REDUCTION IN THE CASE OF IMPERFECT RESIDUE FIELDS

DINO LORENZINI

Let K be a complete field with a discrete valuation v, ring of integers $\mathcal{O}_K$, and maximal ideal (π_K). Let $k := \mathcal{O}_K/(\pi_K)$ be the residue field, assumed to be separably closed of characteristic $p \geq 0$. Let X/K be a smooth proper geometrically irreducible curve of genus $g \geq 1$. Let $\mathcal{X}/\mathcal{O}_K$ denote a regular model of X/K. Let $\mathcal{X}_k = \sum_{i=1}^{v} r_i C_i$ be its special fiber, where C_i/k is an irreducible component of $\mathcal{X}_k$ of multiplicity r_i. Let $e(C_i)$ denote the geometric multiplicity of C_i (see [BLR], 9.1/3). In particular, $e(C_i) = 1$ if and only if C_i/k is geometrically reduced. Any reduced curve C/k is geometrically reduced when k is perfect. Associate to $\mathcal{X}/\mathcal{O}_K$ the field extension $k_{\mathcal{X}}/k$ generated by the following three types of subfields: by the fields $H^0(C, \mathcal{O}_C)$, where C is any irreducible component of $\mathcal{X}_k$; by the fields of rationality of all points P such that P is the intersection point of two components of $\mathcal{X}_k^{red}$; and by the fields of rationality of all points Q that belong to geometrically reduced components and such that Q is not smooth.

We will mostly be interested in this article in the properties of the minimal regular model $\mathcal{X}^{min}$ and the minimal regular model with normal crossings $\mathcal{X}^{nc}$ (see, e.g., [Liu], 10.1.8). The combinatorics of the special fiber of these models when k is imperfect is studied in section 1. Theorem 1.1 and Proposition 1.8 show that for these two models, if $k_{\mathcal{X}} \neq k$ or if there exists a component of $\mathcal{X}_k$ that is not geometrically reduced, then k is imperfect and $p \leq 2g + 1$.

Let A/K be any abelian variety of dimension g. Let L/K denote the extension of K minimal with the property that A_L/L has semi-stable reduction ([Gro], IX.4.1, page 355). It is known that $[L : K]$ is bounded by a constant depending on g only, and that if q is a prime dividing $[L : K]$, then $q \leq 2g + 1$. We let k_L denote the residue field of $\mathcal{O}_L$. It follows that if $k_L \neq k$, then $p \leq 2g + 1$. When k is imperfect and A is the jacobian of X, it is natural to wonder whether there are any relationships between k_L/k and the extension $k_{\mathcal{X}}/k$ introduced above. In this regard, we show in 2.12 that if $k_{\mathcal{X}} \neq k$ or if there exists a component of $\mathcal{X}_k$ that is not geometrically reduced, then $p \mid [L : K]$. We show in 2.4 that this statement cannot be strengthened as follows: if $k_{\mathcal{X}} \neq k$ or if there exists a component of $\mathcal{X}_k$ that is not geometrically reduced, then it is not necessarily the case that $k_L \neq k$ (when $\mathcal{X}$ is either $\mathcal{X}^{nc}$ or $\mathcal{X}^{min}$). The converse to this latter statement, that is, if $k_{\mathcal{X}} = k$ and all components of $\mathcal{X}_k$ are geometrically reduced, then $k_L = k$, is not true in general when $\mathcal{X} = \mathcal{X}^{min}$ (2.13), but we provide some evidence that it could be true when $\mathcal{X} = \mathcal{X}^{nc}$.

Let $\mathcal{A}/\mathcal{O}_K$ denote the Néron model of an abelian variety A/K, with special fiber $\mathcal{A}_k/k$ and group of components $\Phi_{A,K}$. Let A'/K denote the dual abelian variety, with Néron model $\mathcal{A}'/\mathcal{O}_K$ and group of components $\Phi_{A',K}$. Grothendieck's pairing

$$\langle\ ,\ \rangle_K : \Phi_{A,K} \times \Phi_{A',K} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Date: September 5, 2002.

is introduced in [Gro], IX, 1.2. Grothendieck conjectured in [Gro], IX, 1.3, that this pairing is always perfect, but this conjecture was shown in [B-B] to be incorrect in general when the residue field is imperfect (see also [B-L]). It is also shown in [B-B], 3.1, that if $p \nmid [L : K]$, then $\langle\ ,\ \rangle_K$ is perfect. When k is imperfect, it is natural to wonder whether there are any relationships between the properties of L/K and the fact that $\langle\ ,\ \rangle_K$ is perfect. We show in this regard in 2.4 that, even for elliptic curves, the triviality of the extension k_L/k does not in general imply that $\langle\ ,\ \rangle_K$ is perfect.

We produce in section 3 new and easy examples of pairings that are not perfect, including in the case of elliptic curves (2.6, 3.5, 3.9), using a formula of Grothendieck pertaining to the behaviour of the pairing under base change. This formula is recalled at the beginning of section 3 and, among other applications, is also used to give a quick new proof of a variation on a theorem of McCallum. We give an example in 3.6 of two K-isogenous abelian varieties A/K and B/K such that Grothendieck's pairing for A is perfect while Grothendieck's pairing for B is not.

The author warmly thanks A. Bertapelle, S. Bosch and Q. Liu for their insighful comments. Special thanks are due to M. Raynaud for many long and fruitful conversations. This article owes much to his contributions.

1. A bound on the geometric multiplicity

Let X/K be a proper smooth geometrically irreducible cuve of genus $g \geq 1$. Let $\mathcal{X}/\mathcal{O}_K$ be a regular model of X/K, with special fiber $\mathcal{X}_k = \sum_{i=1}^{v} r_i C_i$. We let M denote the intersection matrix $((C_i \cdot C_j))$. Fix a component C of $\mathcal{X}_k$. We let $h^i := h^i(C)$ denote the dimension over k of $H^i(C, \mathcal{O}_C)$. We also let $e := e(C)$ denote the geometric multiplicity of C, and $r := r(C)$ denote its multiplicity. It is shown in [BLR], 9.1/8, that e divides the intersection number $(C \cdot D)$ for any divisor D on $\mathcal{X}_k$. It is clear that h^0 divides e. The integers h^0 and h^1 are related by the adjunction formula

$$C \cdot C + C \cdot \mathcal{K} = 2h^1 - 2h^0,$$

where $\mathcal{K}$ denotes the relative canonical divisor. In particular, $h^0(C)$ divides $h^1(C)$. The same formula applied to $\mathcal{X}_k$ instead of C reads:

$$2g - 2 = \sum_{i=1}^{v} r_i(|C_i \cdot C_i| - 2h^0(C_i) + 2h^1(C_i)). \tag{1}$$

Recall that any component C such that $h^0(C) = |C \cdot C|$ and $h^1(C) = 0$ can be blown down so that the resulting model is again regular (see, e.g., [Chi], 3.1, [Liu], 9.3/8). A regular model is called *minimal* if no component is such that $h^0(C) = |C \cdot C|$ and $h^1(C) = 0$. Recall also that on a minimal regular model, the quantity $|C \cdot C| - 2h^0(C) + 2h^1(C)$, that is, $C \cdot \mathcal{K}$, is always non-negative (see, e.g., [Liu], 9.3/10).

Theorem 1.1. *Let X/K be a proper smooth geometrically irreducible cuve of genus $g \geq 2$. Let $\mathcal{X}/\mathcal{O}_K$ be its regular minimal model, with special fiber $\mathcal{X}_k = \sum_{i=1}^{v} r_i C_i$. Then*

(i) If $e(C_i) > 1$ for some i, then $p \leq 2g + 1$.

(ii) Assume that $p \geq 5$. Then $e(C_i) \leq 5(g-1)$ for all $i = 1, \ldots, v$.

Proof: Fix a component C of $\mathcal{X}_k$ with $p \mid e$. Then $(C \cdot C) < 0$, unless $\mathcal{X}_k = rC$, in which case $(C \cdot C) = 0$. When $(C \cdot C) = 0$, we find that $2g - 2 = r(C \cdot \mathcal{K})$ and, thus, $re \leq 2g - 2$ since e divides $(C \cdot \mathcal{K})$ and $(C \cdot \mathcal{K}) \neq 0$ because $g > 1$. Let us assume from now on that $(C \cdot C) < 0$. We start with three easy lemmata.

Lemma 1.2. *Let p be any prime. The quantity $|C \cdot C| - 2h^0(C)$ is strictly negative only when $h^0(C) = e = |C \cdot C|$, and in this case $re \leq 2g-2$. In particular, $p \leq 2g-2$.*

Proof: We know that $h^0(C) \mid e$. Assume that $h^0(C) \leq e/p$. Then

$$|C \cdot C| - 2h^0(C) \geq e - 2h^0(C) \geq e - 2e/p \geq 0.$$

When $h^0(C) = e$, we find that $|C \cdot C| - 2h^0(C) < 0$ only when $C \cdot C = -e$. It follows from the adjunction formula for $\mathcal{X}_k$ that when $h^0(C) = e = |C \cdot C|$,

$$2g - 2 \geq r|C \cdot C| - 2rh^0(C) + 2rh^1(C) \geq re.$$

Lemma 1.3. *(a) Let p be any prime. Assume that $|C \cdot C| \geq 3e$, or that $|C \cdot C| = 2e$ and $e > h^0(C)$. Then $re \leq 2g - 2$. In particular, $p \leq 2g - 2$.*
(b) Let $p \geq 5$ be prime. If $|C \cdot C| = e$ and $e > h^0(C)$, then $re \leq 10(g-1)/3$. If, in addition, $e = p$, then $e \leq 2g$. In all cases, $p \leq 2g + 1$.

Proof: If $|C \cdot C| \geq 3e$, then

$$2g - 2 \geq r|C \cdot C| - 2rh^0(C) \geq 3re - 2re \geq re.$$

If $|C \cdot C| = 2e$ and $e > h^0(C)$, then

$$2g - 2 \geq r|C \cdot C| - 2rh^0(C) \geq 2re - 2re/p,$$

so $re \leq p(g-1)/(p-1) \leq 2g - 2$.

If $|C \cdot C| = e$ and $e > h^0(C)$, then

$$2g - 2 \geq r|C \cdot C| - 2rh^0(C) \geq re - 2re/p,$$

so $re \leq p(2g-2)/(p-2)$.

Lemma 1.4. *Let $p \geq 5$ be a prime. If a component C of geometric multiplicity e intersects a component C_1 with multiplicity $r_1 \leq 3r/2$ and $h^0(C_1) < e$, then $er \leq 5(g-1)$. When $e = p$, $e \leq 2g + 1$. In all cases, $p \leq 2g + 1$.*

Proof: Note first that

$$|C_1 \cdot C_1| r_1 \geq (C \cdot C_1) r \geq er.$$

Then

$$2g - 2 \geq r_1(|C_1 \cdot C_1| - 2h^0(C_1)) \geq er - 2r_1 e/p \geq e(r - 3r/p).$$

Since we assume that $p \geq 5$, we find that $er \leq 5(g-1)$. When $e = p$, we further find that $e \leq 2g + 1$. When $p^2 \mid e$, it follows from $e \leq 5(g-1)$ that $p \leq g - 1$. □

Let us proceed with the proof of Theorem 1.1. Consider a component C such that $e := e(C)$ is maximum among the geometric multiplicities of the components of $\mathcal{X}_k$. In view of 1.2 and 1.3, we can assume that all components C' of $\mathcal{X}_k$ which have maximal geometric multiplicity satisfy $2h^0(C') = 2e = |C' \cdot C'|$, otherwise 1.1 holds.

