$ of X. Then $\Gamma \simeq \mathbb{P}^1_k$ and meets the other components of $(\mathcal{X}_1)_s$ in at most two points. By Lemma 4.3, any point of Γ lying over a singular point of $(\mathcal{Y}_1)_s$ is a singular point of $(\mathcal{X}_1)_s$. We conclude that $\varphi_1(\Gamma)$ is isomorphic to $\mathbb{P}^1_k$ and meets the other components of $(\mathcal{Y}_1)_s$ in at most two points. So $\varphi_1(\Gamma)$ is mapped to a point of the stable model $\mathcal{Y}$ of Y. Lemma 4.11 implies then that φ_1 induces a morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ which extends f.

COROLLARY 4.7. *Let $f\colon X \to Y$ be a finite morphism of curves over K, with $g(Y) \geqslant 2$. Assume that X admits a stable model $\mathcal{X}$ over $\mathcal{O}_K$. Then Y has a stable model $\mathcal{Y}$ over $\mathcal{O}_K$ and f extends to a morphism $\mathcal{X} \to \mathcal{Y}$.*

Proof. Let L/K be a finite Galois extension with Galois group G such that Y_L admits a stable model $\mathcal{Y}'$ over the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in L. Proposition 4.4 implies that f_L extends to a morphism $\mathcal{X}_{\mathcal{O}_L} \to \mathcal{Y}'$. So f extends to a morphism $\mathcal{X} \to \mathcal{Y} := \mathcal{Y}'/G$ over $\mathcal{O}_K$. Since $\mathcal{X}_s$ is geometrically reduced, so is $\mathcal{Y}_s$. Hence $\mathcal{Y}_{\mathcal{O}_L}$ is normal (1.1.). The morphism $\mathcal{Y}' \to \mathcal{Y}_{\mathcal{O}_L}$ is finite and birational, thus it is an isomorphism. So $\mathcal{Y}$ is stable. Thus Y admits a stable model over $\mathcal{O}_K$.

We may also use Néron models to prove that Y has a stable model over $\mathcal{O}_K$. Indeed, a curve of genus $\geqslant 2$ admits a stable model if and only if the Néron model of its Jacobian has semi-Abelian reduction ([D-M], Theorem 2.4). Since the Jacobian $\mathrm{Jac}(X)$ of X is isogenous to a product $\mathrm{Jac}(Y) \times A$, Corollary 7 in [BLR], Section 7.3, implies that the Néron model of $\mathrm{Jac}(Y)$ is semi-Abelian.

Remark 4.8. Let us consider in this remark the case where $g(Y) = 1$. If Y has potentially good reduction, then Proposition 4.4. and Corollary 4.7. still hold if one replaces 'stable model of Y' by 'smooth model of Y'. The proof is exactly the same.

If Y has potentially multiplicative reduction and $f\colon X \to Y$ is any cover of Y such that $g(X) \geqslant 2$ and X has a stable model over $\mathcal{O}_K$, then Y has multiplicative reduction over $\mathcal{O}_K$ already. This can be seen easily using Jacobians. However, it may happen that the morphism f cannot be extended to a morphism between the stable model of X and a semi-stable model of Y. Let us consider the following example.

Let K be complete with algebraically closed residue field k and $p \neq 2$. Let $G := \mathbb{Z}/n\mathbb{Z}$, with n prime to p. By gluing together two suitable G-covers of $\mathbb{P}^1_k$, we can construct a semi-stable curve $\mathcal{X}_s$ over k with a faithful action of G, and such that (1) $\mathcal{X}_s$ has two smooth irreducible components Γ_1 and Γ_2 that intersect at exactly two points, (2) G leaves fixed these points as well as each Γ_i, and (3) $g(\Gamma_1) \geqslant 1$, $g(\Gamma_2) = 0$, and $g(\Gamma_1/G) = 0$. Let $\mathcal{X}'_s = \mathcal{X}_s/G$. Then $\mathcal{X}'_s$ is the union of two smooth rational lines that intersect transversally in two distinct points. Consider the étale map of degree two from $\mathcal{X}'_s$ to a rational curve with a node. Let us denote this latter curve by $\mathcal{Y}_s$.

Our aim is to lift the composition $\mathcal{X}_s \to \mathcal{Y}_s$ to a finite morphism of models. To do so, we will use some results of Saïdi. Let $\mathcal{Y}$ be a semi-stable model of a Tate elliptic curve Y/K, with special fiber isomorphic to $\mathcal{Y}_s$. After making a ramified extension if necessary, we can use [Said], 5.7, first to lift the morphism $\mathcal{X}'_s \to \mathcal{Y}_s$ to a finite morphism $\mathcal{X}' \to \mathcal{Y}$ of semi-stable models, and then to lift the morphism $\mathcal{X}_s \to \mathcal{X}'_s$ to a finite morphism of semi-stable models $\mathcal{X} \to \mathcal{X}'$ (the scheme $\mathcal{X}$ is normal by 1.1). Let $\varphi: \mathcal{X} \to \mathcal{Y}$ denote the composition, and let $f = \varphi_K$. Let $\mathcal{Z}$ be the stable model X over $\mathcal{O}_K$. Its special fiber consists in the curve Γ_1 with two points identified.

We claim that the morphism f cannot be extended to a morphism from $\mathcal{Z}$ to any semi-stable model of Y. To prove this fact, let $\mathcal{Y}'$ be any semi-stable model of Y. Let us show first that if a morphism $\mathcal{Z} \to \mathcal{Y}'$ exists, then $\mathcal{Y}' = \mathcal{Y}$. Indeed, $\mathcal{Y}'_s$ is irreducible. Let ξ_1, ξ, η, and η', denote respectively the generic points of Γ_1, $\mathcal{Z}_s$, $\mathcal{Y}_s$ and $\mathcal{Y}'_s$. Then $\mathcal{O}_{\mathcal{X},\xi_1} = \mathcal{O}_{\mathcal{Z},\xi}$ dominates both $\mathcal{O}_{\mathcal{Y},\eta}$ and $\mathcal{O}_{\mathcal{Y}',\eta'}$. Thus $\mathcal{O}_{\mathcal{Y},\eta} = \mathcal{O}_{\mathcal{Y}',\eta'}$ and $\mathcal{Y}' = \mathcal{Y}$ (Lemma 4.11). If a morphism $\mathcal{Z} \to \mathcal{Y}$ exists, then it is finite. Thus $\mathcal{Z}$ is the integral closure of $\mathcal{Y}$ in $K(X)$ and, hence, $\mathcal{X} = \mathcal{Z}$, which is impossible.

Remark 4.9. As pointed out by R. Coleman, the following variation on Proposition 4.4(a) holds without the assumption that $g(Y) \geqslant 2$. Let $\mathcal{X}$ and $\mathcal{Y}$ be semi-stable curves over $\mathcal{O}_K$ with smooth generic fibers X and Y. Let $f: X \to Y$ be a finite morphism. Then after a suitable extension $\mathcal{O}_L$ of $\mathcal{O}_K$, there exists a semi-stable model $\mathcal{X}'/\mathcal{O}_L$ dominating $\mathcal{X}_{\mathcal{O}_L}$ such that f extends to a morphism $\mathcal{X}' \to \mathcal{Y}_{\mathcal{O}_L}$.

To prove this statement, we may assume that K is complete. Let $\mathcal{Z}$ be a model of X which dominates both $\mathcal{X}$ and $N(\mathcal{Y}, K(X))$. Then after a finite extension $\mathcal{O}_L/\mathcal{O}_K$, there is a semi-stable model $\mathcal{X}'/\mathcal{O}_L$ of X_L which dominates $\mathcal{Z}_{\mathcal{O}_L}$ (see reference just before 1.8). Thus $\mathcal{X}'$ dominates $\mathcal{X}_{\mathcal{O}_L}$ and f extends to $\mathcal{X}' \to \mathcal{Y}_{\mathcal{O}_L}$.

COROLLARY 4.10. *Let $f: X \to Y$ be a finite morphism of curves over K. Assume that $g(Y) \geqslant 1$, and that X admits a smooth model $\mathcal{X}$. Then Y admits a smooth model $\mathcal{Y}$, and f extends to a finite morphism $\mathcal{X} \to \mathcal{Y}$.*

Proof. Assume that $g(Y) \geqslant 2$. Let $\mathcal{Y}$ be the stable model of Y over $\mathcal{O}_K$, which exists by Corollary 4.7. Let $\varphi: \mathcal{X} \to \mathcal{Y}$ be the morphism which extends f. Since $\mathcal{X}_s$ is irreducible, φ is finite. Thus Proposition 4.4(b), implies that $\mathcal{Y}_s$ is smooth, and the corollary is proved when $g(Y) \geqslant 2$.

Let us now present a different argument that also applies to the case where $g(Y) = 1$. Let $J(X)$ (resp. $J(Y)$) be the Jacobian of X (resp. of Y). Then $J(X)$ has good reduction, and so does $J(Y)$ ([S-T], Section 1, Corollary 2). Since $\mathcal{X}$ is smooth, it has a section over some étale extension of $\mathcal{O}_K$ ([EGA], IV.17.16.3 (ii)). Let $\mathcal{Y}$ be the minimal regular model of Y over $\mathcal{O}_K$. Then by composition $\mathcal{Y}$ also has an étale quasi-section. So the identity component of the Néron model of $J(Y)$ over $\mathcal{O}_K$ is isomorphic to $\mathrm{Pic}^{\circ}_{\mathcal{Y}/\mathcal{O}_K}$ ([BLR], Section 9.5, Remark 5), $\mathcal{Y}$ is semi-stable, all the irreducible components of $\mathcal{Y}_s$ are smooth and the graph of $\mathcal{Y}_s$ is a tree (loc. cit. Section 9.3, Corollary 12(c)). Since $g(Y) \geqslant 1$ and the graph is a tree, $\mathcal{Y}_s$ contains an irreducible component Δ of positive genus. As before, we may assume K complete. In particular, the ring $\mathcal{O}_K$ is excellent. Consider the normalization $\mathcal{X}' := N(\mathcal{Y}, K(X))$ of $\mathcal{Y}$ in $K(X)$. Then Δ is dominated by some irreducible component Γ of $\mathcal{X}'_s$ of positive geometric genus. Since $\mathcal{X}$ is smooth, Γ is the unique component of $\mathcal{X}'_s$ with positive geometric genus, and thus Δ is the unique such component of $\mathcal{Y}_s$. Hence, $\mathcal{Y}_s = \Delta$ and $\mathcal{Y}$ is smooth. Any irreducible component of $\mathcal{X}'_s$ dominates Δ, and thus has positive geometric genus. Therefore, $\mathcal{X}'_s = \Gamma$. Since $\mathcal{X}$ is smooth and $g(X) \geqslant 1$, $\mathcal{X}$ is the unique model of X with integral special fiber of positive geometric genus. Hence, $\mathcal{X}' = \mathcal{X}$ and we have a finite morphism $\mathcal{X} \to \mathcal{Y}$.

Remark 4.11. The fact that the good reduction of X implies that of Y is already known, and can be found in the literature in [Lge], Section 3, Lemma 1, in the case where $X(K) \neq \emptyset$, and in [Y-M], théorème 2.2, in the general case; see also [N-S], Lemma 5.1.

The hypothesis that $g(Y) \geqslant 1$ is necessary in the statement of the corollary. Indeed, here is a counterexample to Corollary 4.10 when $Y = \mathbb{P}^1_K$. Let $Z \to Y$ to the cover corresponding to the field extension $K(Y) = K(y) \to K(Z) = K(Y)[z]/(z^2 - yz + t)$, where t is an uniformizing parameter of K. Then $Z \simeq \mathbb{P}^1_K$ since Z has rational points at infinity. Consider the model $\mathcal{Y} = \operatorname{Spec}\mathcal{O}_K[y] \cup \operatorname{Spec}\mathcal{O}_K[1/y] \simeq \mathbb{P}^1_{\mathcal{O}_K}$ of Y. Put $\mathcal{Z} = N(\mathcal{Y}, K(Z))$. Then $\mathcal{Z}$ is regular, and $\mathcal{Z}_s$ is the union of two projective lines of self-intersection -1. One can contract one of these lines, with contraction morphism $\mathcal{Z} \to \mathcal{W}$, so that $\mathcal{W} \simeq \mathbb{P}^1_{\mathcal{O}_K}$.

Let $\mathcal{X} \to \mathcal{W}$ be a finite cover with $\mathcal{X}$ smooth over $\mathcal{O}_K$. Since $\mathcal{W}_K = Z$, one obtains a cover $X := \mathcal{X}_K \to Y$ by composition. We claim that this cover cannot be extended to smooth models. Assume for simplicity that $g(X) > 0$. Assume that $X \to Y$ extends to a cover of smooth models $\mathcal{X}' \to \mathcal{Y}'$. Then $\mathcal{X}' = \mathcal{X}$ by uniqueness of the smooth model of X over $\mathcal{O}_K$. Let ξ (resp. ξ') be the generic point of $\mathcal{Y}_s$ (resp. of $\mathcal{Y}'_s$). Then $\mathcal{O}_{\mathcal{Y},\xi}$ and $\mathcal{O}_{\mathcal{Y}',\xi'}$ are discrete valuation rings with same quotient field $K(Y)$ and both dominated by the valuation ring induced by the generic point of $\mathcal{X}_s$, so they are equal. This implies that $\mathcal{Y}' = \mathcal{Y}$ and $\mathcal{X} = N(\mathcal{Y}, K(X))$. But $N(\mathcal{Y}, K(X))_s$ has at least two components since the same is true for $N(\mathcal{Y}, K(Z))_s = \mathcal{W}_s$, so there is contradiction.

Remark 4.12. Corollary 4.10. holds even when $g(Y) = 0$ if $f: X \to Y$ is Galois with Galois group G: in this case the quotient $\mathcal{X}/G$ is smooth (Proposition 1.6.). Unfortunately, the general case of Corollary 4.10. cannot be deduced from this Galois case. Indeed, let $X \to Y$ be any separable cover. Let $\hat{X} \to Y$ be the Galois closure of X over Y. Assume that X has good reduction. One can ask whether $\hat{X}$ also has good reduction. The answer to this question is negative in general. Consider for instance the cover $X \to Y = \mathbb{P}^1_K$ in Remark 4.11 Let H be the subgroup $\operatorname{Gal}(K(\hat{X})/K(X))$ of G. If $\hat{X}$ had good reduction, then the cover $X \to Y$ would extend to a Galois cover of smooth models $\hat{\mathcal{X}}/H \to \hat{\mathcal{X}}/G$. But we saw in Remark 4.11 that this is impossible.

Remark 4.13. Assume that Y has a smooth model $\mathcal{Y}$ over $\mathcal{O}_K$. Let $X \to Y$ be an étale Galois cover of group G. A theorem of Grothendieck states that if the order of

G is prime to the residue characteristic $p = \mathrm{char}(k)$, then after a base change of the ground field, $N(\mathcal{Y}, K(X))$ is smooth and étale over $\mathcal{Y}$. This can be deduced easily from Theorem 2.3. If p divides the order of G, then the situation is more complex. See [Ray], especially Section 3, for further information. In particular, it is possible to construct examples where X may not have potentially good reduction.

Finally, let us state the following lemma which pertains to the problem of extending finite covers. Given a morphism $f: X \to Y$ and a model $\mathcal{Y}$ of Y, the process of normalization produces a model $N(\mathcal{Y}, K(X))$ of X that is in general finite over $\mathcal{Y}$. But in general, given a model $\mathcal{X}$ of X, it is not possible to construct a model $\mathcal{Y}$ and a finite morphism $\mathcal{X} \to \mathcal{Y}$. The following lemma shows the existence of a model $\mathcal{Y}$ and a rational map $\mathcal{X} \to \mathcal{Y}$ with some finiteness and surjectivity properties.

LEMMA 4.14. *Let $f: X \to Y$ be a finite cover of curves over K. Let $\mathcal{X}$ and $\mathcal{Y}'$ be models of X and Y over $\mathcal{O}_K$, respectively. Then there is a model $\mathcal{Y}$ of Y over $\mathcal{O}_K$ which dominates $\mathcal{Y}'$ and such that f extends to a rational map $\mathcal{X} \to \mathcal{Y}$ which is quasi-finite in codimension* 1*. If K is Henselian, then there is a model $\mathcal{Y}''$ of Y over $\mathcal{O}_K$ such that f extends to a rational map $\mathcal{X} \to \mathcal{Y}''$ which is quasi-finite and surjective in codimension* 1.

Proof. Let ξ be the generic point of an irreducible component of $\mathcal{X}_s$. Then $\mathcal{O}' := \mathcal{O}_{\mathcal{X},\xi} \cap K(Y)$ is a valuation ring of $K(Y)$. Let us show first that there is a model $\mathcal{Z}$ of Y such that the center of $\mathcal{O}'$ in $\mathcal{Z}$ is a generic point of $\mathcal{Z}_s$.

Let p be the center of $\mathcal{O}'$ in $\mathcal{Y}'$. The residue field of $\mathcal{O}'$ is a sub-extension of finite index of $k(\xi)$, so it has transcendental degree 1 over k. We can easily deduce that $\mathcal{O}'$ is a prime divisor of $\mathcal{Y}'$ of center p in the sense of Zariski (see [Art], Section 5). By a theorem of Zariski, after a suitable blow-up $\mathcal{Z}' \to \mathcal{Y}'$, the center of $\mathcal{O}'$ in $\mathcal{Z}'$ will be a generic point of $\mathcal{Z}'_s$ (op. cit., Theorem 5.2). The normalization $\mathcal{Z}$ of $\mathcal{Z}'$ satisfies the required condition. By induction, one constructs a model $\mathcal{Y}$ of Y dominating $\mathcal{Y}'$ and such that the local ring of any one-codimensional point of $\mathcal{X}$ dominates the local ring of an one-codimensional point of $\mathcal{Y}$.

If K is Henselian, one can contract the irreducible components of $\mathcal{Y}_s$ which are not dominated by a component of $\mathcal{X}_s$ ([BLR], Section 6.7, Proposition 4). After such a contraction, the new model $\mathcal{Y}''$ fulfills the required conditions.

5. Models that Dominate Regular Models

In this preparatory section, we study some properties of the fibers of a birational morphism $\mathcal{Z} \to \mathcal{Y}$ when $\mathcal{Y}$ is assumed to be regular. As we shall see, these fibers behave as if they belonged to the special fiber of a regular model of $\mathbb{P}^1_K$ (compare with [Li2], Section 3.2). The results of this section will be used to prove the main results of the next two sections. The reader may skip this section and refer to it as necessary while reading the following sections.

Let $\mathcal{Z}$ be a model. We denote by $r(\Gamma)$ the multiplicity of a vertical divisor Γ in $\mathcal{Z}$. Let $\mathcal{Z}'$ be another model of $\mathcal{Z}_K$ and let $\pi\colon\mathcal{Z}'\to\mathcal{Z}$ be a morphism of models. Let $\widetilde{\Gamma}$ be the strict transform of Γ in $\mathcal{Z}'$. When Γ is smooth, $\mathcal{Z}'\to\mathcal{Z}$ induces an isomorphism $\widetilde{\Gamma}\to\Gamma$, and $r(\widetilde{\Gamma})=r(\Gamma)$. To simplify our notation when no confusion may result, the strict transform $\widetilde{\Gamma}$ will be denoted again by Γ. We shall say that a point x of Γ is an *interior point* of Γ in $\mathcal{Z}$ if $\mathcal{Z}_s$ is irreducible at x. If $\pi\colon\mathcal{Z}'\to\mathcal{Z}$ is any birational morphism and $z\in\mathcal{Z}$, we may denote the fiber $\pi^{-1}(z)$ simply by $\mathcal{Z}'_z$.

Let $\mathcal{W}/\mathcal{O}_K$ be a normal model of a curve W over a discrete valuation field K. Let w_0 be a closed point of $\mathcal{W}_s$. Set $W_+(w_0):=\{P\in W\mid\overline{\{P\}}\cap\mathcal{W}_s=\{w_0\}\}$, where $\overline{\{P\}}$ denotes the Zariski closure of $\{P\}$ in $\mathcal{W}$. Note that $W_+(w_0)$ depends on the choice of the model $\mathcal{W}$, even though this dependence is not explicitly indicated in our notation. Note also that if K is Henselian, then $\overline{\{P\}}\cap\mathcal{W}_s$ is alway reduced to a single point.