Consider the following graph G: its components are the irreducible components of $\mathcal{X}_k$, and two components C_i and C_j are linked in G by one edge if and only if $(C_i \cdot C_j) > 0$. Let G' denote the subgraph of G consisting of all the components of maximal geometric multiplicity e (and self-intersection $2h^0$). Let $G(C)$ denote the connected component of G' that contains C. Suppose that $G(C)$ contains w components. Consider now the $w \times w$ principal minor $M(C)$ of M defined as follows: $M(C) = ((C_i \cdot C_j))$, where C_i, C_j belong to $G(C)$. Since $g > 1$, we cannot have $w = v$, so the minor $M(C)$ defines a negative definite quadratic form. By hypothesis, each entry in $M(C)$ is divisible by e. Thus, $M(C)/e$ is an integer matrix whose coefficients on the main diagonal are all equal to -2, and which is definite negative. Such matrices are well-known, and are listed for instance in [Des], 4.6. They correspond to the Dynkin diagrams A_n, D_n, E_6, E_7, and E_8. In particular, $G(C)$ is a tree with at most one node. We may, without loss of generality, assume that C is a component of $G(C)$ such that $r(C)$ is minimal among the multiplicities of all components of $G(C)$.

The only case where 1.1 is not yet proven is the case where C intersects no components C_1 of $\mathcal{X}_k$ with $r_1 \leq 3r/2$ and $h^0(C_1) < e$. We may thus assume that if a component C_1 intersects C, then either $3r/2 < r_1$, or $h^0(C_1) = e$. If C meets a component C_1 with $h^0(C_1) = e$, we may assume that $|C_1 \cdot C_1| = 2e$, otherwise 1.1 holds. Then, C_1 belongs to $G(C)$ and by minimality of r, $r \leq r_1$. It follows that we may assume that any component C_1 that meets C is such that $r_1 \geq r$. Since $|C \cdot C| = 2e$, we find from the relation

$$|C \cdot C| r = \sum_{C_j \neq C} (C \cdot C_j) r_j$$

that C can only be of three types: either

(a) C intersects in $\mathcal{X}_k$ only a single component C_1, and $(C \cdot C_1) = e$ and $r_1 = 2r$, or

(b) C intersects in $\mathcal{X}_k$ only a single component C_1, and $(C \cdot C_1) = 2e$, $r = r_1$, and $h^0(C_1) = e = e(C_1)$, or

(c) C intersects in $\mathcal{X}_k$ exactly two components C_1 and C_2, and $(C \cdot C_1) = e = (C \cdot C_2)$, $r = r_1 = r_2$, and $h^0(C_1) = e = h^0(C_2)$.

In case (c), we note that we may also assume that $2e = |C_i \cdot C_i|$ for $i = 1, 2$, otherwise 1.1 is proved. If all components of $\mathcal{X}_k$ of maximal geometric multiplicity are of type (c), then the special fiber consists only of such components, and it follows that $g = 1$, contradicting our hypothesis. Thus, we may assume that there is a component C as in (a) or (b).

Consider first the case (b) where the component C_1 is such that $(C \cdot C_1) = 2e$, $r = r_1$, and $h^0(C_1) = e$. Since e is maximum, $e = e(C_1)$. We may also assume that $(C_1 \cdot C_1) = 2e$, otherwise 1.1 is proved. Then C and C_1 are the only components of $\mathcal{X}_k$, and the genus is 1, contradicting our hypothesis.

Consider now the case (a) where the component C_1 is such that $(C \cdot C_1) = e$ and $r_1 = 2r$. If C_1 only meets C, then the intersection matrix of $\mathcal{X}_k$ (with vector of multiplicities) should be

$$\begin{pmatrix} -2e & e \\ e & -e/2 \end{pmatrix} \begin{pmatrix} r \\ 2r \end{pmatrix},$$

which cannot occur because $|C_1 \cdot C_1| = e/2$ must be an integer, and we assume that $p \geq 3$. Thus, C_1 meets at least one other component C_2. Hence, we find that

$$r_1|C_1 \cdot C_1| \geq r(C \cdot C_1) + r_2(C_1 \cdot C_2) \geq er + fh^0(C_1),$$

for some integer $f \geq 1$. Suppose first that $h^0(C_1) < e$. Then

$$\begin{array}{rcl} 2g-2 & \geq & r_1(|C_1 \cdot C_1| - 2h^0(C_1)) \\ & \geq & (er + h^0(C_1)) - 4rh^0(C_1) \\ & \geq & er - (4r-1)e/p \\ & = & e(r - \dfrac{4r-1}{p}) \\ & \geq & e(\dfrac{r+1}{5}). \end{array}$$

The last inequality is true only if $p \geq 5$, which we assume. This allows us to conclude that $e \leq 5(g-1)$. When $e = p$, we find that $e \leq 2g+1$. Thus, in all cases, we found that $p \leq 2g+1$.

Suppose now that $h^0(C_1) = e$. Again, we may assume that $(C_1 \cdot C_1) = 2e$, since otherwise 1.1 is proved. We now return to the tree $G(C)$. We just showed that we may assume in this case that it has at least two vertices. To conclude the proof of 1.1, we now proceed to a case by case analysis for each type of tree $G(C)$. The component C is a terminal vertex of $G(C)$. We define a subtree $G'(C)$ of $G(C)$ as follows: $G'(C)$ contains C, which is at distance 0 from C in $G(C)$. A component C_n of $G(C)$ that is at distance n from C in $G(C)$ belongs to $G'(C)$ if the unique component C_{n-1} of $G(C)$ at distance $n-1$ from C that meets C_n is such that all components of $\mathcal{X}_k$ that meet C_{n-1} are in $G(C)$. By construction, any vertex D of $G'(C)$ that meets a component of $\mathcal{X}_k$ that is not in $G(C)$, say D_1, is a terminal vertex of $G'(C)$. When $G'(C)$ has a node, we will be able to show that there exist such components D and D_1 with $r(D) \geq r(D_1)$. We will conclude the proof of 1.1 in this case by applying Lemma 1.4.

The first case that we will consider, though, is the case where $G'(C)$ is a chain $\{C, C_1, \ldots, C_s\}$ of length $s+1$, $(s \geq 1)$. By construction, there exists a component D_1 not in $G(C)$ that meets C_s. We obtain the following intersection matrix (and vector of multiplicities):

$$\begin{pmatrix} -2e & e & & & & \\ e & -2e & e & & & \\ & e & \ddots & \ddots & & \\ & & \ddots & -2e & e & \\ & & & e & -d & f \\ & & & & f & \ddots \end{pmatrix} \begin{pmatrix} r \\ 2r \\ \vdots \\ (s+1)r \\ (s+2)r \\ \vdots \end{pmatrix},$$

where $d = |D_1 \cdot D_1|$, f may be 0, and such that either $|D_1 \cdot D_1| \neq 2e$, or $h^0(D_1) < e$. If $|D_1 \cdot D_1| \neq 2e$ and $h^0(D_1) = e$, then 1.2 and 1.3 show that 1.1 holds. When $h^0(D_1) < e$, we find that

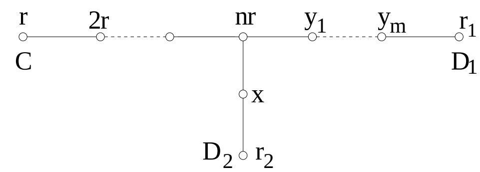
$$\begin{array}{rcl} 2g-2 & \geq & r(D_1)(|D_1 \cdot D_1| - 2h^0(D_1)) \\ & \geq & r(C_s)(C_s \cdot D_1) - 2(s+2)rh^0(D_1) \end{array}$$

$$\begin{aligned} &\geq\ e((s+1)r - \frac{2(s+2)r}{p}) \\ &\geq\ 3er(\frac{s+2}{5}). \end{aligned}$$

Again, we also find that when $e = p$, then $e \leq 2g+1$. Thus, in all cases, $p \leq 2g+1$.

It remains to discuss the following possibilities where the graph $G'(C)$ has a node.

Case 1



(where $n \geq 2$). Represented above are the vertices of $G'(C)$ and their multiplicities, as well as the vertices D_1 and D_2 where $G'(C)$ meets the rest of the special fiber that is not in $G(C)$. We have the relations

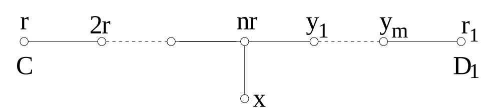
$$\begin{aligned} 2nr &= (n-1)r + x + y_1 \\ 2x &\geq nr + r_2 \\ 2y_1 &\geq nr + y_2, \end{aligned}$$

where we let $y_2 := r_1$ if $m = 1$. (The last two lines above are inequalities only, because we do not specify with what multiplicities D_1 and D_2 meet $G'(C)$, or if other components meet the components of multiplicities x and y_m.) Thus

$$(x - r_2) + (y_1 - y_2) \geq (n-1)r > 0.$$

Therefore, either $(x - r_2) > 0$ or $(y_1 - y_2) > 0$. If $y_1 - y_2 > 0$, we find that $y_1 > y_2 > \cdots > y_m > r_1$. If $h^0(D_1) = e$, then $|D_1 \cdot D_1| \neq 2e$ by construction since $D_1 \notin G(C)$. Thus we can apply 1.2 and 1.3 to conclude. If $h^0(D_1) < e$, then we conclude using 1.4 since $y_m > r_1$. When $x - r_2 > 0$, we argue similarly using D_2 instead of D_1.

Case 2

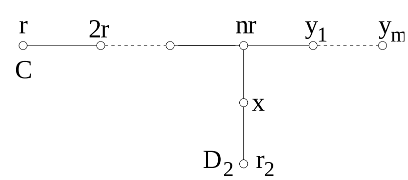


(where $n \geq 2$). This case and Case 3 are similar to Case 1, except that now $G'(C)$ meets the fiber minus the components in $G(C)$ in only one component. We have the relations

$$\begin{aligned} 2nr &= (n-1)r + x + y_1 \\ 2x &= nr \\ 2y_1 &\geq nr + y_2, \end{aligned}$$

where we let $y_2 := r_1$ if $m = 1$. Thus, $y_1 - y_2 \geq nr - y_1 = (n-2)r/2 \geq 0$. Since $y_1 \geq y_2$, we find that $y_1 \geq y_2 \geq \cdots \geq y_m \geq r_1$ and we conclude as in Case 1.

Case 3

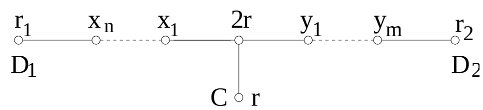


We have the relations

$$\begin{array}{rcl} 2nr & = & (n-1)r + x + y_1 \\ 2x & \geq & nr + r_2. \end{array}$$

Thus $x - r_2 \geq nr - x = y_1 - r$. By minimality of r, $y_1 - r \geq 0$. We conclude as in Case 1 using the fact that $x \geq r_2$.

Case 4

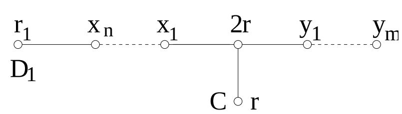


We have the relation $4r = x_1 + y_1 + r$. Thus, either $x_1 < 2r$ or $y_1 < 2r$. Assume without loss of generality that $x_1 < 2r$. Then

$$2r > x_1 > x_2 > \cdots > x_n > r_1.$$

We conclude as in Case 1 using $x_n > r_1$.

Case 5



From the relations $y_{m-1} = 2y_m, \ldots, 2y_2 = y_1 + y_3$, we find that $y_1 = my_m$. From $4r = x_1 + r + y_1$, we find that $y_1 \leq 2r$, since $x_1 \geq r$ by minimality of r. Hence, $y_m \leq 2r/m$. Thus, the minimality of r forces $m \leq 2$. The case $m = 1$ is such that $y_1 \mid 2r$, so $y_1 = r$. This case is treated in Case 2 above with $n = 2$. The case $m = 2$ gives $y_1 = 2y_2$, $2y_1 = 2r + y_2$, so $3y_2 = 2r$. But $y_2 < r$ contradicts the minimality of r and this case does not happen. This concludes the proof of Theorem 1.1.

Example 1.5 It is natural to wonder what is the best possible bound in 1.1 (ii). We present below examples with $e(C) = 2g + 2$. Consider the curve X/K given by the equation

$$y^2 = f(x) = \pi(x^e + \pi(a_{e-1}x^{e-1} + \cdots + a_1 x) - b)$$

with $a_i \in \mathcal{O}_K$, $b \in \mathcal{O}_K^*$, $x^e - \overline{b}$ irreducible in $k[x]$, and $e = p^r$ for some $r \geq 1$. This curve is such that $e = 2g+1$ if p is odd, and $e = 2g+2$ if $p = 2$. When p is odd, the reduction consists of three irreducible components C_1, C_2, and C_3, of multiplicity 1, 2, and 1, respectively. The first two components are smooth rational curves over k, and the component C_3 is a smooth rational curve over $k(\sqrt[e]{\overline{b}})$, so that $h^0(C_3) = e$. The associated intersection matrix is:

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -(e+1)/2 & e \\ 0 & e & -2e \end{pmatrix}.$$

Using the results of Raynaud [BLR], 9.6, we find that $\Phi_{\mathrm{Jac}(X),K} = (0)$.

When $p = 2$, the reduction consists of two irreducible components C_1 and C_2, of multiplicity 2 and 1, respectively. The first component is a smooth rational curve over k, and the second component is a smooth rational curve over $k(\sqrt[e]{b})$, so $h^0(C_2) = e$. The associated intersection matrix is:

$$\begin{pmatrix} -e/2 & e \\ e & -2e \end{pmatrix}.$$

The results of Raynaud [BLR], 9.6, cannot be applied to compute $\Phi_{\mathrm{Jac}(X),K}$.

To prove the above claims, start with the model $\mathcal{X}_0$ obtained by glueing

$$\mathrm{Spec}\ \mathcal{O}_K[x,y]/(y^2 - f(x))$$

with

$$\mathrm{Spec}\ \mathcal{O}_K[u,v]/(v^2 + \pi - \pi^2(a_{e-1}u + \cdots + a_1u^{e-1}) - \pi bu^e) \quad \text{when} \quad p = 2,$$

and with

$$\mathrm{Spec}\ \mathcal{O}_K[u,v]/(v^2 + \pi u - \pi^2(a_{e-1}u^2 + \cdots + a_1u^e) - \pi bu^{e+1}) \quad \text{when} \quad p \geq 3.$$

The special fiber of $\mathcal{X}_0$ is an irreducible component of multiplicity 2, with one singular point if $p = 2$, namely the point corresponding to the ideal $(\pi, y, x^e - b)$. When p is odd, it has two singular points, namely $(\pi, y, x^e - b)$ and the reduction of the point at infinity of X/K, (π, u, v). Blowing up the singular points resolve the singularities and produces the desired model.

Example 1.6 Consider the following 'potential' special fiber, consisting in three components C_1, C_2, C_3 satisfying the following properties (we do not know if this data can arise as the special fiber of some curve X/K). Let $e = p^s$, $p \geq 5$ and $s \geq 2$. The intersection matrix and vector of multiplicities are

$$\begin{pmatrix} -2e & e & 0 \\ e & -e(p+1)/2p & e/p \\ 0 & e/p & -2e/p \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

We assume moreover that $h^0(C_1) = e$, and $h^0(C_2) = h^0(C_3) = e/p$. We set $h^1(C_i) = 0$ for $i = 1, 2, 3$. Then the adjunction formula shows that $g - 1 = \frac{p-3}{2p}e$. Hence, we find that $e = 2g - 2 + \frac{6}{p-3}(g-1)$. Thus, if such a data can correspond to the special fiber of a curve, then Theorem 1.1 is sharp when $p = 5$.

Remark 1.7 The geometric multiplicity $e(C)$ of a component C of the regular minimal model of a curve of genus 1 is not bounded by an absolute constant. For further information on the reduction of curves of genus 1, see [LLR].

As a complement to Theorem 1.1, we show:

Proposition 1.8. *Let X/K be a proper smooth geometrically irreducible curve of genus $g \geq 1$. Let $\mathcal{X}/\mathcal{O}_K$ be its regular minimal model.*

(i) Let C/k be a geometrically reduced component of $\mathcal{X}_k$. Let $P \in C$ be a point that is not smooth. If $k(P) \neq k$, then $p \leq 2g + 1$.

(ii) Let C and D be two distinct geometrically reduced components of $\mathcal{X}_k$. Then $(C \cdot D) \leq g + 1$. In particular, if C and D meet in a point P such that $k(P) \neq k$, then $p \leq g + 1$.

Proof: Assume that C is geometrically reduced, but not smooth. Let $\tilde{C}/k$ denote the normalization of C/k. Then $h^1(C) \geq h^1(\tilde{C})$ and $h^1(\tilde{C}) \geq (p-1)/2$ ([Tat], Cor. 1). Since $2g-2 \geq |C \cdot C| - 2h^0(C) + 2h^1(C) \geq 2h^1(C) - 2$, (i) follows.

Let us now prove (ii). Let r and r' denote the multiplicities of C and D, respectively. It follows from $\mathcal{X}_k \cdot C = \mathcal{X}_k \cdot D = 0$ that $r|C \cdot C| \geq r'(C \cdot D)$ and $r'|D \cdot D| \geq r(C \cdot D)$. Hence,

$$\begin{aligned} 2g-2 &\geq r(|C \cdot C| - 2h^0(C)) + r'(|D \cdot D| - 2h^0(D)) \\ &\geq (r+r')[(C \cdot D) - 2], \end{aligned}$$

so that $(C \cdot D) \leq g+1$. If C and D meet in a point P such that $k(P) \neq k$, then $[k(P):k] \mid (C \cdot D)$, and (ii) follows. □

Corollary 1.9. *Let X/K be a proper smooth geometrically irreducible curve of genus $g \geq 1$. Then*

(i) $k_{\mathcal{X}^{min}} \subseteq k_{\mathcal{X}^{nc}}$.
(ii) If $k_{\mathcal{X}^{min}} \neq k$, then $p \leq 2g+1$.
(iii) If $k_{\mathcal{X}^{min}} = k$ and $k_{\mathcal{X}^{nc}} \neq k$, then $p(p-1) \leq 2g$.
(iv) If $p > 2g+1$, then $k_{\mathcal{X}^{min}} = k_{\mathcal{X}^{nc}} = k$, and all components of $\mathcal{X}^{min}$ and $\mathcal{X}^{nc}$ are geometrically reduced.
(v) If $p(p-1) > 2g$ and $k_{\mathcal{X}^{min}} = k$, then $k_{\mathcal{X}^{min}} = k_{\mathcal{X}^{nc}}$, and all components of $\mathcal{X}_k^{min}$ are geometrically reduced if and only if all components of $\mathcal{X}_k^{min}$ are geometrically reduced.

Proof: Consider the natural contraction map $\mathcal{X}^{nc} \to \mathcal{X}^{min}$, obtained by a sequence of blow-ups of points. A blow-up of a regular point P has the following properties: the exceptional divisor E is a smooth rational curve over $k(P)$, and $h^0(E) = [k(P):k] = E \cdot E$. Part (i) follows immediately from these facts. Part (ii) when $k_{\mathcal{X}^{min}} \neq k$ is an immediate consequence of 1.1 and 1.8. Let us now prove Part (iii). Consider any intermediate blow-up $\mathcal{Y} \to \mathcal{Z}$ of a point P in $\mathcal{Z}$, with

$$\mathcal{X}^{nc} \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{X}^{min}.$$

If P is a k-rational point, intersection of two or more components that are smooth at P, then the exceptional divisor E is a smooth rational line over k, which intersects all components of $\mathcal{Y}$ at k-rational points. In particular, $k_{\mathcal{Y}} = k_{\mathcal{Z}}$. Thus, when $k_{\mathcal{X}^{min}} = k$, in order for $k_{\mathcal{X}^{nc}} \neq k$, we find that there always exists a blow-up $\mathcal{Y} \to \mathcal{Z}$ of a k-rational point P such that one of the components C passing through P is not smooth, and the exceptional divisor E intersects a component D of $\mathcal{Y}_k$ in a point Q that is not rational. Then $E \cdot D$, which equals the multiplicity μ of P in the image C of D (see, e.g., [Liu], exer. 9.2/9, (d)), is divisible by $[k(Q):k]$. Hence,

$$p(p-1) \leq 2h^1(D) + \mu(\mu-1) = 2h^1(C) \leq 2g.$$

(See [Liu], exer. 9.2/12 (b), for the equality above.)

To prove (iv), we note that it follows from (ii) and (iii) that $k_{\mathcal{X}^{min}} = k_{\mathcal{X}^{nc}} = k$. It follows from 1.1 that all components of $\mathcal{X}^{min}$ are geometrically reduced. An exceptional component of $\mathcal{X}^{nc} \to \mathcal{X}^{min}$ that is not geometrically reduced is obtained as a blow-up of a point that is not k-rational. If such a point existed in one of the intermediate blow-ups of the contraction $\mathcal{X}^{nc} \to \mathcal{X}^{min}$, then (iii) would show that $p(p-1) \leq 2g$, which is a contradiction. Part (v) is left to the reader.

Remark 1.10 In the theory of reduction of curves in the case of perfect residue fields, one finds two finiteness statements:

(i) Given $g \geq 2$ and $m \geq 0$, there exists only finitely many reduction types of curves of genus g with chains of length at most m (see for instance [Des], Théorème 4.5).

(ii) Let X/K be a curve whose jacobian A/K has toric rank t_K equal to zero. Let u_K denote the unipotent rank of A/K. Then $|\Phi_{A,K}| \leq 2^{2u_K}$. More precisely,

$$\sum_{q \text{ prime}} \mathrm{ord}_q(|\Phi_{A,K}|)(q-1) \leq 2u_K$$

(see [Lo1], 2.4, and also [BLR], 9.6/9, for a weaker statement).

It is likely that statements analogue to (i) and (ii) are also true in the case where the residue field is imperfect.