LEMMA 5.1. *Let K be Henselian with algebraically closed residue field k. Let $\mathcal{X}/\mathcal{O}_K$ be a regular model of a curve X/K. Fix a closed point $x\in\mathcal{X}_s$. Denote by $\Gamma_1,\dots,\Gamma_n$ the irreducible components of $\mathcal{X}_s$ containing x, with multiplicities $r_1,\dots,r_n$, respectively.*

(a) *Let $P\in X_+(x)$. Then $[K(P):K]\in\sum_{i=1}^n r_i\mathbb{N}$.* ($\mathbb{N}$ *denotes the set of positive integers*.)
(b) *Assume that $\mathcal{X}$ is a good model as in* 1.8. *Then there exists a point $P\in X_+(x)$ such that $[K(P):K]=\sum_{i=1}^n r_i$.*
(c) *Assume that $\mathcal{X}$ is a good model and that x belongs to two distinct components Γ_1 and Γ_2 of $\mathcal{X}_s$. Let $\pi\colon\mathcal{Z}\to\mathcal{X}$ be a morphism of models of X such that $\Gamma:=\pi^{-1}(x)$ is irreducible of multiplicity r_1+r_2 and meets the other components of $\mathcal{Z}_s$ in two distinct points, and such that π is an isomorphism over $\mathcal{X}\setminus\{x\}$. Then π is the blow-up of $\mathcal{X}$ with (reduced) center x. In particular, $\mathcal{Z}$ is regular.*

Proof. Consider the closed subscheme $\overline{\{P\}}$ as a divisor on $\mathcal{X}$. Then $[K(P):K]=\overline{\{P\}}\cdot\mathcal{X}_s=\sum_i r_i(\overline{\{P\}}\cdot\Gamma_i)$. This proves (a).

(b) Since $\mathcal{X}$ is a good model, $n=1$ or 2. Assume first that $n=1$. Let $u\in\mathcal{O}_{\mathcal{X},x}$ be a local equation of Γ_1 at x. Since x is regular in Γ_1, the maximal ideal of $\mathcal{O}_{\Gamma_1,x}=\mathcal{O}_{\mathcal{X},x}/(u)$ is generated by the image of some $h\in\mathcal{O}_{\mathcal{X},x}$. Thus (u,h) is the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$. Let $\mathcal{D}$ be an irreducible component of $\mathrm{div}(h)$. Then $\mathcal{D}$ intersects Γ_1 transversally at x, and thus the generic point P of $\mathcal{D}$ belongs to $X_+(x)$. It is easy to check that $K(P)$ has degree r_1 over K.

Consider now the case $n=2$. Let $\mathcal{X}'\to\mathcal{X}$ be the blow-up with center x. Denote by E the exceptional component. Then E has multiplicity r_1+r_2 (Lemma 1.4.). Let x_1 be an interior point of E. From the case $n=1$, we get a point $P\in X_+(x_1)\subseteq X_+(x)$ with $[K(P):K]=r_1+r_2$.

(c) Let $\widetilde{\mathcal{Z}}\to\mathcal{Z}$ be the minimal desingularization of $\mathcal{Z}$. Denote by $\widetilde{\Gamma}$ the strict transform of Γ in $\widetilde{\mathcal{Z}}$. The composition $\widetilde{\mathcal{Z}}\to\mathcal{X}$ is a birational morphism between

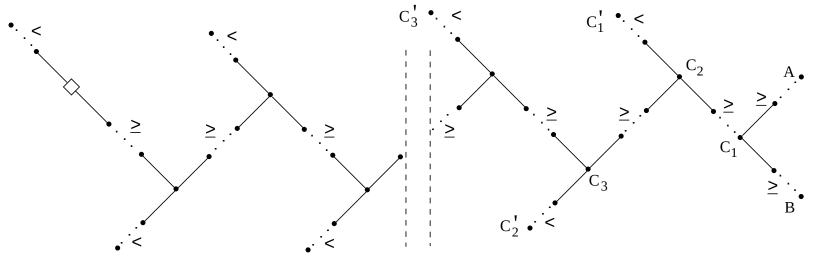

Figure 1.

two regular schemes and in thus a sequence of blow-ups. The first blow-up is the blow-up of the point x and thus its exceptional fiber Δ is a component of multiplicity $r_1 + r_2$. Any component of $\widetilde{\mathcal{Z}}_s$ lying over x has multiplicity in $r_1\mathbb{N} + r_2\mathbb{N}$. The only components of $\widetilde{\mathcal{Z}}_s$ that have multiplicity equal to $r_1 + r_2$ are components that are obtained by blowing up smooth points on components of multiplicity $r_1 + r_2$.

Since the model $\mathcal{Z}$ is obtained by contracting all the components of $\widetilde{\mathcal{Z}}_s$ lying over x except $\widetilde{\Gamma}$, and since Γ meets the rest of the fiber in two distinct points, we see that $\widetilde{\Gamma}$ can only be the component Δ. Since $\widetilde{\mathcal{Z}}$ is the minimal desingularization of $\mathcal{Z}$, none of the added components are rational of self-intersection -1 in $\widetilde{\mathcal{Z}}$. On the other hand, $\widetilde{\mathcal{Z}}$ is not a minimal regular model, and thus $\widetilde{\Gamma}$ must be an exceptional divisor. Intersection theory on $\widetilde{\mathcal{Z}}$ shows then that the multiplicity of any component that intersects $\widetilde{\Gamma}$ is strictly smaller than r_1+r_2. Hence, $\widetilde{\mathcal{Z}} = \mathcal{Z}$, and $\mathcal{Z}$ is the blow-up of $\mathcal{X}$ along x.

In the sequel, we will need the following terminology used in graph theory. Recall that a *vertex* of the *dual graph* associated to a curve represents an irreducible component of the curve, and that two vertices are linked by as many edges as the number of intersection points of the corresponding irreducible components. Recall that the degree $d(v)$ of a vertex v of a graph $\mathcal{G}$ is the number of edges of $\mathcal{G}$ attached to v. When $d(v) = 1$, the vertex is called *terminal*, and when $d(v) \geqslant 3$, the vertex is called a *node*. A terminal chain of a graph $\mathcal{G}$ attached to a vertex v is a connected component of $\mathcal{G} \setminus \{v\}$ that contains a terminal vertex but does not contain any node.

LEMMA 5.2. *Assume that k is algebraically closed. Let $\mathcal{Y}/\mathcal{O}_K$ be a normal model of a curve Y/K, and let $\pi : \mathcal{Z} \to \mathcal{Y}$ be a proper birational morphism with $\mathcal{Z}$ normal. Fix a closed point $y \in \mathcal{Y}_s$ such that $\pi^{-1}(y)$ is one-dimensional.*

(a) *Assume that y is a rational singularity. Then each irreducible component of $\pi^{-1}(y)$ is isomorphic to $\mathbb{P}^1_k$. If $\mathcal{Z}$ is regular, then the graph of $\pi^{-1}(y)$ is a tree. In general, the graph of $\pi^{-1}(y)$ may contain loops, but only because of the existence of the following configuration of curves. Let $\Gamma_1, \ldots, \Gamma_n$ be a sequence of $n \geqslant 2$ distinct components of $\pi^{-1}(y)$ such that $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$, $i = 1, \ldots, n-1$, and $\Gamma_n \cap \Gamma_1 \neq \emptyset$. Then there is a single point x in $\pi^{-1}(y)$ such that $\Gamma_i \cap \Gamma_{i+1} = \{x\}$, $i = 1, \ldots, n-1$, and $\Gamma_n \cap \Gamma_1 = \{x\}$.*

(b) *Assume that $\mathcal{Y}$ is a good model and that $\mathcal{Z}$ is regular. Then $\mathcal{Z}$ is also a good model and $\pi^{-1}(y)$ contains at least one exceptional divisor Γ. Then Γ meets the other components of $\mathcal{Z}_s$ in at most two points. Moreover, if Γ is unique, then the graph of $\pi^{-1}(y)$ can be described as in Figure 1.*

Each vertex $\bullet$ in Figure 1 represents a smooth rational curve. The symbol $\square$ indicates the position of the unique curve of self-intersection -1. The special fiber $\mathcal{Y}_s$ meets $\pi^{-1}(y)$ in one or both components A and B, and nowhere else. The symbols $\geqslant$ and $<$ next to a chain indicate that the multiplicities decrease (resp. strictly increase) along that chain when read from left to right. We have $r(C_1) = r(C_1')$, $r(C_2) > r(C_1)$, $r(C_2) = r(C_2')$, etc.

(c) *Keep the hypothesis of (b). Let $\Gamma_1, \dots, \Gamma_m$ be the exceptional divisors of $\mathcal{Z}$ contained in $\pi^{-1}(y)$. Then* $\max\{r(\Gamma_i) \mid 1 \leqslant i \leqslant m\}$ *is also the maximum of the multiplicities of the components of $\pi^{-1}(y)$.*

Proof. (a) Since y is a rational singularity, $R^1\pi_*\mathcal{O}_{\mathcal{Z}} = 0$ ([Lip2], 1.1 and 1.2 (2)). Let C be any connected reduced curve over k contained in $\pi^{-1}(y)$. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{Z}}$ be the sheaf of ideals defining C. Then the sequence $0 \to \mathcal{I} \to \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_C \to 0$ is exact. The long exact cohomology sequence gives an exact sequence

$$R^1\pi_*\mathcal{O}_{\mathcal{Z}} \to R^1\pi_*\mathcal{O}_C \to R^2\pi_*\mathcal{I}.$$

The last group vanishes since the fibers of π have dimension at most 1. Thus $\mathrm{H}^1(C, \mathcal{O}_C) = 0$. If $\mathcal{Z}$ is regular, then intersection theory on $\mathcal{Z}$ shows that each component of C is a projective line, and that the dual graph of C is a tree. When $\mathcal{Z}$ is not regular, let $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ denote the minimal desingularization of $\mathcal{Z}$, and denote by η the composition $\widetilde{\mathcal{Z}} \to \mathcal{Y}$. The curve $\pi^{-1}(y)$ is obtained by contracting components of the curve $\eta^{-1}(y)$, whose dual graph is a tree. The last statement of (a) follows.

(b) The morphism $\mathcal{Z} \to \mathcal{Y}$ consists of successive blow-ups of closed points, so $\mathcal{Z}$ is good. Since y is a regular point, $\pi^{-1}(y)$ must contain an exceptional divisor Γ. One can contract Γ and get a new regular model $\mathcal{Z}_1$ dominating $\mathcal{Y}$. All the components of $\mathcal{Z}_s$ that meet Γ then meet each other at a same point in $\mathcal{Z}_1$. But $\mathcal{Z}_1$ is a good model, so there are at most two components in $\mathcal{Z}_s$ which intersect Γ. Assume that Γ is the unique exceptional divisor of $\pi^{-1}(y)$. Then exactly one of the components of $\pi^{-1}(y)$ that meet Γ becomes the unique exceptional divisor of $(\mathcal{Z}_1)_y$. Now it is easy to check by induction on the number of components that, in case where Γ is the unique exceptional divisor in $\pi^{-1}(y)$, the shape of the divisor $\pi^{-1}(y)$ is as indicated in Figure 1.

(c) We proceed by induction on the number n of components of $\pi^{-1}(y)$. If $n = 1$, the statement holds. If $n > 1$, contract Γ_1 to get a new model $\mathcal{Z}_1$ and a factorization $\mathcal{Z} \to \mathcal{Z}_1 \to \mathcal{Y}$. Then the exceptional divisors of $(\mathcal{Z}_1)_y$ are $\Gamma_2, \dots, \Gamma_m$, and possibly the image of a component, say $\Gamma_0 \subset \pi^{-1}(y)$, that intersects Γ_1 in $\mathcal{Z}$. Since $r(\Gamma_0) \leqslant r(\Gamma_1)$, (c) holds.

We now apply the above lemma to study the exceptional locus of the minimal desingularization of a singular point of $\mathcal{Z}$.

LEMMA 5.3. *Assume that K is henselian and k is algebraically closed. Let $\mathcal{Y}/\mathcal{O}_K$ be a good model (see* 1.8*) of a curve Y/K, and let $\pi : \mathcal{Z} \to \mathcal{Y}$ be a proper birational morphism with $\mathcal{Z}$ normal. Fix a closed point $y \in \mathcal{Y}_s$ such that $\pi^{-1}(y)$ is one-dimensional.*

(a) *Let Γ be a component of $\pi^{-1}(y)$ of multiplicity r and let z be an interior point of Γ in $\mathcal{Z}$. Then z is singular in $\mathcal{Z}$ if and only if there exists $Q \in Y_+(z)$ such that $[K(Q) : K] < r$. Assume now that z is singular in $\mathcal{Z}$ and let $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ denote the minimal desingularization of z. Then $\widetilde{\mathcal{Z}}_z$ consists of a chain of vertical divisors. Moreover, as one moves away from the component Γ along the chain, the multiplicities of the components are strictly decreasing.*

(b) *Let Γ be a component of $\pi^{-1}(y)$. Assume that Γ meets a component Γ_1 in z_1 and a second component Γ_2 in z_2, with $z_1 \neq z_2$. If the multiplicities of Γ_1 and Γ_2 are strictly smaller than $r(\Gamma)$, then all the interior points of Γ are regular in $\mathcal{Z}$.*

(c) *Let z be a point of $\pi^{-1}(y)$, singular in $\mathcal{Z}$, and which belongs to exactly two components Γ_1 and Γ_2 of $\mathcal{Z}_s$. Let $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ be the minimal desingularization of z. Then each component of $\widetilde{\mathcal{Z}}_z$ has multiplicity less than or equal to* $\max\{r(\Gamma_1), r(\Gamma_2)\}$.

Proof. We shall use the following common construction in the proofs of (a)–(c). Let z be a singular point of $\mathcal{Z}$. Let $\mathcal{Z} \to \mathcal{Z}_0$ be the contraction of all the components of $\pi^{-1}(y)$ except for the components that contain z. This contraction is an isomorphism in a neighborhood of z. Let $\lambda\colon \mathcal{W} \to \mathcal{Z}_0$ be the minimal desingularization of $\mathcal{Z}_0$. Let $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ be the minimal desingularization of $z \in \mathcal{Z}$. Then $\widetilde{\mathcal{Z}}$ is isomorphic to $\mathcal{W}$ in a neighborhood of $\widetilde{\mathcal{Z}}_z$. Moreover, $\widetilde{\mathcal{Z}}_z$ is isomorphic to $\lambda^{-1}(z)$. Thus it is enough to prove the statements (a)–(c) for $\lambda^{-1}(z)$. Denote by $\eta\colon \mathcal{W} \to \mathcal{Y}$ the composition $\mathcal{W} \to \mathcal{Z}_0 \to \mathcal{Y}$. Since the regular model $\mathcal{W}$ is the minimal desingularization, $\mathcal{W}$ has the property that the only possible exceptional divisors contained in $\eta^{-1}(y)$ are the strict transforms of the components of $\pi^{-1}(y)$ that meet z. Note that since $\mathcal{W}$ is not the minimal model, at least one of these strict transforms must be exceptional.

(a) Assume that z is singular, and consider the morphism η introduced above. Then Γ is the unique exceptional divisor contained in $\eta^{-1}(y)$. Since z is an isolated point, we find, using Figure 1 in the case of the morphism $\eta\colon \mathcal{W} \to \mathcal{Y}$, that the only possibility for $\lambda^{-1}(z)$ is to correspond to the terminal chain attached to the exceptional curve Γ. This proves the statement about $\widetilde{\mathcal{Z}}_z$ (which is isomorphic to $\lambda^{-1}(z)$).

Let Δ be a component of $\lambda^{-1}(z)$. Then, as we just showed, $r(\Delta) < r$. Let $Q \in Y$ be a closed point of degree $[K(Q) : K] = r(\Delta)$ specializing to an interior point of Δ (use 5.1(b), since $\mathcal{W}$ is a good model). Then $Q \in Y_+(z)$ and $[K(Q) : K] < r$. If z is regular, then Lemma 5.1(a) implies that $[K(Q) : K] \geqslant r$ for all $Q \in Y_+(z)$.

(b) Assume that there exists a singular interior point $z \in \Gamma \subset \mathcal{Z}_s$. Let $\mathcal{W}$ be the model constructed using $z \in \mathcal{Z}$ as above. Then Γ is the unique exceptional curve on $\eta^{-1}(y)$, and thus Lemma 5.2(b) implies that Γ can meet the rest of $\eta^{-1}(y)$ in at most two points. Since z is singular, Γ certainly meets $\eta^{-1}(y)$ at z. To obtain a contradiction, we will show that each z_i gives rise to a component of $\mathcal{W}_s$ that meets Γ in z_i. We need to consider two cases. First, if $\Gamma_i \subset \pi^{-1}(y)$, then there is a point $Q_i \in Y$ of degree $r(\Gamma_i)$ that specializes to an interior of $\Gamma_i \subset \mathcal{Z}$ (use 5.1(b), since the curve Γ_i contains a point smooth in $(\mathcal{Z}_s)_{\text{red}}$ and regular in $\mathcal{Z}$). So Part (a) shows that $z_i \in \Gamma$ is singular in $\mathcal{Z}_0$ since Q_i specializes to $z_i \in \mathcal{Z}_0$ and $r(\Gamma_i) < r(\Gamma)$. Hence, a component of $\mathcal{W}_s$ meets Γ in z_i. Second, if Γ_i is not a component of $\pi^{-1}(y)$, then it is not contracted in $\mathcal{Z}_0$. Thus again a component of $\mathcal{W}_s$ meets Γ in z_i.

(c) Consider the model $\mathcal{W}$ associated to z as above. The only possible exceptional divisors in $\mathcal{W}$ are Γ_1 and Γ_2. Thus (c) is a consequence of Lemma 5.2(c) applied to $\mathcal{W} \to \mathcal{Y}$.

6. Normalization of Regular Models

Let $f: X \to Y$ be a cover of smooth, proper, and geometrically connected curves over K. It is natural to wonder whether there exist regular models $\mathcal{X}$ and $\mathcal{Y}$ of X and Y respectively, and a finite morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ such that $\varphi_K = f$. Note that in this case $\mathcal{X} = N(\mathcal{Y}, K(X))$ since φ is finite. We shall say that f *can be extended to a cover of regular models* if such a cover $\varphi: \mathcal{X} \to \mathcal{Y}$ exists.

This question was considered by Abhyankar in [Ab1]. Let $f: \mathcal{X}' \to \mathcal{Y}'$ be a finite morphism of normal schemes of dimension 2. The morphism f is said to have the property of simultaneous resolution of singularities if there exist a regular scheme $\mathcal{X}$ and a birational morphism $\pi_X: \mathcal{X} \to \mathcal{X}'$, a regular scheme $\mathcal{Y}$ and a birational morphism $\pi_Y: \mathcal{Y} \to \mathcal{Y}'$, and a finite morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ such that $f \circ \pi_X = \pi_Y \circ \varphi$. Given any cyclic group G of order $n > 3$ and any field k of characteristic prime to n, Abhyankar gave examples of G-covers of normal surfaces $\mathcal{X}' \to \mathcal{Y}'$ over k which do not satisfy the property of simultaneous resolution of singularities. In this section, we give local obstructions in some cases to a positive solution to the extension problem (6.2 and 6.4), and then use these local obstructions to construct global examples of cyclic étale morphisms of curves $f: X \to Y$ of degree $n > 3$ where the extension problem has a negative answer (6.6). We treat the case of covers of degree 2 and 3 in the next section.

Throughout this section, we assume that K is complete and k algebraically closed.

6.1. THE LOCAL CASE

Let us recall some facts about intersection theory in the following special case.