2. Relations between the fields k_L/k and $k_{\mathcal{X}}/k$

Theorem 2.1. *Let X/K be a proper smooth geometrically irreducible curve of genus $g \geq 1$. Let $\mathcal{X}$ denote either $\mathcal{X}^{min}$ or $\mathcal{X}^{nc}$. If $k_{\mathcal{X}} \neq k$ or if there exists a component of $\mathcal{X}_k$ that is not geometrically reduced, then $p \mid [L:K]$.*

Proof: Let us assume that $p \nmid [L:K]$, which implies that L/K is a cyclic Galois extension with Galois group $\mathrm{Gal}(L/K) = <\sigma>$. Under this hypothesis, we can describe a regular model with normal crossings $\mathcal{X}$ of X/K over $\mathcal{O}_K$ using the quotient/desingularization construction. The model $\mathcal{X}$ is such that all components are smooth, and all intersection points are k-rational. Since $\mathcal{X}^{nc}$ and $\mathcal{X}^{min}$ are obtained from $\mathcal{X}$ be a series of contractions, the statement of the theorem follows.

Let us briefly recall now the quotient/desingularization construction. Let $\mathcal{Y}/\mathcal{O}_L$ be the minimal regular model of X_L/L. The map σ induces a canonical morphism $X_L \to X_L$ over the map $\sigma : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$. Since X_L is the generic fiber of $\mathcal{Y}$, the map σ induces a birational proper map $\mathcal{Y} \to \mathcal{Y} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\mathcal{O}_L)$ over $\mathrm{Spec}(\mathcal{O}_L)$. By the universal property of a minimal model, this map extends to a morphism from $\mathcal{Y}$ to $\mathcal{Y} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\mathcal{O}_L)$ over $\mathrm{Spec}(\mathcal{O}_L)$. Since $\mathcal{Y}$ is reduced and separated, this extension is unique. Hence, there exists then a unique automorphism $\tau : \mathcal{Y} \to \mathcal{Y}$ over the automorphism $\sigma : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_L)$.

Let $G := <\tau>$, with $\tau : \mathcal{Y} \to \mathcal{Y}$ lifting $\sigma : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_L)$. The following fact is standard: Since $\mathcal{Y}/\mathcal{O}_L$ is projective, the quotient $\mathcal{Z} := \mathcal{Y}/G$ can be constructed in the usual way by glueing together the rings of invariants of G-invariant affine open sets of $\mathcal{Y}$. The scheme $\mathcal{Z}/\mathcal{O}_K$ is normal and, hence, its singular points are closed points of its special fiber. We let $f : \mathcal{Y} \longrightarrow \mathcal{Z}$ denote the quotient map.

The normal scheme $\mathcal{Z}$ has quotient singularities. A desingularization $\nu : \mathcal{X} \to \mathcal{Z}$ leads to a regular model $\mathcal{X}/\mathcal{O}_K$ of X/K. When L/K is tame and $\mathcal{Y}$ is the minimal semi-stable regular model of X_L, the quotient singularities of $\mathcal{Z}$ are well-understood, due to the fact that the action of an automorphism at a fixed point can be linearized (see, e.g., [J-M], 2.4). We recall the properties of the resolutions of cyclic quotient singularities below, closely following Viehweg's article [Vie]. We refer the reader to his work for more details (see also e.g., [Lip], 206-212, or [Pin], 12-15, or [CES], section 2). Unfortunately, none of the references above discuss cyclic quotient singularities in the generality needed in this article, namely for discrete valuation rings $\mathcal{O}_K$ with possibly imperfect residue fields and possibly of mixed characteristic. We recall below the statements on cyclic quotient singularities needed in this article. The results quoted below are mostly proven in the literature only in equicharacteristic $p \geq 0$ and with algebraically closed residue fields. We believe that when L/K is tame, these results can be proved in the general case using arguments similar to the ones found in the literature to prove these statements under more restrictive hypotheses.

Since $\mathcal{Y}/\mathcal{O}_L$ is a semi-stable model of X_L/L, $\mathcal{Y}_k = \bigcup Y_i$ is reduced and is a divisor with normal crossings. Each irreducible component Y_i has at worst ordinary double points as singularities. All singular points of $\mathcal{Y}_k$ are defined over k (see, for instance, 10.3/7 in [Liu]). The ramification locus of a tamely ramified morphism of smooth curves $Y_i \to Z_j$ is always defined over k (see, for instance, [L-L], 3.3).

2.2 ([Vie, section 6] Let $y \in \mathcal{Y}$ be a closed point. Let $I_y := \{\tau \in G \mid \sigma(y) = y\}$. We shall call a point $y \in \mathcal{Y}$ with $I_y \neq \{id\}$ a ramification point, and the image of this point in $\mathcal{Z}$ will be called a branch point. Let $\{z_1, \ldots, z_d\}$ denote the set of closed branch points of the morphism $\mathcal{Y}_k \to \mathcal{Z}_k$. The map $\mathcal{Y} \to \mathcal{Z}$ is etale outside the preimage $\mathcal{Y}'$ of $\mathcal{Z} \setminus \mathcal{Z}_{sing}$ since any closed point in $\mathcal{Y}'$ has trivial inertia ([Gro2], Exp. V, 2.2). A ramification point y is either a singular point of $\mathcal{Y}_k$, and in this case it is k-rational, or it is a smooth point of a component Y_i and a ramification point of the restriction of the map $\mathcal{Y} \to \mathcal{Z}$ to Y_i. As we mentioned above, such a point is also k-rational. It follows that the completion of the local ring of $\mathcal{Y}$ at y is of the form $\mathcal{O}_L[[u]]$ or $\mathcal{O}_L[[u, v]]/(uv - \pi_L)$ (see, e.g., [Liu] 10.3/22 (b)), as in the classical case. The action of σ can then be linearized, and the singularity of the ring of invariants can be described explicitly. It follows that $\{z_1, \ldots, z_d\}$ is the set $\mathcal{Z}_{sing}$ of singular points of $\mathcal{Z}$, and there exists a regular scheme $\mathcal{X}/\mathcal{O}_K$ and a proper birational morphism $\nu : \mathcal{X} \to \mathcal{Z}$ such that ν induces an isomorphism between $\mathcal{X} - \{\nu^{-1}(\mathcal{Z}_{sing})\}$ and $\mathcal{Z} - \{\mathcal{Z}_{sing}\}$. Moreover, for any $z \in \mathcal{Z}_{sing}$, $\nu^{-1}(z)$ is a connected chain of rational curves, where we call *chain of rational curves on $\mathcal{X}$* a divisor D such that:

1. $D = \bigcup_{i=1}^{q} E_i$, E_i smooth and rational curve over k, for $i = 1, \ldots, q$.
2. $(E_i \cdot E_{i+1}) = 1$ for all $i = 1, \ldots, q - 1$ and $(E_i \cdot E_j) = 0$ for all $j \neq i + 1$. Moreover, $(E_i \cdot E_i) \leq -2$ for all i. Let us call E_1 and E_q the end-components of the chain.

If the preimage of a singular point z in the normalization of $\mathcal{Y}_k$ consists of a single point, then z belongs to a single component E of the chain $\nu^{-1}(z)$, and this component is an end-component. Moreover, if z belongs to the component Z_i of $\mathcal{Z}_k$, then Z_i meets E with normal crossings.

If the preimage of a singular point z in the normalization of $\mathcal{Y}_k$ consists of two distinct points, then z belongs to two components Z_i and Z_j of $\mathcal{Z}_k$. The chain $\nu^{-1}(z)$ meets Z_i with normal crossings at one of its end-component, and it meets Z_j with normal crossings at the other end-component. (Viehweg states in 8.1.d) on page 306 of [Vie] that the model $\mathcal{X}$, obtained by taking the quotient of $\mathcal{Y}$ and then resolving the singularities, has normal crossings.)

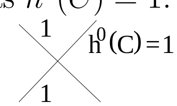
Let Z_j be an irreducible component of $\mathcal{Z}$, and let Y_i be a component of $\mathcal{Y}$, preimage of Z_j under the map $\mathcal{Y} \to \mathcal{Z}$. We claim that Z_j is geometrically reduced. Indeed, let $V_j \subset Z_j$ be an affine open set whose affine preimage U_i under $Y_i \to Z_j$ is smooth and allows us to consider V_j as the quotient of U_i by a finite group H. We only need to show that V_j is smooth or, equivalently, that $V_j \times_k \overline{k}$ is normal. This latter fact follows from the fact that taking quotients commutes with flat base change, and the fact that the quotient of a normal affine scheme is normal. Hence, the component Z_j/k are geometrically reduced. All singular points of $\mathcal{Z}_k^{red}$ are k-rational since these singularities are images of k-rational points in $\mathcal{Y}_k$. Hence, all singular points of $\mathcal{Z}$ have k as their residue field. Since the resolution of the singularities of $\mathcal{Z}$ does not introduce exceptional components that are not geometrically reduced, and since all intersection points on exceptional components are k-rational, we find that $k_{\mathcal{X}} = k$, and all components of the model $\mathcal{X}$ are geometrically reduced.

The proof of Theorem 2.1 is now easy. The contraction $\mathcal{X} \to \mathcal{X}'$ of a smooth rational curve over k of self-intersection -1 produces a regular scheme such that all components are geometrically reduced, and all singular points of $\mathcal{X}'_k$ are k-rational. Hence, $k_{\mathcal{X}^{min}} = k_{\mathcal{X}^{nc}} = k$, and all components of $\mathcal{X}_k^{min}$ and $\mathcal{X}_k^{nc}$ are geometrically reduced. □

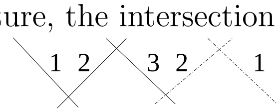
We show below in 2.4 that 2.1 cannot be strengthened to state: *If $k_{\mathcal{X}} \neq k$ or if there exists a component of $\mathcal{X}_k$ that is not geometrically reduced, then $k_L \neq k$ (when $\mathcal{X}$ is either $\mathcal{X}^{nc}$ or $\mathcal{X}^{min}$).* We consider the converse of this statement in 2.9 and 2.13.

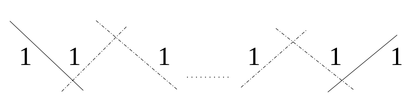
2.3 Let us recall here for the convenience of the reader the types of reduction of elliptic curves that are not classical Kodaira types. Assume first that $p = 2$. In the notation of [Sz], the special fiber of the types X_1, Y_1, and K_1, are irreducible. Each has a singular point, defined over a quadratic extension of k for X_1 and Y_1, and over k for K_1. The type K_1 is the only one that does not have normal crossings (see 2.13).

In the diagrams below, the type X_2 in [Sz] is described as T_0. A segment represents a smooth projective line defined over k, a dotted segment represents a smooth projective line defined over an inseparable extension of k of degree p (the same extension for all such components in a given diagram), and a segment adorned with the symbols $h^0(C) = 1$ represents a component of multiplicity 1 that is not geometrically reduced, but has $h^0(C) = 1$.

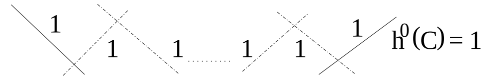
The case Y_2: 

In this above picture, the intersection point is rational over k.

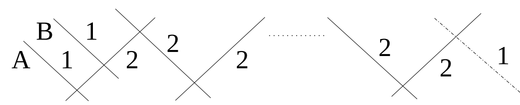
The case Y_3: 

The case K_{2n}: 

The reduction K_2 consists of two smooth rational curves over k meeting in a point P defined over a quadratic extension of k.