Let $\mathcal{W}$ and $\mathcal{Z}$ be local regular schemes of dimension 2, with closed points w and z, respectively. Let $\varphi\colon \mathcal{W} \to \mathcal{Z}$ be a dominant finite morphism. Consider two irreducible divisors Γ_1, Γ_2 in $\mathcal{W}$. Let $\Delta_i := \varphi(\Gamma_i)$. Assume that $\Delta_1 \neq \Delta_2$ and $\varphi^{-1}(\Delta_1) = \Gamma_1$. Define $\varphi^*\Delta_1 := e_{\Gamma_1/\Delta_1}\Gamma_1$, where e_{Γ_1/Δ_1} is the ramification index of Γ_1 over Δ_1, and $\varphi_*\Gamma_2 := [k(\Gamma_2) : k(\Delta_2)]\Delta_2$. The projection formula is the equality $\Delta_1 \cdot \varphi_*\Gamma_2 = [k(w) : k(z)]\varphi^*\Delta_1 \cdot \Gamma_2$. Since this equality is a local property on $\mathcal{Z}$, the reader will have no difficulty in proving it by adapting to the local case the proof given for arithmetic surfaces in [Lan], III, Theorem 4.1. It follows from the definitions that the above formula is equivalent to

$$(\Delta_1 \cdot \Delta_2)\deg\varphi = e_{\Gamma_1/\Delta_1}e_{\Gamma_2/\Delta_2}[k(w) : k(z)](\Gamma_1 \cdot \Gamma_2). \tag{1}$$

In particular, if $k(w) = k(z)$ and $\Gamma_1 \cdot \Gamma_2 = \Delta_1 \cdot \Delta_2 = 1$, then

$$\deg\varphi = e_{\Gamma_1/\Delta_1}e_{\Gamma_2/\Delta_2}. \tag{2}$$

Recall that a sequence of irreducible components $\Delta_1, \dots, \Delta_n$ of a model $\mathcal{Z}$ is called a *chain* if $\Delta_i \cap \Delta_{i+1}$ is a single point for all $i \leqslant n-1$ and if $\Delta_i \cap \Delta_j = \emptyset$ whenever $|i-j| \geqslant 2$.

PROPOSITION 6.2. *Let* $\mathcal{W} = \operatorname{Spec}(\mathcal{O}_K[[u, v]]/(uv - t))$. *Let G be a group of $\mathcal{O}_K$-automorphisms of $\mathcal{W}$ of order $n \geqslant 4$. Assume that G does not permute the two irreducible components of $\mathcal{W}_s$. Then for any proper birational morphism $\mathcal{Y} \to \mathcal{W}/G$ with $\mathcal{Y}$ regular, the normalization $N(\mathcal{Y}, K(\mathcal{W}))$ is a singular scheme.*

Proof. Since the components of $\mathcal{W}_s$ are not permuted by G, $(\mathcal{W}/G)_s$ has two irreducible components Δ' and Δ''. The quotient $\mathcal{W}/G$ is semi-stable over $\mathcal{O}_K$ (Proposition 1.6). Thus, if $\mathcal{W}/G$ were regular, then the components Δ' and Δ'' would intersect transversaly, and this would contradict the projection formula recalled above. Hence, $\mathcal{W}/G$ is singular. Assume that there exists $\mathcal{Y} \to \mathcal{W}/G$ as in the statement of the proposition with $\mathcal{X} := N(\mathcal{Y}, K(\mathcal{W}))$ regular. We can choose $\mathcal{Y}$ to be minimal with respect to this property.

In the first part of this proof, let us show that $\mathcal{Y}_s$ consists of a chain of irreducible components. Decompose the morphism $\mathcal{Y} \to \mathcal{W}/G$ into a sequence of modifications $\mathcal{Y} = \mathcal{Y}_q \to \mathcal{Y}_{q-1} \to \cdots \to \mathcal{Y}_0 \to \mathcal{W}/G$, where each $\mathcal{Y}_i$ is regular, $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is the blow-up along a closed point in $(\mathcal{Y}_i)_s$, and $\mathcal{Y}_0 \to \mathcal{W}/G$ is the minimal desingularization of $\mathcal{W}/G$. Since $\mathcal{W}/G$ is semi-stable, $(\mathcal{Y}_0)_s$ is a chain of rational curves $\Delta' =: \Delta_1, \Delta_2, \dots, \Delta_\ell := \Delta''$, crossing transversaly. Let $\mathcal{X}_0 := N(\mathcal{Y}_0, K(\mathcal{W}))$, and let $\varphi\colon \mathcal{X}_0 \to \mathcal{Y}_0$ be the canonical morphism.

We claim that $\varphi^{-1}(\Delta_i)$ is irreducible for all i, and that $\varphi^{-1}(\Delta_1), \dots, \varphi^{-1}(\Delta_\ell)$ is a chain of components of $(\mathcal{X}_0)_s$. Denote by Γ' and Γ'' the (smooth) components of $\mathcal{W}_s$. Then (up to renumbering) we can assume that $\varphi^{-1}(\Delta_1) = \Gamma'$ and $\varphi^{-1}(\Delta_\ell) = \Gamma''$. Fix an integer i with $2 \leqslant i \leqslant \ell - 1$. Let Γ_i be a component of $(\mathcal{X}_0)_s$ lying over Δ_i. Using repeatedly 4.2 (or its geometric form in Lemma 4.3(a)), we can construct a sequence of components $\Gamma_1, \dots, \Gamma_i, \dots, \Gamma_\ell$, with $\varphi(\Gamma_j) = \Delta_j$

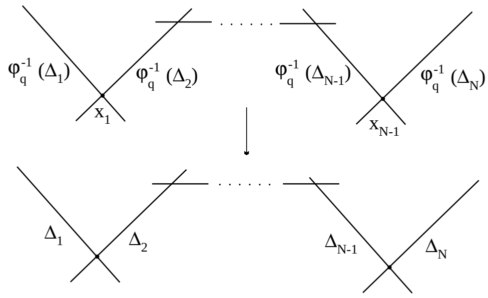

Figure 2.

and $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$. In particular, $\Gamma_1 = \Gamma'$ and $\Gamma_\ell = \Gamma''$. Denote by $\pi\colon \mathcal{X}_0 \to \mathcal{W}$ the natural morphism. Since $\mathcal{W}$ is regular at its closed point w, Lemma 5.2(a) provides information on the dual graph of the curve $\pi^{-1}(w)$. In particular, two distinct components Γ_r and Γ_s cannot intersect in more than one point. Since $(\mathcal{Y}_0)_s$ is a chain, Lemma 5.2(a) implies that the sequence $\Gamma_1, \dots, \Gamma_\ell$ must also be a chain. If there exists a different component Γ'_i of $(\mathcal{X}_0)_s$ lying over Δ_i, then we can construct a different chain of components $(\Gamma'_j)_{1\leqslant j\leqslant \ell}$ with $\Gamma'_1 = \Gamma'$ and $\Gamma'_\ell = \Gamma''$. Moreover, since $\mathcal{X}_0 \to \mathcal{W}$ is an isomorphism when restricted to Γ', the components Γ_2 and Γ'_2 intersect Γ' at the same point; hence, $\Gamma_2 \cap \Gamma'_2 \neq \emptyset$. Similarly, $\Gamma_{\ell-1} \cap \Gamma'_{\ell-1} \neq \emptyset$. If two such chains existed, then the dual graph of $\pi^{-1}(w)$ would contain a true loop, and this would contradict Lemma 5.2(a). The claim is thus proved.

Note that each component $\varphi^{-1}(\Delta_j)$, $j = 2, \dots, \ell - 1$, has multiplicity greater than one: the minimal desingularization $\widetilde{\mathcal{X}_0}$ of $\mathcal{X}_0$ is obtained by a sequence of blow-ups from $\mathcal{W}$, and any component in $(\widetilde{\mathcal{X}_0})_s$ has multiplicity greater than one except for the components corresponding to Γ' and Γ''. Denote by x_j the intersection of $\varphi^{-1}(\Delta_j)$ and $\varphi^{-1}(\Delta_{j+1})$. We claim that $\varphi^{-1}(\Delta_j) \setminus \{x_{j-1}, x_j\}$ is contained in the regular locus of $\mathcal{X}_0$. Indeed, consider the model $\mathcal{X}_{0j}$ where all the components $\varphi^{-1}(\Delta_i)$ of $\mathcal{X}_0$ have been contracted except for $i = 1$, j, and ℓ. Since $\varphi^{-1}(\Delta_1)$ and $\varphi^{-1}(\Delta_\ell)$ have multiplicity 1, our claim follows from Lemma 5.3(b) applied to $\mathcal{X}_{0j}$. Now the minimality of $\mathcal{Y}$ implies that $\mathcal{Y}_1 \to \mathcal{Y}_0$ is the blow-up along an intersection point of $(\mathcal{Y}_0)_s$. In particular, $(\mathcal{Y}_1)_s$ is still a chain of irreducible components. Repeating the same arguments for $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_q = \mathcal{Y}$ proves that $\mathcal{Y}_s$ is a chain.

Renumbering the components if necessary, we write $\mathcal{Y}_s = \cup_{1\leqslant j\leqslant N}\Delta_j$ with $\Delta_1 = \Delta'$, $\Delta_N = \Delta''$ and $\Delta_j \cap \Delta_{j+1}$ is a single point. Let φ_q be the canonical map $\mathcal{X} \to \mathcal{Y}$. Using similar arguments as above, we see that $\varphi_q^{-1}(\Delta_j)$ is irreducible, and that $\varphi_q^{-1}(\Delta_j) \cap \varphi_q^{-1}(\Delta_{j+1})$ is a single point x_j. The special fibers $\mathcal{X}_s$, $\mathcal{Y}_s$ are represented in Figure 2.

Denote by m_j the multiplicity of Δ_j in $\mathcal{Y}_s$ and by r_j that of $\varphi_q^{-1}(\Delta_j)$ in $\mathcal{X}_s$. Since $\mathcal{Y}$ is obtained by successive blow-ups of closed points starting from $\mathcal{Y}_0$, $\mathcal{Y}_s$ is a divisor with normal crossings. Similarly, the same is true for $\mathcal{X}_s$. The projection formula (2) applied to the localization of $\mathcal{X}$ at x_j implies that $n =$

$(r_j/m_j)(r_{j+1}/m_{j+1})$. Since $m_1 = r_1 = 1$, we see that $r_j = m_j$ if j is odd and $r_j = nm_j$ if j is even. Since $m_N = r_N = 1$, N must be an odd number. We will show by induction on j that, for all $1 \leqslant j \leqslant (N-1)/2$,

$$m_{2j+1} \geqslant \max\{m_{2j-1}, 2m_{2j}\}. \tag{3}$$

Since $r_2 = nm_2$ divides $r_1 + r_3 = 1 + m_3$ (intersection theory on the regular scheme $\mathcal{X}$), we have $m_3 \geqslant 2m_2$. Thus, the inequality (3) is true for $j = 1$. Assume that it is true for j and that $j+1 \leqslant (N-1)/2$. For the same reason as before, $m_{2j+3} \geqslant nm_{2j+2} - m_{2j+1}$. But $m_{2j+2} \geqslant m_{2j+1} - m_{2j}$ (intersection theory on $\mathcal{Y}$), so

$$m_{2j+3} \geqslant (n-1)m_{2j+1} - nm_{2j} \geqslant (n/2-1)m_{2j+1} \geqslant m_{2j+1},$$

(the last inequality holds because $n \geqslant 4$). On the other hand, $2m_{2j+3} \geqslant m_{2j+3} + m_{2j+1} \geqslant nm_{2j+2}$, so $m_{2j+3} \geqslant 2m_{2j+2}$, and the inequality (3) is proved for all $j \leqslant (N-1)/2$. In particular, since N is odd, we find that $m_N \geqslant 2$. This is a contradiction.