The cases K'_{2n} and K_{2n+1} $(n \geq 1)$: 

In the case K_{2n+1}, the last component (with $h^0(C) = 1$) has a k-rational point. In case K'_{2n}, the last component does not have a rational point. All intersection points are defined over the same quadratic extension of k. Note that the reduction K_3 has the same picture as the reduction Y_2, but in the case K_3, the intersection point is not rational over k.

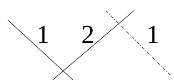
The case T_n ($n \geq 0$, with $n{+}4$ components): 

We report in the following table the degree of $k_{\mathcal{X}^{min}}/k$ for each type of reduction.

T	X_1	Y_1	Y_2	Y_3	K_1	K_{2n}	K_{2n+1}	K'_{2n}	T_n
$[k_{\mathcal{X}^{min}} : k]$	2	2	1	2	1	2	2	2	2

.

The case where $p = 3$ is considerably easier, as there are only two types of reduction that are not classical Kodaira types. The reduction Z_1 has a unique irreducible component, with a singular point defined over a cubic extension of k.

The case Z_2: 

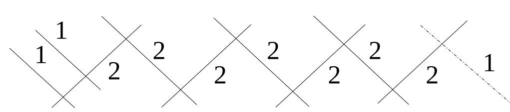
Both cases have $[k_{\mathcal{X}^{min}} : k] = 3$ and $|\Phi| = 1$.

Example 2.4 We exhibit below an example of an elliptic curve X/K such that: 1) $\mathcal{X}^{min} = \mathcal{X}^{nc}$, 2) $\mathcal{X}^{min}_k$ has a component that is not geometrically reduced, 3) Grothendieck's pairing for X is not perfect, and 4) $k_L = k$.

Thus, this example shows that it may happen that $k_L = k$ even though $\mathcal{X}^{min}_k$ has a component that is not geometrically reduced and Grothendieck's pairing for X is not perfect.

Lemma 2.5. *Assume that* $\pi = 2$, *and* $b \in \mathcal{O}^*_K$, $\overline{b} \notin (k^*)^2$. *Let* X/K *be the elliptic curve given by* $y^2 = x^3 + \pi x^2 + b\pi^9$. *Then*

(i) X/K *has reduction over* K *given by*



The last component on the right is isomorphic to $\mathbb{P}^1/k(\sqrt{\overline{b}})$. *All other components are smooth rational curves over* k. *All intersection numbers are equal to* 1, *except for the last two components, where the intersection number is* 2. *We find that* $\Phi_K = \mathbb{Z}/2\mathbb{Z}$ *and* $\langle\ ,\ \rangle_K$ *is trivial.*

(ii) X/K *has good reduction over* $L = K(\sqrt{\pi})$, *so* $k_L = k$.

Proof: To determine the reduction over K, we may use [Sz] 3.8 (the reduction has type T_6 in the notation of [Sz]). Alternatively, we may follow Tate's Algorithm as given in [Sil] up to Step 7 on page 374, and make the necessary adjustments to find the desired reduction. Since the minimal regular model of X/K has only two smooth components of multiplicity 1, and since the Néron model of X/K is obtained as the smooth locus of $\mathcal{X}/\mathcal{O}_K$, we find that $\Phi_K = \mathbb{Z}/2\mathbb{Z}$. The pairing $\langle\ ,\ \rangle_K$ is computed below:

Lemma 2.6. *Let X/K be an elliptic curve with reduction of type T_n. Then $\Phi_{X,K} = \mathbb{Z}/2\mathbb{Z}$, and Grothendieck's pairing $\langle\ ,\ \rangle_K$ is trivial.*

Proof: Label the components of the special fiber by $A, B, \ldots$, as on the picture of the type T_n in 2.3. Let M denote the intersection matrix associated with T_n and this ordering. Let R denote the vector of multiplicities, transpose of $(1, 1, 2, \ldots, 2, 1)$. The group $\Phi_{X,K}$ can be identified with a subgroup of the group $\mathrm{Ker}({}^tR)/\mathrm{Im}(M)$ (see, e.g., [B-L], 2.2). Let τ denote the image of the vector $(1, -1, 0, \ldots, 0)$ in $\mathrm{Ker}({}^tR)/\mathrm{Im}(M)$. Using 3.7 in [Lo3], we find that τ corresponds to a generator of $\Phi_{X,K}$. The value $\langle\tau, \tau\rangle_K = 0$ is computed using [B-L], 4.6 and 5.1. $\square$

Over the field $L = K(\sqrt{\pi})$, the equation

$$y^2 = x^3 + \pi x^2 + b\pi^9$$

is not minimal. We can divide it by π^3, make the appropriate change of variables, and get a new equation

$$y^2 = x^3 + x^2 + b\pi^6.$$

Changing variables again, we obtain the equation

$$y^2 + \pi xy = x^3 + b\pi^6$$

which is not minimal. We can divide by π^6 and make a last change of variables to obtain the equation

$$y^2 + xy = x^3 + b.$$

Since $\overline{b} \neq 0$, the equation $y^2 + xy = x^3 + \overline{b}$ defines an elliptic curve over k.

Example 2.4 shows that in general $k_{\mathcal{X}^{nc}}$ is not contained in k_L. More precisely, 2.4 shows that:

(i) It is not true in general that if $\mathcal{X}_k^{nc}$ has a component C with $e(C) > 0$, then $k_L \neq k$, and
(ii) It is not true in general that if a component C of $\mathcal{X}_k^{nc}$ is such that the field $H^0(C, \mathcal{O}_C)$ is not equal to k, then k_L contains a subfield isomorphic to $H^0(C, \mathcal{O}_C)$.

We now present an example to show that, in general, k_L is not contained in $k_{\mathcal{X}^{nc}}$.

Example 2.7 Suppose that there exists a field K of equicharacteristic 2 and an elliptic curve E/K with good reduction over $\mathcal{O}_K$ having a group of automorphisms over K of order divisible by 8. Suppose also that there exists a Galois extension L/K of degree 8 with Galois group isomorphic to the quarternion group Q_2 and such that $[k_L : k] = 8$. (We will show below that such a data exists.) Assuming the existence of such a data, choose an injective homomorphism

$$\mathrm{Gal}(L/K) \longrightarrow \mathrm{Aut}(E/K).$$

This injection defines an element in the set $H^1(G_{\overline{K}/K}, \mathrm{Aut}(E))$, which we use to twist E/K to obtain a new elliptic curve E'/K. The curve E'/K has potentially good reduction, being isomorphic to E_L/L over L, but does not have good reduction over K since it is the twist of an elliptic curve that has good reduction over K. A similar argument over any subfield $K \subsetneq F \subsetneq L$ shows that L/K is the extension of K minimal with the property that E'_L/L has good reduction. By construction,

$[k_L : k] = 8$. The reader may now check, using the list of possible reductions of elliptic curves when $p = 2$ given in [Sz] (see 2.3), that the field $k_{\mathcal{X}^{nc}}$ associated with any of these reductions has degree at most 2 over k. Hence, k_L is not contained in $k_{\mathcal{X}^{nc}}$. We now turn to show the existence of the necessary data.

Suppose that $\mathcal{O}_F$ is a discrete valuation ring with maximal ideal (π). Let F'/F be any Galois extension with Galois group G, given by an Eisenstein equation

$$f(w) = w^{p^n} + a_{p^n-1}w^{p^n-1} + \cdots + a_1 w + \pi.$$

Let u be an undeterminate, and set $K := F(u)$ and $\mathcal{O}_K := \mathcal{O}_F[u, \frac{\pi}{u^{p^n}}]_{(u)}$. The automorphisms of $F[w]/(f)$ over F (i.e., the elements of G) clearly extend to automorphisms of $F(u)[w]/(f)$ over $F(u)$. Let A denote the integral closure of $\mathcal{O}_K$ in $F(u)[w]/(f)$. Then A contains $\frac{w}{u}$, since

$$f(w)/u^{p^n} = (\frac{w}{u})^{p^n} + \cdots + \frac{\pi}{u^{p^n}} \in \mathcal{O}_K[\frac{w}{u}]$$

is an integral equation for $\frac{w}{u}$ over $\mathcal{O}_K$. The residue field k of $\mathcal{O}_K$ is $(\mathcal{O}_F/\pi)(\frac{\pi}{u^{p^n}})$. Thus, the residue field k_A of A at the maximal ideal containing $((\frac{w}{u})^{p^n} - \frac{\pi}{u^{p^n}}, u)$ is inseparable of degree p^n over k. By construction, the field of fractions of A is Galois over $F(u)$ with Galois group G.

Returning to the data needed for Example 2.7, we note that it is well known that there exist discrete valuation rings $\mathcal{O}_F$ and Galois extensions F'/F with Galois group the quaternions Q_2. We may thus apply the above construction to obtain such an extension L/K with $[k_L : k] = [L : K]$. Let now K'/K be any finite Galois extension with trivial residue extension. Then $L' := K'L$ is such that $\mathrm{Gal}(L'/K') = \mathrm{Gal}(L/K)$ and $[k_{L'} : k] = [k_L : k]$. Consider now the curve E/K with $y^2+y = x^3$. There exists a finite Galois extension K'/K with trivial residue extension such that the group of automorphism of $E_{K'}/K'$ over K' has order divisible by 8. (To prove this claim, use for instance explicit equations for the automorphisms.) Hence, passing from L/K to L'/K' if necessary, we find that the data needed for Example 2.7 exists. □

Consider the statement:

2.8 If $k_{\mathcal{X}} = k$ and all components of $\mathcal{X}_k$ are geometrically reduced, then $k_L = k$.

We provide below some evidence that this statement is true for $\mathcal{X}^{nc}$. It is false for $\mathcal{X}^{min}$ (2.13).

Proposition 2.9. *Let X/K be a proper smooth geometrically irreducible curve of genus $g \geq 1$. Let $\mathcal{X} = \mathcal{X}^{nc}$ be its minimal regular model with normal crossings. Assume that $k_{\mathcal{X}} = k$ and that all components of $\mathcal{X}_k$ are geometrically reduced. Let F/K be any finite extension such that $[k_F : k] = [F : K]$. Then X_F/F has semistable reduction over $\mathcal{O}_F$ if and only if X/K has semistable reduction over $\mathcal{O}_K$.*

Remark 2.10 Suppose that $K \neq L$. Under the hypotheses of 2.8, we find that 2.9 implies that $[k_L : k] < [L : K]$. The conclusion of 2.8 is that $[k_L : k] = 1$. This conclusion holds true when $[L : K]$ is a prime number, a fact that we shall exploit in 2.12 to prove that 2.8 is true in some cases.

Proof of 2.9: Let us first note the following reduction step. The extension k_F/k is obtained by adjoining to k a finite number of elements ζ_i that are purely inseparable over k. Thus the extension F/K contains a sequence of subextensions $K \subset K(z_1) \subset$

$\cdots \subset K(z_1, \ldots, z_s) = F$, where $z_i \in \mathcal{O}_F$ for all i, and its minimal polynomial over $K(z_{i-1})$ reduces modulo π to an irreducible expression of the form $z_i^p - t$, for some t in $k_{K(z_{i-1})}$. We are thus reduced to proving the statement of 2.9 for extensions of the form $F = K(z)$, where z is integral over $\mathcal{O}_K$ and whose image generates k_F/k. Let $f(x) \in \mathcal{O}_K[x]$ denote its minimal polynomial over $\mathcal{O}_K$. Note that $\mathcal{O}_F = \mathcal{O}_K[z]$, since the maximal ideal of $\mathcal{O}_K[z]$ is principal (it equals (π)).