6.3. THE GLOBAL CASE

LEMMA 6.4. *Let X be a curve of genus $g(X) \geqslant 1$ over K. Denote by $\mathcal{X}_0$ its minimal regular model over $\mathcal{O}_K$. Let G be a finite subgroup of* $\mathrm{Aut}_K(X)$. *Assume that there exists $x \in (\mathcal{X}_0)_s$ such that in a neighborhood of x, $\mathcal{X}_0$ is semi-stable and not smooth, and the inertia group I_x does not permute the irreducible components of $(\mathcal{X}_0)_s$ passing through x. Then if $|I_x| \geqslant 4$, the cover $X \to X/G$ cannot be extended to a finite cover of regular models.*

Proof. Denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{\mathcal{X}_0,x}$. Then the $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{O}}_x$ of $\mathcal{O}_{\mathcal{X}_0,x}$ is isomorphic to $\mathcal{O}_K[[u,v]]/(uv-t)$, I_x acts faithfully on $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$ and does not permute the irreducible components of $\mathrm{Spec}(\widehat{\mathcal{O}}_x \otimes k)$. Let y be the image of x in $\mathcal{Y}_0 := \mathcal{X}_0/G$ and let $\widehat{\mathcal{O}}_y$ be the $\mathfrak{m}_y$-adic completion of $\mathcal{O}_{\mathcal{Y}_0,y}$. Then $\mathrm{Spec}(\widehat{\mathcal{O}}_x) \to \mathrm{Spec}(\widehat{\mathcal{O}}_y)$ is a Galois cover with group I_x.

Assume that $X \to X/G$ can be extended to a finite morphism $\mathcal{X} \to \mathcal{Y}$ between regular models. Then $\mathcal{X}$ dominates $\mathcal{X}_0$, so $\mathcal{Y}$ dominates $\mathcal{Y}_0$. The fiber product $\widehat{\mathcal{Y}} := \mathcal{Y} \times_{\mathcal{Y}_0} \mathrm{Spec}(\widehat{\mathcal{O}}_y)$ is regular since it is formally smooth over $\mathcal{Y}$, and the projection $\widehat{\mathcal{Y}} \to \mathrm{Spec}(\widehat{\mathcal{O}}_y)$ is proper and birational. The fiber product $\mathcal{X} \times_{\mathcal{Y}_0} \mathrm{Spec}(\widehat{\mathcal{O}}_y)$ is regular (formally smooth over $\mathcal{X}$), and finite over $\mathcal{Y} \times_{\mathcal{Y}_0} \mathrm{Spec}(\widehat{\mathcal{O}}_y) = \widehat{\mathcal{Y}}$. Since $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$ is a connected component of $\mathcal{X}_0 \times_{\mathcal{Y}_0} \mathrm{Spec}(\widehat{\mathcal{O}}_y)$, this implies that $N(\widehat{\mathcal{Y}}, K(\widehat{\mathcal{O}}_x))$ is regular. But this is impossible by Proposition 6.2.

Remark 6.5. One can give a stronger conclusion under the assumption of the above lemma: *for any extension of discrete valuation fields L/K, the cover $X_L \to Y_L$ cannot be extended to a cover of regular models.*

Actually, $(\mathcal{X}_0)_{\mathcal{O}_L}$ is semi-stable on a neighborhood of the point x' lying over x. Consider the minimal desingularization $\psi\colon \mathcal{X}_1 \to (\mathcal{X}_0)_{\mathcal{O}_L}$. One can see that no irreducible component of $(\mathcal{X}_1)_s$ lying over a component of $(\mathcal{X}_0)_s$ passing through x is exceptional. Thus $\mathcal{X}_1$ is isomorphic to the minimal regular model of X_L over $\mathcal{O}_L$ in a neighborhood of x'. We may then apply Lemma 6.4. to the data $(X_L, G, x', I_{x'})$.

PROPOSITION 6.6. *Let K be a complete discrete valuation field with algebraically closed residue field k. Let $n \geqslant 4$ be any integer prime to $p = \mathrm{char}(k) \geqslant 0$. Then for any $g \geqslant 0$, there exists a cyclic cover $X \to Y$ of order n, with $g(Y) = g$, which cannot be extended to a finite cover of regular models. Moreover, $X \to Y$ can be chosen to be étale if $g \geqslant 1$.*

Proof. Denote by $G = \langle\sigma\rangle$ a cyclic group of order n. In the case $g = 0$, it is possible to construct explicitly the desired covers $X \to \mathbb{P}^1_K$. Let $P(u) \in \mathcal{O}_K[u]$ be a monic polynomial such that $\widetilde{P}(u)$, its image in $k[u]$, is separable and does not vanish at 0. Assume moreover that n and $\deg(P)+p$ are coprime (when the residue characteristic is 0, $p > 1$ will denote a fixed integer coprime to n). Let X/K be the curve whose function field $K(X)/K$ is the field of fractions of the domain $K[u,v]/(v^n - (u^p - t^{pn-n}u)P(u))$. The curve X has genus $(n-1)(\deg(P) + p - 1)/2$. Let $\zeta_n \in K$ is a primitive nth root of unit. Let σ acts on $K(X)$ as $\sigma(u) = u$ and $\sigma(v) = \zeta_n v$. The quotient X/G is isomorphic to $\mathbb{P}^1_K$. The minimal regular model $\mathcal{X}_0$ of X over $\mathcal{O}_K$ is semi-stable. Indeed, $(\mathcal{X}_0)_s$ consists of two smooth components; one of the components is the normalization of the reduction of $v^n - (u^p - t^{pn-n}u)P(u)$ modulo t, which has genus equal to $(n-1)\deg(P)/2$. The other component is obtained as follows. Make a change of variables $u = t^n u_1$ and $v = t^p v_1$. Then $v_1^n = (u_1^p - u_1)P(t^n u_1)$. This equation modulo (t) gives rise to a smooth curve of genus $(n-1)(p-1)/2$. Since the genus of these two components add up to the genus of X, we see that the reduction of X s stable and that the components can intersect only in a single point x. (Note that we can also use Theorem 3.9 to determine the stable model of X). To check that x is a regular point in $\mathcal{X}_0$, we need only to note that its image under the quotient map of the stable model is a point of multiplicity n. With the notation of Lemma 6.4, we find that $I_x = G$. So $X \to \mathbb{P}^1_K$ cannot be extended to a cover of regular models.

Now assume that $g \geqslant 1$. Let C/k be a smooth proper curve of genus $g-1$. Fix two distinct points $y_1, y_2 \in C$. There exist $D \in \mathrm{Pic}^0(C)$ and $f \in k(C)$ such that $\mathrm{div}(f) = y_1 - y_2 + nD$. Consider the G-cover $E \to C$ defined by $k(C)[v]/(v^n - f)$. It is totally ramified at y_1 and y_2, and étale elsewhere. Let $\mathcal{X}_s/k$ be the stable curve obtained by gluing together the preimages in E of y_1 and y_2. Let x denote the unique singular point of $\mathcal{X}_s$. The action of G on E induces an action on $\mathcal{X}_s$. The morphism $\mathcal{X}_s \to \mathcal{X}_s/G$ is étale away from x, and we claim that this cover is of Kummer type (see [Said], 5.6(iii) and [Said2], 2.1). Indeed, let x_i be the preimage of y_i in E. Then v is a parameter of E at x_1, while v^{-1} is a parameter of E at x_2. Let σ be a generator of $\mathrm{Gal}(k(E)/k(C))$. Then there exists an n-th root of unit ξ_n such that $\sigma(v) = \xi_n v$ and $\sigma(v^{-1}) = \xi_n^{-1}v^{-1}$. Since x_1 and x_2 are the points of E lying over $x \in \mathcal{X}_s$, this shows that $\mathcal{X}_s \to \mathcal{X}_s/G$ is of Kummer type.

Let $\mathcal{Y}_s := \mathcal{X}_s/G$ and let y be the singular point of $\mathcal{Y}_s$. As in [Said], 6.3, one can lift $\mathcal{Y}_s$ to a semi-stable curve $\mathcal{Y}/\mathcal{O}_K$ such that y is a singular point of $\mathcal{Y}$ of multiplicity n. (As stated, [Said], 6.3 applies only when $\mathcal{Y}_s$ has smooth components. However, a slight modification of the proof of 6.3 leads to a proof of the desired lifting.) Now $\mathcal{X}_s \to \mathcal{Y}_s$ can be lifted to a G-cover $\mathcal{X} \to \mathcal{Y}$ of semi-stable curves of $\mathcal{O}_K$ (see [Said], 5.7). Note that the lifting in 5.7 can be made to $\mathcal{O}_K$ directly, and not to an extension of $\mathcal{O}_K$ because the multiplicity of y is a multiple of the order of the inertia group at x). We have $I_x = G$ and the multiplicity of y in $\mathcal{Y}$ is $n = |I_x|$. Since the multiplicity of x in $\mathcal{X}$ is that of y divided by $|I_x|$ (see the proof in [Ray], Proposition 5, Premier cas), the point x is regular in $\mathcal{X}$. We may thus use Lemma 6.4 to conclude that $\mathcal{X}_K \to \mathcal{Y}_K$ cannot be extended to a finite cover of regular models. Finally, by construction, the arithmetical genus of $\mathcal{X}_K$ and $\mathcal{Y}_K$ are $(g-1)n+1$ and g, respectively. Using Hurwitz's formula, we see that the morphism $\mathcal{X}_K \to \mathcal{Y}_K$ is étale.

7. Cyclic Covers of Degree 2 or 3

The counterexamples presented in the previous section are all cyclic covers of degree at least 4. In this section, we treat the case of cyclic covers of degree 2 or 3. In [Ab2], Theorems 9 and 10, Abhyankar proved that if $f\colon \mathcal{X}' \to \mathcal{Y}'$ is any cyclic cover of degree 2 or 3 between normal algebraic surfaces over a field k whose characteristic is prime to $\deg(f)$, then there are suitable desingularizations $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{Y} \to \mathcal{Y}'$ of $\mathcal{X}'$ and $\mathcal{Y}'$ such that f extends to a cyclic cover $\varphi\colon \mathcal{X} \to \mathcal{Y}$. In Theorem 7.3, we prove this result for normal models of curves, without any assumption on the residue characteristic. The method of proof that we use applies only to relative curves over an one-dimensional base. However, a version of this theorem ought to hold in the more general setting of cyclic covers $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of degree 2 or 3 between two normal excellent schemes of dimension two.

LEMMA 7.1. *Let A be a Noetherian factorial local ring and let G be a finite group of automorphisms of A. Let $B := A^G$. Assume that A is finite over B. Let $\mathfrak{q}$ be a prime ideal of height 1 in B, and let $\mathfrak{p}$ be a prime ideal of A lying over $\mathfrak{q}$. If the extension $B_\mathfrak{q} \to A_\mathfrak{p}$ has trivial residue field extension, then $\mathfrak{q}$ is a principal ideal. If in addition $B/\mathfrak{q}$ is regular, then B is regular.*

Proof. Since A/B is finite, the prime $\mathfrak{p}$ is of height 1, and is thus principal, generated by an element v. We claim that $\mathfrak{q}$ is generated by $u := \mathrm{Norm}_{A/B}(v)$. Let $S := B \setminus \mathfrak{q}$. We may apply the theory of Dedekind domains to the extension $S^{-1}A/B_\mathfrak{q}$. In particular, since the extension of residue fields induced by $A_\mathfrak{p}/B_\mathfrak{q}$ is trivial, we conclude that $\mathrm{Norm}_{S^{-1}A/B_\mathfrak{q}}(\mathfrak{p}) = \mathfrak{q}B_\mathfrak{q} = (u)$. Hence, any element b of $\mathfrak{q}$ can be written as $b = \alpha u \beta$, with $\alpha, \beta \in B$ and $\beta \notin \mathfrak{q}$. Considering the factorization of $\beta b = \alpha u$ in A, we conclude that $b = \gamma u$ for some $\gamma \in A$. Since both b and u

belong to B, γ is fixed by G and is thus in B, so $\mathfrak{q} = uB$. Finally, if $B/\mathfrak{q}$ is regular, then B is regular since $\mathfrak{q}$ is principal.

The following corollary is an immediate consequence of Lemma 7.1.

COROLLARY 7.2. *Let $\mathcal{X}$ be a regular model over $\mathcal{O}_K$ of a curve X/K. Fix a closed point $x \in \mathcal{X}_s$. Let G be a finite group acting on $\mathcal{X}/\mathcal{O}_K$ and fixing x. Denote by $\varphi\colon \mathcal{X} \to \mathcal{X}/G$ the canonical morphism. Let Γ be an irreducible component of $\mathcal{X}_s$ that passes through x. If $\varphi(x)$ is regular in $\varphi(\Gamma)$ and if $r(\Gamma) = |G| r(\varphi(\Gamma))$, then $\varphi(x)$ is regular in $\mathcal{X}/G$.*

THEOREM 7.3. *Let $\mathcal{O}_K$ be a Dedekind domain with perfect residue fields. Let $f\colon X \to Y$ be a Galois cover of curves over K, with Galois group G. Let $\mathcal{W}/\mathcal{O}_K$ be a regular model of X/K. Assume that G acts on $\mathcal{W}$. If, for every closed point $x \in \mathcal{W}_s$, the inertia group I_x has order at most 3, then f can be extended to a G-cover of regular models.*

Proof. Let $\mathcal{Y}_0$ be a good model of Y (see 1.8) which dominates $\mathcal{W}/G$. Let $\mathcal{X}_0$ be the minimal good model of X dominating $N(\mathcal{Y}_0, K(X))$. Then G acts on $\mathcal{X}_0$. Denote by S_0 the (finite) set of points of $\mathcal{X}_0$ whose images in $\mathcal{X}_0/G$ are singular. If S_0 is empty, Theorem 7.3 holds. If S_0 is not empty, consider the blow-up $\mathcal{X}_1 \to \mathcal{X}_0$ along S_0. Then $\mathcal{X}_1$ is a good model, and G acts on $\mathcal{X}_1$ since S_0 is globally fixed by G. Define similarly the set $S_1 \subset \mathcal{X}_1$ relatively to the quotient $\mathcal{X}_1 \to \mathcal{X}_1/G$. One defines in this way a sequence of blow-ups $\mathcal{X}_m \to \mathcal{X}_{m-1}$ and a sequence of subsets $S_m \subset \mathcal{X}_m$. Note that S_m is contained in the preimage of S_{m-1}. We will prove that S_1 is empty if all nontrivial inertia groups have order 2, and that S_1 or S_2 is empty if at least one inertia group has order 3.

We may assume that K is Henselian and k algebraically closed. Indeed, all of the above operations (taking regular models, blow-ups, quotients) commute with étale base change, and $\mathcal{O}_K$ is assumed to have perfect residue fields.

Let us note now that to prove Theorem 7.3, it is sufficient to consider the cases where $|G| = 2$ or 3. Since $\mathcal{Y}_0$ dominates $\mathcal{W}/G$, $\mathcal{X}_0$ dominates $\mathcal{W}$. Let $x_0 \in S_0$, let w be its image in $\mathcal{W}$. Then $I_{x_0} \subseteq I_w$. Thus $H := I_{x_0}$ has order 2 or 3. Consider the quotient $\mathcal{X}_m/H$. Denote by x_0' the image of x_0 in $\mathcal{X}_0/H$, and by η_m the canonical map $\mathcal{X}_m/H \to \mathcal{X}_0/H$. Then it is easy to check that $\mathcal{X}_m/H \to \mathcal{X}_m/G$ is étale in a neighborhood of $\eta_m^{-1}(x_0')$. Thus in a neighborhood of the preimage of x_0 by $\mathcal{X}_m \to \mathcal{X}_0$, the quotient map $\mathcal{X}_m \to \mathcal{X}_m/G$ behaves exactly like $\mathcal{X}_m \to \mathcal{X}_m/H$. Since the study of singularities in the quotient is a local problem, we can replace G by H, and suppose that $S_0 = \{x_0\}$, $|G| \leqslant 3$.

The group G fixes x_0 as well as each component of $\mathcal{X}_s$ passing through x_0; indeed, there are at most two components through x_0, and if G permuted the components, then $|G| = 2$, and Lemma 3.7 would imply that the image of x_0 is regular. Let $\mathcal{Y} := \mathcal{X}_0/G$ and let $\psi\colon \mathcal{X}_0 \to \mathcal{Y}$ be the canonical morphism. We will distinguish two cases.

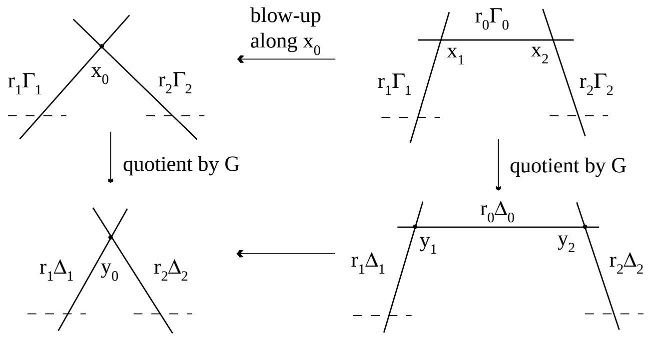

Figure 3.

First case Suppose that x_0 belongs to two irreducible components Γ_1 and Γ_2 of $(\mathcal{X}_0)_s$. Let $r_i := r(\Gamma_i)$ and $\Delta_i := \psi(\Gamma_i)$. Then Lemma 5.2(a) shows that Δ_i is a smooth rational line, and Corollary 7.2 shows then that $r(\Delta_i) = r_i$ (since $\psi(x_0)$ is assumed to be singular, and $|G|$ is prime). Consider the blow-up $\pi : \mathcal{X}_1 \to \mathcal{X}_0$ with center x_0 (Figure 3 below). Then $\pi^{-1}(x_0)$ consists of a divisor Γ_0 of multiplicity $r_0 = r_1 + r_2$, and Γ_0 is stable under G. Let x_i, $i = 1, 2$ denote the intersection $\Gamma_i \cap \Gamma_0$. If the image Δ_0 of Γ_0 in $\mathcal{X}_1/G$ has multiplicity $r_0/|G|$, then $\mathcal{X}_1/G$ is regular in a neighborhood of Δ_0 (Corollary 7.2), and thus S_1 is empty. So suppose that $r(\Delta_0) = r_0$. Denote by y_i the intersection point $\Delta_i \cap \Delta_0$, for $i = 1, 2$. The points y_1 and y_2 are singular (use the projection Formula 6.1).

Let $\mathcal{Y}_1 \to \mathcal{X}_1/G$ be the minimal desingularization of $\mathcal{X}_1/G$ at y_1 and y_2 (Figure 4 below). Let Δ be any component of $(\mathcal{Y}_1)_s$ lying over y_1, and let Γ be a component of $N(\mathcal{Y}_1, K(X))_s$ lying over Δ. Then Γ is mapped to the regular point $x_1 \in \Gamma_1 \cap \Gamma_0$ of $\mathcal{X}_1$. Thus $r(\Gamma) \in r_1\mathbb{N} + r_0\mathbb{N}$. On the other hand, Lemma 5.3(a) implies that $r(\Delta) \leqslant r_0$. So $r(\Gamma) = |G|r(\Delta)$. Hence $N(\mathcal{Y}_1, K(X)) \to \mathcal{Y}_1$ is totally ramified over Δ, in particular it induces a bijection between the set of the irreducible components of $N(\mathcal{Y}_1, K(X))_{x_1}$ and the set of irreducible components of $(\mathcal{Y}_1)_{y_1}$.

Let Δ_3 be a component of $(\mathcal{Y}_1)_s$ lying over y_1 that intersects Δ_0. Let Γ_3 be its preimage in $N(\mathcal{Y}_1, K(X))$. Define similarly Δ_4 and Γ_4. Then $r(\Gamma_i) = |G|r(\Delta_i)$ for $i = 3, 4$.