Lemma 2.11. *The model $\mathcal{X}' := \mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_F$ is regular.*

Proof: Consider any closed point $P \in \mathcal{X}_k$ with residue field k. Then the preimage of P under $\mathcal{X}' \to \mathcal{X}$ consists of a unique point P' which is regular with residue field k_F. Indeed, let (u, v) denote the maximal ideal of $\mathcal{O}_{\mathcal{X},P}$. Then the ring $\mathcal{O}_{\mathcal{X},P} \otimes_{\mathcal{O}_K} \mathcal{O}_K[x]/(f(x))$ is a local ring with maximal ideal generated by u and v.

Since $k_{\mathcal{X}} = k$, every component of $\mathcal{X}_k$ is smooth, because a regular point with residue field k is smooth (see, e.g., [BLR], 2.2/15). Since k is separably closed, every component of $\mathcal{X}_k$ has a k-rational point (see, e.g., [BLR], 2.2/13). Thus, every component of $\mathcal{X}'_{k_F}$ has a regular point, and we find that $\mathcal{X}'$ has at most finitely many singular (closed) points. In particular, $\mathcal{X}'$ is normal. Consider a minimal resolution $\mathcal{Z} \to \mathcal{X}'$ of the singularities of $\mathcal{X}'$. Write $\mathcal{X}_k = \sum r_i C_i$, and let $D_i := C_i \times_k k_F$, so that $\mathcal{X}'_{k_F} = \sum r_i D_i$. Denote by $D_i^* \subset \mathcal{Z}_{k_F}$ the strict transform of $D_i \subset \mathcal{X}_{k_F}$ in $\mathcal{Z}$. Note that if $P \in D_i^* \cap D_j^*$, then the intersection number of D_i^* and D_j^* at P is equal to the intersection number of C_i and C_j at the image of P. The equality $(\mathcal{Z}_{k_F} \cdot D_i^*) = 0 = (\mathcal{X}_k \cdot C_i)$ shows that $|D_i^* \cdot D_i^*| \geq |C_i \cdot C_i|$. Let $\mathcal{K}_{\mathcal{Z}}$ and $\mathcal{K}$ denote the canonical divisors of $\mathcal{Z}$ and $\mathcal{X}$. It follows from the adjunction formula for C_i and for D_i^* that $0 \leq C_i \cdot \mathcal{K} \leq D_i^* \cdot \mathcal{K}_{\mathcal{Z}}$. If $\mathcal{X}'$ is not regular, write $\mathcal{Z}_{k_F} = \sum r_i D_i^* + \sum s_i E_i$. Using the formula

$$2g - 2 = \mathcal{Z}_{k_F} \cdot \mathcal{K}_{\mathcal{Z}} = \mathcal{X}_k \cdot \mathcal{K}$$

and the last inequality above, we find that $|D_i^* \cdot D_i^*| = |C_i \cdot C_i|$ and $s_i = 0$ for all i. Hence, $\mathcal{X}'$ is regular. □

It follows from this lemma that $\mathcal{X}'$ is semistable if and only if $\mathcal{X}$ is semistable. Proposition 2.9 follows. Slighlty more can be said for elliptic curves.

Proposition 2.12. *Let X/K be an elliptic curve. Then 2.8 is true for $\mathcal{X}^{min}$ when $p = 3$ (and, hence, also for $\mathcal{X}^{nc}$), and is true for $\mathcal{X}^{nc}$ when $p = 2$ and $j(X) \in \mathcal{O}_K^*$.*

Proof: We first note that every elliptic curve X/K when $p = 3$ has a minimal Weierstrass model of the form $y^2 = x^3 + a_2x^2 + a_4x + a_6$ (see [Sz], 3.9). It is also well known that $3 \mid [L : K]$ if and only if the splitting field F/K of the extension given by the polynomial $f(x) = x^3 + a_2x^2 + a_4x + a_6$ has degree divisible by 3, since the points of order 2 are defined over an extension of L of degree a power of 2 ([Gro], IX, 4.7). Assume that $k_L \neq k$. Then $f(x)$ is irreducible and no points of order 2 are defined over K. Thus the reduction cannot be of types III, III^*, I_0, and I_n^*. Consider the remaining cases listed in the algorithm. In each case, the Newton polygon is a straight segment. When $v(a_6) = 1, 2, 4$, or 5, we have reductions II, IV, IV^*, and II^*, respectively. In these cases, since $3 \nmid v(a_6)$ and this polygon is a single segment (no intermediate vertices allowed), then the associated extension is Eisenstein and $k_L = k$. The remaining two cases are when $v(a_6) = 0$ or 3 and $a_6\pi^{-v(a_6)}$ is not a cube in k. These cases give the reductions Z_1 and Z_2, respectively, and in these two cases, $[k_{\mathcal{X}^{min}} : k] = 3$.

When $p = 2$ and $j(X) \in \mathcal{O}_K^*$, then X_L/L has good reduction, and the reduction of X_L is an elliptic curve with non-zero j-invariant. Hence, the automorphism group of the reduction is cyclic of order 2 and, thus, $[L : K] = 2$. That 2.8 is true for $\mathcal{X}^{nc}$ in this case follows immediately from 2.9 and 2.10.

Example 2.13 We exhibit below an example of an elliptic curve X/K with $\mathcal{X}_k^{min}$ irreducible, geometrically reduced, and $k_{\mathcal{X}^{min}} = k$, but such that $k_L \neq k$. Thus this example shows that 2.8 is false in general for $\mathcal{X} = \mathcal{X}^{min}$.

Assume that $\pi = 2$, and $a \in \mathcal{O}_K^*$, $\overline{a} \notin (k^*)^2$. Let X/K be the elliptic curve given by $y^2 = x^3 + ax^2 + \pi$. Then this equation defines $\mathcal{X}^{min}$, with a reduction of type K_1. The special fiber $\mathcal{X}_k^{min}$ has a singularity at the maximal ideal (π, x, y), which is k-rational. So $k_{\mathcal{X}^{min}} = k$.

Let F/K be any finite extension. It follows from the algorithm in [Sz2], 5.3, that any elliptic curve of the form $y^2 = x^3 + ax^2 + b\pi_F^s$, with $a, b \in \mathcal{O}_F^*$, $\overline{a} \notin (k_F^*)^2$, and $s \geq 1$, has reduction of type K_n or K_n'. Consider now the curve X/K. It follows that X_F/F does not have semistable reduction over any extension F/K such that $k_F = k$, since when $k_F = k$, then $\pi_K = b\pi_F^s$ for some unit b in $\mathcal{O}_F$, $s \geq 1$, and $\overline{a} \notin (k_F^*)^2$. Thus, $k_L \neq k$.

Note that the morphism $\mathcal{X}^{nc} \to \mathcal{X}^{min}$ consists of a single blow-up, and $\mathcal{X}_k^{nc}$ is the union of two smooth rational lines over k meeting in a point P with $[k(P) : k] = 2$. Hence, we find that $k_{\mathcal{X}^{nc}} \neq k$, so this example cannot be used to show that 2.8 is false when $\mathcal{X} = \mathcal{X}^{nc}$.

Remark 2.14 We discussed in this section possible relationships between the fields $k_{\mathcal{X}^{min}}$ and k_L associated with X/K. Recall that the extension L/K is related to the fields of definition of the points of order $\ell \neq p$ in $\mathrm{Jac}(X)/K$. It is natural to wonder whether the components of $\mathcal{X}_k^{nc}$ that are not geometrically reduced are linked to the extension k_F/k, where F/K is minimal with the property that all torsion points in $\mathrm{Jac}(X)/K$ of order p are defined over F. As the following example shows, there seems to be no obvious relationship between these objects.

Let $p = 2$ and let us return to the example introduced in 2.5. Assume that $\pi = 2$, and let $r \geq 8$, $b \in \mathcal{O}_K^*$, $\overline{b} \notin (k^*)^2$. Then the curve $y^2 = x^3 + \pi x^2 + b\pi^r$ has reduction similar to the curve introduced in 2.5, but with $r - 2$ components of multiplicity 2, instead of 7 as in 2.5 (i). (To see this fact, use [Sz], 3.8.) In particular, the minimal model has a component that is not geometrically reduced. Moreover, Φ_K has order 2 and Grothendieck's pairing is trivial (2.6). Consider now the extension F, given in our case as the splitting field of $x^3+\pi x^2+b\pi^r$. The discriminant of such a polynomial is given by the formula $\delta := -4\pi^3(b\pi^r) - 27(b\pi^r)^2 = -\pi^{r+5}b(1 + 27b\pi^{r-5})$. We have $k_F = k$ if and only if the extension $K(\sqrt{\delta})/K$ has a trivial residue field extension. The expression $1 + 27b\pi^{r-5}$ is always a square in K if $r \geq 8$. Thus, $k_F = k$ if and only if r is even. In particular, when $r \geq 8$ is even, $\mathcal{X}^{nc}$ contains a component that is not geometrically reduced, but $k_F = k$.

3. Grothendieck's pairing under base change

The pairing $\langle\ ,\ \rangle_K$ behaves very nicely under extensions of the ground field. Let F/K be any finite extension. Denote by $\Phi_{A,F}$ the group of components of the Néron model of A_F/F. Let $e_{F/K}$ denote the ramification index of F/K, with $e_{F/K}[k_F : k] = [F : K]$. Let $\gamma : \Phi_{A,K} \to \Phi_{A,F}$ and $\gamma' : \Phi_{A',K} \to \Phi_{A',F}$ denote the natural maps induced by the base change map from $\mathcal{A} \times_{\mathcal{O}_K} \mathcal{O}_F$ to the Néron model of A_F/F, and from $\mathcal{A}' \times_{\mathcal{O}_K} \mathcal{O}_F$ to the Néron model of A'_F/F, respectively. The following key formula can be infered from [Gro], VIII, (7.3.5.2) and (7.3.1.2). Let $x \in \Phi_{A,K}$ and $y \in \Phi_{A',K}$. Then

$$\langle \gamma(x), \gamma'(y) \rangle_F = e_{F/K} \langle x, y \rangle_K. \tag{2}$$

Our next theorem, as well as several new examples given below where Grothendieck's pairing is not perfect, are immediate consequences of this formula.

Grothendieck's pairing is known to be perfect when the residue field is perfect and K is of mixed characteristic [Beg], when the residue field is finite [McC], or when the residue field is perfect and A has potentially purely toric reduction [Bos]. The case of jacobians is discussed in [B-L]. Grothendieck gave some indications on how to prove the perfectness of the pairing in certain cases, namely on the ℓ-part of $\Phi_{A,K} \times \Phi_{A',K}$ with ℓ prime to p, as well as in the semi-stable reduction case, see [Gro], IX, 11.3 and 11.4, and [Ber], [Wer], for full proofs. Grothendieck's pairing is likely to be always perfect when the residue field is perfect.

Theorem 3.1. *Let A/K be an abelian variety. Assume that Grothendieck's pairing $\langle \ , \ \rangle_K$ is perfect. Then, for any finite extension F/K, the kernel $\Psi_{K,F}$ of the map $\gamma : \Phi_{A,K} \to \Phi_{A,F}$ is killed by $e_{F/K}$.*

Proof: Let $x \in \Psi_{K,F}$ and $y \in \Phi_{A',K}$. Then

$$\langle e_{F/K} x, y \rangle_K = e_{F/K} \langle x, y \rangle_K = \langle \gamma(x), \gamma'(y) \rangle_F = 0.$$

Since $\langle \ , \ \rangle_K$ is perfect, $e_{F/K} x = 0$.

McCallum's Theorem states that $\Psi_{K,F}$ is killed by $[F : K]$, even when $\langle \ , \ \rangle_K$ is not perfect (see [ELL]). We show in 3.5 that the conclusion of Theorem 3.1 does not hold when $\langle \ , \ \rangle_K$ is not assumed to be perfect. It would be interesting to know whether the converse of 3.1 holds in general.

Another application of Formula (2) above is the following proposition. Let A/K be an abelian variety of dimension g. Let a_K, t_K, u_K, denote respectively the abelian, toric, and unipotent ranks of the connected component of zero of the special fiber $\mathcal{A}^0_k/k$ of the Néron model of A/K. Let F/K denote a totally ramified extension of degree ℓ. Let a_F, t_F and u_F be the corresponding integers for the Néron model of A_F/F. Recall that if ℓ is a prime, $\ell \neq p$, and $t_K = 0$, then $\mathrm{ord}_\ell(\Phi_{A,K})(\ell - 1) \leq 2u_K$ ([Lo2], 2.15). Let H_ℓ denote the ℓ-part of any finite abelian group H.

Proposition 3.2. *Let $\ell \neq p$ be prime and k be algebraically closed. Assume that A/K has a polarization of degree prime to ℓ, and has potentially good reduction. Assume also that $\Phi_{A,K,\ell}$ is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^s$ with $(s+1)(\ell-1) > 2u_K$. Then $2a_F \geq 2a_K + s(\ell - 1)$, and if a prime q divides $[L : K]$, then $q \leq \ell$.*

In particular, if $s(\ell - 1) = 2u_K$, then $L = F$ and, thus, L/K is tame and $\Phi_{A,K} = (\mathbb{Z}/\ell\mathbb{Z})^s$.

Proof: Since $\ell \neq p$, we know that the ℓ-part of Grothendieck's pairing on groups of components is perfect. We have $\Phi_{A,K}/\Psi_{K,F} \hookrightarrow \Phi_{A,F}$ by definition. Using (2), we find that $\langle \ , \ \rangle_F$ is trivial when restricted to the image of $(\Phi_{A,K}/\Psi_{K,F})_\ell \times (\Phi_{A,K}/\Psi_{K,F})_\ell$. We are going to show that $\Phi_{A,K,\ell} = \Psi_{K,F,\ell}$. Assume that $(\Phi_{A,K}/\Psi_{K,F})_\ell \neq (0)$. Then, since $\langle\ ,\ \rangle_F$ is perfect, we find that the ℓ-part of $\Phi_{A,F}$ is not equal to $(\Phi_{A,K}/\Psi_{K,F})_\ell$. Then, since $t_F = 0$,

$$\begin{array}{rcl} 2g - 2a_F & = & 2u_F \\ & \geq & \mathrm{ord}_\ell(\Phi_{A,F})(\ell-1) \\ & \geq & (\ell-1) + [\mathrm{ord}_\ell(\Phi_{A,K}) - \mathrm{ord}_\ell(\Psi_{K,F})](\ell-1), \end{array}$$

where the first inequality follows from [Lo2], 2.15. The hypothesis that A/K has a polarization of degree prime to ℓ is needed to be able to apply [Lo2], 3.1, (5) and (10), to obtain the bound:

$$(\ell-1)\mathrm{ord}_\ell(\Psi_{K,F}) \leq 2a_F - 2a_K.$$

This bound is likely to hold without this hypothesis. It follows that

$$\begin{array}{rcl} 2u_K & = & 2g - 2a_F + 2a_F - 2a_K \\ & \geq & 2g - 2a_F + (\ell-1)\mathrm{ord}_\ell(\Psi_{K,F}) \\ & \geq & (s+1)(\ell-1), \end{array}$$

contradicting our hypothesis. Thus, $\Phi_{A,K,\ell} = \Psi_{K,F,\ell}$. Then

$$s(\ell-1) = \mathrm{ord}_\ell(\Psi_{K,F})(\ell-1) \leq 2a_F - 2a_K,$$

as desired. The claim on the degree of L/K follow immediately from [Lo1], 3.1. When $s(\ell-1) = 2u_K$, we find that $g = a_F$, so A/K achieves good reduction over F. Since $[L:K]$ kills $\Phi_{A,K}$ [ELL], we find that $\Phi_{A,K} = (\mathbb{Z}/\ell\mathbb{Z})^s$. □.

Remark 3.3 When $\ell = 2$, 3.2 applies when $\Phi_{A,K,2} = (\mathbb{Z}/2\mathbb{Z})^{2u_K}$, in which case all points of order 2 on A are defined over K. Then the fact that $[L:K] = 2$ is true without the assumption of potential good reduction ([S-Z], 7.2).

When $\ell = 3$, 3.2 applies when $\Phi_{A,K,3} = (\mathbb{Z}/3\mathbb{Z})^{u_K}$, and the fact that $[L:K] = 3$ follows under a different hypothesis from [S-Z], 7.5 with 7.1.

Example 3.4 McCallum asked whether the group $\Psi_{K,F}$ is killed by the exponent of $\mathrm{Gal}(F/K)$. The reader will find in [ELL] a long example showing that the answer to this question is negative. This example is such that $\mathrm{Gal}(F/K)$ is an elementary abelian p-group. In the examples provided below, the p-part of $\mathrm{Gal}(F/K)$ can be arbitrarily specified. These examples seem to indicate that there is no obvious relationship between the exponent of $\Psi_{K,F}$ and the exponent of $\mathrm{Gal}(F/K)$.

Assume that k is algebraically closed. Let F/K be any Galois extension. Let $\mathbb{G}_m/K$ denote the multiplicative group. Let $R_{F/K}\mathbb{G}_{m,F}$ denote the Weil restriction of $\mathbb{G}_m/F$. The universal property of the Weil restriction implies the existence of a canonical closed immersion $\mathbb{G}_{m,K} \to R_{F/K}\mathbb{G}_{m,F}$. Let S/K be the quotient torus $R_{F/K}\mathbb{G}_{m,F}/\mathbb{G}_{m,K}$. Corollary 4.3 in [L-L] states that the group of components $\Phi_{S,K}$ of the Néron model of S/K is cyclic of order $[F:K]$. Let Λ/K be a lattice in S/K, in the sense of [B-X], 1.1. Then S/Λ is an abelian variety A/K, with rigid analytic uniformization $\Lambda \to S \to A$ ([B-X], 1.2). To show the existence of a desired lattice Λ in S, we may proceed as follows. We first pick a lattice Λ_0 in $\mathbb{G}_m/K$ so that the quotient $\mathbb{G}_m/\Lambda_0$ is an elliptic curve. Then we may consider the Weil restriction $R_{F/K}\Lambda_0$ in $R_{F/K}\mathbb{G}_{m,F}$. Finally, we choose as Λ the natural quotient $R_{F/K}\Lambda_0/\Lambda_0$.

Proposition 5.3 in [B-X] shows that the natural map $\Phi_{S,K} \to \Phi_{A,K}$ is injective. The proof of 1.7 in [L-L] shows that the image of $\Phi_{S,K}$ is a subgroup of the group

$\Psi_{K,F}$. Thus, when the p-part of the Galois group of F/K is not cyclic, we find that the exponent of $\mathrm{Gal}(F/K)$ does not kill $\Psi_{K,F}$.

Example 3.5 The example below will show that when $\langle\ ,\ \rangle_K$ is not perfect, the conclusion of Theorem 3.1 does not hold in general. This example is also a new example of an abelian variety with $\langle\ ,\ \rangle_K$ not perfect.

Let $p = 2$. Consider an abelian variety A/K of dimension g which has good ordinary reduction over $\mathcal{O}_K$ (i.e., $\mathcal{A}_k$ is an ordinary abelian variety). Assume in addition that every point of order 2 in $\mathcal{A}_k(k)$ lifts to a point of order 2 in $A(K)$. Pick a Galois extension L/K of degree 2 with associated residue field extension k_L/k. Consider the twist B/K of A/K obtained from the natural map $\mathrm{Gal}(L/K) \to \{\pm \mathrm{id}\} \subset \mathrm{Aut}(A/K)$. The abelian variety B/K has purely additive reduction K. Indeed, $A \times B$ is isogenous over K to the Weil restriction $R_{L/K}(A_L)$ ([Mil], Prop.7), and this Weil restriction has abelian rank over K equal to the abelian rank of A/K since the kernel of the norm map $R_{L/K}(A_L) \to A$ has unipotent reduction (see, e.g., [ELL], proof of Thm. 1).

Since the points of order 2 of $A(K)$ are invariant under the action of $\mathrm{Gal}(L/K)$, we find that $B[2](K)$ contains a subgroup C of order 2^g. Indeed, consider an isomorphism $\rho : B_L \to A_L$ defined over L. Then the map $c : G \to \mathrm{Aut}(A/K)$ given by $c_\sigma := \rho^\sigma \circ \rho^{-1}$ is the cocycle giving the twist B/K. If P is a point of order 2 in $A(K)$, then $\rho^{-1}(P)$ has order 2 in $B(\overline{K})$. To show that $\rho^{-1}(P)$ belongs to $B(K)$, we note that since P is a fixed point of the inverse map, we must have $c_\sigma(P) = P$ for all σ. Thus, $\rho^{-1}(P) = (\rho^{-1}(P))^\sigma$ for all σ, and $\rho^{-1}(P) \in B(K)$. We claim that the natural reduction map $\mathrm{red} : B(K) \to \Phi_{B,K}$ is not trivial when restricted to C, which implies that $\Phi_{B,K} \neq (0)$. Indeed, let $\mathcal{B}_k^0$ denote the connected component of the special fiber of the Néron model of B/K over $\mathcal{O}_K$. The group scheme $\mathcal{B}_k^0$ is unipotent. If $\mathrm{red}(C) \subseteq \mathcal{B}_k^0$, we find that the image of C under the reduction map $B(L) \to \mathcal{B}_{L,k_L}$, where $\mathcal{B}_L/\mathcal{O}_L$ is the Néron model of B_L/L, is trivial, contradicting the hypothesis that the points of C reduce to the points of order 2 in $\mathcal{B}_{L,k_L} = \mathcal{A}_k \times_k k_L$. Thus, $\Phi_{B,K} \neq (0)$.

Now choose L/K such that $e_{L/K} = 1$. By construction, the map $\Phi_{B,K} \to \Phi_{B,L} = (0)$ is not injective. Hence, $\mathrm{Ker}(\Phi_{B,K} \to \Phi_{B,L})$ is not killed by $e_{L/K}$. Thus, $\langle\ ,\ \rangle_K$ is not perfect.