Note that $\mathcal{Y}_1$ is a desingularization of y_0, and that the only divisor of $(\mathcal{Y}_1)_s$ lying over y_0 that could possibly be exceptional is Δ_0. Since $r(\Delta_0) = r_1 + r_2 > \max\{r(\Delta_1), r(\Delta_2)\}$, Lemma 5.3(c) shows that Δ_0 is in fact exceptional. So $r(\Delta_0) = r_1 + r_2 = r(\Delta_3) + r(\Delta_4)$. When $|G| = 2$, the equality $|G|(r_1 + r_2) = r(\Gamma_3) + r(\Gamma_4)$ is impossible, and thus S_1 is empty. When $|G| = 3$, this equality is only possible if $r(\Delta_3) = (r_1 + r_0)/3$ and $r(\Delta_4) = (r_2 + r_0)/3$. In this case, $r(\Gamma_3) = r_1 + r_0$ and $r(\Gamma_4) = r_2 + r_0$. Lemma 5.1(c) implies then that Γ_3 and Γ_4 are obtained by blowing-up $\mathcal{X}_1$ at x_1 and x_2, respectively. Let $\mathcal{X}_2 \to \mathcal{X}_1$ be the blow-up with center $\{x_1, x_2\}$, represented in Figure 5.

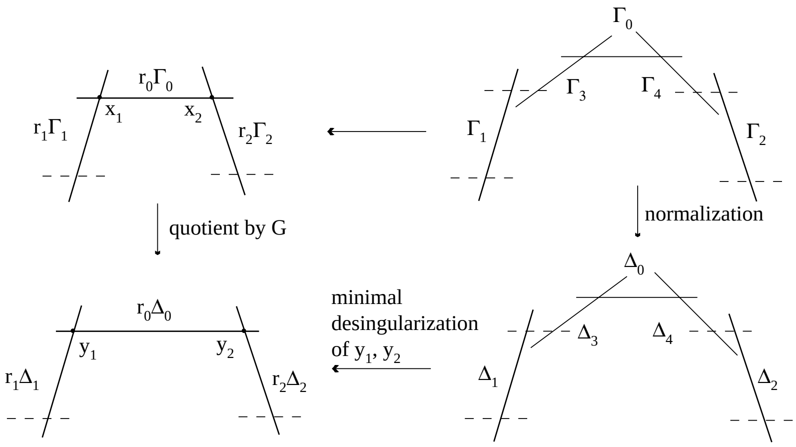

Figure 4.

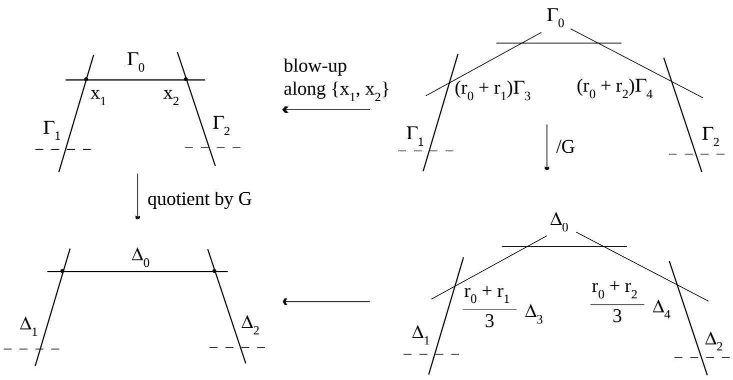

Figure 5.

It follows from Lemma 5.3(b) that $\mathcal{X}_2/G$ is regular at all interior points of Δ_0, and Corollary 7.2. implies that $\mathcal{X}_2/G$ is regular at all points of $\Delta_3 \cup \Delta_4$. This achieves the proof of the theorem in the first case.

Second case. Assume that x_0 is contained in a unique component Γ_0 of $(\mathcal{X}_0)_s$, with multiplicity r_0. Let $\rho\colon \mathcal{Y}_1 \to \mathcal{X}_0/G$ be the minimal desingularization of y_0. Let Δ_0 denote the image of Γ_0 in $\mathcal{Y}_1$. As in the previous case, we may assume that $r(\Delta_0) = r_0$. According to Lemma 5.3(a), the preimage of y_0 in $\mathcal{Y}_1$ consists of a chain of components $\Delta_1, \Delta_2, \ldots$, thus $(\mathcal{Y}_1)_s$ has the form represented in Figure 6 with $r(\Delta_{i+1}) < r(\Delta_i)$ for all $i \geqslant 0$. As in the first case (Figure 4), the preimage Γ_i of Δ_i in $N(\mathcal{Y}_1, K(X))$ is irreducible, $r(\Gamma_i) \in r_0\mathbb{N}$, and $N(\mathcal{Y}_1, K(X)) \to \mathcal{Y}_1$ is totally ramified over Δ_i. Write $r(\Gamma_i) = a_i r_0$, $r(\Delta_i) = a_i r_0/|G|$ with $|G| > a_1 > a_2 > \cdots$. Moreover, if $a_j = 1$, then $r(\Gamma_j) = r_0$, and Lemma 5.3(a) implies that the interior points of Γ_j are regular in $N(\mathcal{Y}_1, K(X))$.

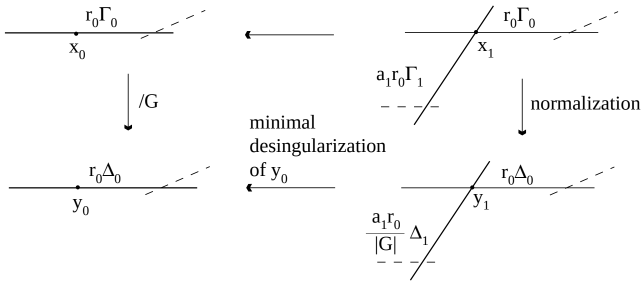

Figure 6.

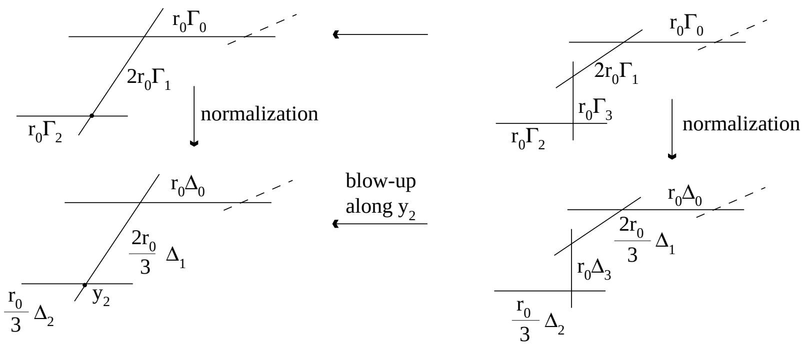

Figure 7.

Let us show that $x_1 \in \Gamma_1 \cap \Gamma_0$ is regular. Let $P \in X_+(x_1)$, and let $Q = \psi(P) \in Y$. Then $[K(P) : K] \in r_0\mathbb{N}$, and $[K(Q) : K] \in a_1 r_0\mathbb{N}/|G| + r_0\mathbb{N}$. Since $[K(P) : K(Q)] = 1$ or $|G|$, and since $|G|$ is prime, it is easy to see that $[K(P) : K] > a_1 r_0$. Lemma 5.3(c) implies that x_1 is regular.

If $a_1 = 1$, then $\rho^{-1}(y_0) = \Delta_1$, and in this case we have proved above that Γ_1 is contained in the regular locus of $N(\mathcal{Y}_1, K(X))$. It follows that $N(\mathcal{Y}_1, K(X)) \to \mathcal{X}_0$ is the blow-up of $\mathcal{X}_0$ with center x_0. Since $a_1 = 1$ is automatically true when $|G| = 2$, Theorem 7.3 is proven in this case.

It remains to treat the case where $|G| = 3$, $a_1 = 2$ and $a_2 = 1$. We proved already that the interior points of Γ_2 are regular. Lemma 5.3(b) implies that the interior points of Γ_1 are also regular. The projection formula (6.1) shows that the intersection point of Γ_2 and Γ_3 is singular. Let $y_2 \in \Delta_1 \cap \Delta_2$. Let $\mathcal{Y}_2 \to \mathcal{Y}_1$ be the blow-up with center y_2. Denote by Δ_3 its exceptional divisor and by Γ_3 the preimage of Δ_3 in $N(\mathcal{Y}_2, K(X))$. Then Γ_3 is irreducible.

Let us show that $r(\Gamma_3) = r(\Delta_3) = r_0$. Let $y_3 = \Gamma_3 \cap \Gamma_1$. If $r(\Gamma_3) = 3r(\Delta_3)$, the projection formula 6.1 shows that the point y_3 is singular. Consider the minimal desingularization $\mathcal{Z}$ of $N(\mathcal{Y}_2, K(X))$ at y_3. Any component of $\mathcal{Z}$ above y_3 has multiplicity ar_0, with $a \leqslant 3$ (Lemma 5.3.(c)). Moreover, $ar_0/|G|$ or $ar_0 \in$

$2r_0/3\mathbb{N} + r_0\mathbb{N}$. We see that we must have $ar_0 \in 2r_0/3\mathbb{N} + r_0\mathbb{N}$, and that the only possibility is $a = 3$ et $3r_0 = 3(2r_0/3) + r_0$. Consider now the component Γ of $\mathcal{Z}$ above y_3 that meets Γ_1. Then Γ has multiplicity $3r_0$, and meets one component of multiplicity $2r_0$ and, say, j components of multiplicity $3r_0$. Since $\mathcal{Z}$ is regular above y_3, the self-intersection of Γ is equal to $(3jr_0 + 2r_0)/3r_0$. Since the self-intersection is an integer, we have obtained a contradiction, and we find that $r(\Gamma_3) = r(\Delta_3) = r_0$.

As noted above already, Lemma 5.3.(a) implies, since $r(\Gamma_3) = r_0$, that the interior points of Γ_3 are regular in $N(\mathcal{Y}_2, K(X))$. Two arguments similar to the one given above in the case of x_1 show that the intersection points of Γ_3 with Γ_2 and Γ_1 are regular. It follows that $N(\mathcal{Y}_2, K(X))$ is regular and equal to $\mathcal{X}_2$ (notation as at the beginning of the theorem). This concludes the proof of Theorem 7.3.

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