Remark 3.6 In the above example, the group of components of the abelian variety $R_{L/K}(A_L)$ is isomorphic to the group $\Phi_{A_L,L}$ and, thus, is trivial. In particular, Grothendieck's pairing for $R_{L/K}(A_L)$ is perfect. On the other hand, Grothendieck's pairing for $A \times B$ is not perfect, as the example above shows, even though $A \times B$ and $R_{L/K}(A_L)$ are isogenous.

Our last example is an example where $\langle\ ,\ \rangle_K$ is perfect while $\langle\ ,\ \rangle_F$ is not perfect, even though the natural map $\Phi_{A,K} \to \Phi_{A,F}$ is an isomorphism. We start with a couple of preliminary lemmata. Let K be a field of mixed characteristic, with $v_K(p) = p^s$, $s \geq 0$. Let X/K denote the plane projective curve

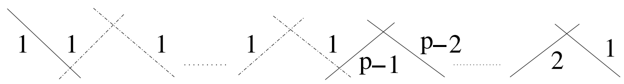
$$y^p z = x^{p+1} + a x^p z + b\pi^{p^r} z^{p+1}, \tag{3}$$

such that $r > 0$, $a, b \in \mathcal{O}_K^*$, and with $\overline{a} \notin (k^*)^p$ and $\overline{b} \in (k^*)^p$. This curve is smooth of genus $p(p-1)/2$ with a rational point at infinity. We describe below the minimal regular model $\mathcal{X}/\mathcal{O}_K$ of X/K in terms of r, s. It turns out that this model can be obtained by a sequence of blow-ups

$$\mathcal{X}_0 \leftarrow \mathcal{X}_1 \leftarrow \cdots \leftarrow \mathcal{X}_{n-1} \leftarrow \mathcal{X} = \mathcal{X}_n,$$

where $\mathcal{X}_0$ is the closure of X in $\mathbb{P}^2_{\mathcal{O}_K}$, and $\mathcal{X}_{i+1}$ is obtained by blowing up a single point in the exceptional divisor of the map $\mathcal{X}_i \to \mathcal{X}_{i-1}$.

Let us denote by K_m the following reduction type (the meaning of the segments is as in 2.3). The integer m represents the number of dotted segments, each representing a smooth rational line over the extension $k[\sqrt[p]{\overline{a}}]$. The first $m+2$ components on the left intersect with multiplicity p. The last $p-1$ components on the right intersect with multiplicity 1.



Lemma 3.7. *Let X/K be given by the equation $y^p = x^{p+1} + ax^p + b\pi^{p^r}$ as in* (3). *Then the reduction is of type K_m with $m = p^{r-1} + (p^s - 1)/(p-1)$.*

Sketch of Proof: The computation of the reduction of X/K is not difficult, and we shall only illustrate it on one example, when $r = 1$ and $s = 1$. We start with

$$y^p = x^{p+1} + ax^p + b\pi^p.$$

We blow up (π, x, y), and look in the chart

$$\left(\frac{y}{\pi}\right)^p = \pi\left(\frac{x}{\pi}\right)^{p+1} + a\left(\frac{x}{\pi}\right)^p + b.$$

Since $\overline{b} \in (k^*)^p$, we may change coordinates ($u := x/\pi, v := y/\pi - c$ with $\overline{c}^p = \overline{b}$) to obtain a new equation

$$v^p + pv^{p-1}c + \cdots + pvc^{p-1} = \pi u^{p+1} + au^p.$$

The singular point is (π, u, v). Using our hypothesis that $p = \alpha\pi^p$ with $\alpha \in \mathcal{O}_K^*$, we blow up this point and look in the chart

$$\left(\frac{v}{\pi}\right)^p + \cdots + \alpha\pi\left(\frac{v}{\pi}\right)c^{p-1} = \pi^2\left(\frac{u}{\pi}\right)^{p+1} + a\left(\frac{u}{\pi}\right)^p.$$

The new special fiber is the curve $V^p = \overline{a}U^p$, which is not geometrically reduced. We blow up

$$V^p + \cdots + \alpha\pi V c^{p-1} = \pi^2 U^{p+1} + aU^p$$

at (π, U, V) and look in the chart

$$(*) \qquad \left(\frac{V}{U}\right)^p U^{p-2} + \cdots + \alpha\frac{\pi}{U}\frac{V}{U}c^{p-1} = \left(\frac{\pi}{U}\right)^2 U^{p+1} + aU^{p-2}.$$

The exceptional divisor is $U = 0$. When $p = 2$, the model is regular, with exceptional divisor a smooth conic. When $p > 2$, the exceptional divisor consists of two affine lines $\frac{\pi}{U} = 0$ and $\frac{V}{U} = 0$. Let us look at the ring

$$A = \mathcal{O}_K\left[U, \frac{\pi}{U}, \frac{V}{U}\right] / \left(\frac{\pi}{U}U - \pi\right).$$

The line $\frac{\pi}{U} = 0$ in the exceptional divisor is given by the ideal $\mathfrak{P} := (\frac{\pi}{U}, U)$, while $\frac{V}{U} = 0$ is given by $\mathfrak{P}' := (\frac{V}{U}, U)$.

In $A_{\mathfrak{P}}$, $\mathfrak{P}A_{\mathfrak{P}}$ is principal, generated by U. The equation $(*)$ shows that $\left(\frac{\pi}{U}\right) = (\beta A_{\mathfrak{P}})^{p-2}$. It follows that, since $\pi = \frac{\pi}{U} \cdot U$, the multiplicity of the component $\left(U, \frac{\pi}{U}\right)$ in the special fiber of $\mathcal{X}_k$ is $p-1$. Similarly, the multiplicity of $\left(U, \frac{V}{U}\right)$ is 1.

The blow-up of the singular point of the chart $(*)$ produces again an exceptional divisor with two components, one of multiplicity $p-2$ and the other of multiplicity 2. The process terminates in the type K_2, as desired. This concludes the sketch of the proof of Lemma 3.7.

Lemma 3.8. *Let X/K be given by the equation $y^p = x^{p+1} + ax^p + b\pi^{p^r}$ as in (3), with $A = \mathrm{Jac}(X)$. Then $\Phi_{A,K} = \mathbb{Z}/p\mathbb{Z}$. Let $P \in X(K)$ be a rational point reducing in $\mathcal{X}_k$ on the first component of K_m of multiplicity 1. Let $Q \in X(K)$ be a rational point reducing in $\mathcal{X}_k$ on the last component of K_m of multiplicity 1. Then $P - Q$ in $\mathrm{Jac}(X)(K)$ reduces to a generator τ in $\Phi_{A,K}$.*

Sketch of Proof: The computation of $|\Phi_K|$ when the reduction is K_m is best done as follows. First the intersection matrix associated with the graph K_m defines as in [B-L], section 2, a group $\Phi(K_m)$. As noted at the end of [B-L], section 2, we have an exact sequence

$$0 \to \Phi_K \to \Phi(K_m) \to (\mathbb{Z}/p\mathbb{Z})^m \to 0$$

The order of $|\Phi(K_m)|$ can be computed easily using the formula in [BLR], 9.6/7. We find that $|\Phi(K_m)| = p^{m+1}$. Thus $\Phi_K \cong \mathbb{Z}/p\mathbb{Z}$.

To prove that τ is a generator, consider the intersection matrix M associated with K_m (numbering the components from left to right) as well as the vectors S and T below satisfying the relation $MS = pT$ (all other entries in the matrix M on the left are assumed to be 0):

$$\begin{pmatrix} -p & p & & & & & & & & \\ p & -2p & p & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & p & -2p & p & & & & & \\ & & & p & -p^2 & p & & & & \\ & & & & p & -2 & 1 & & & \\ & & & & & 1 & -2 & 1 & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & 1 & -2 & 1 \\ & & & & & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} -m \\ -m+1 \\ \vdots \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \\ p-2 \\ p-1 \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \\ \\ \vdots \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

Using the description of Φ_K given by Raynaud (see, e.g., [BLR], chap. 9, or [B-L], section 2), we conclude that τ has order p in Φ_K if and only if there does not exist an integer vector S' such that $MS' = T$. That $MS' \neq T$ for any integer vector S' is obvious since all coefficients of the first line of M are divisible by p. This concludes the proof of the lemma. Let us now state the following example.

Proposition 3.9. *Let A/K denote the jacobian of X/K, given by the equation $y^p = x^{p+1} + ax^p + b\pi^{p^r}$ as in (3). Let F/K be any extension with $p = e_{F/K} = [F : K]$. Let Φ_K and Φ_F denote the groups of components of the Néron models of A/K and*

A_F/F, respectively. Then $\Phi_K \cong \mathbb{Z}/p\mathbb{Z} \cong \Phi_F$ and the natural map $\Phi_K \to \Phi_F$ is bijective. It follows that $\langle\ ,\ \rangle_F$ is not a perfect pairing.

Proof: Let us show that the natural map $\Phi_K \to \Phi_F$ is an isomorphism when $[F : K] = p = e_{F/K}$. Lemma 3.7 shows that the reduction of X/K is of type K_m, with $m = p^{r-1} + (p^s - 1)/(p-1)$. We find, using 3.7 again, that the reduction of X_F/F is $K_{m'}$ with $m' = p^r + (p^{s+1} - 1)/(p-1)$. Indeed, in F, with uniformizer π_F, we have $\pi_K = c\pi_F^p$ for some $c \in \mathcal{O}_F^*$. Hence, the equation of X_F/F is

$$y^p = x^{p+1} + ax^p + bc^{p^r}\pi_F^{p^{r+1}},$$

and $\overline{bc^{p^r}} \in (k^*)^p$ since $\overline{b} \in (k^*)^p$. We conclude from 3.8 that $\Phi_K \cong \mathbb{Z}/p\mathbb{Z} \cong \Phi_F$. Let $\mathcal{X}/\mathcal{O}_K$ denote the model of X/K of type K_m and let $\mathcal{X}'/\mathcal{O}_F$ denote the model of X_F/F of type $K_{m'}$. Let $P \in X(K)$ be a rational point reducing in $\mathcal{X}_k$ on the first component of K_m of multiplicity 1. Let $Q \in X(K)$ be a rational point reducing in $\mathcal{X}_k$ on the last component of K_m of multiplicity 1. It is easy to check that P and Q still reduce to the first (resp. the last) component of multiplicity 1 of $\mathcal{X}'$. Since $P, Q \in X(F)$ are still such that $P - Q$ reduces to a generator of Φ_F (3.8 applied to Φ_F), we conclude that $\Phi_K \to \Phi_F$ is an isomorphism.

To show that $\langle\ ,\ \rangle_F$ is trivial, we use formula (2). Let $x, y \in \Phi_F$. Let x', y' be preimages of x, y under $\Phi_K \to \Phi_F$. Then, since $\langle x', y'\rangle_K$ has order at most $p = e_{F/K}$ in $\mathbb{Q}/\mathbb{Z}$, we find that $\langle x, y\rangle_F = e_{F/K}\langle x', y'\rangle_K = 0$.

Example 3.10 We show below that it is possible for the pairing $\langle\ ,\ \rangle_K$ to be perfect with $\Phi_K \neq (0)$, while after an extension F/K with $e_{F/K} = [F : K]$, the pairing $\langle\ ,\ \rangle_F$ is not perfect, even though the map $\Phi_{A,K} \to \Phi_{A,F}$ is an isomorphism.

Let us consider the same curve X/K as in 3.9, with $r = 1$. We showed in 3.9 using elementary means that $\langle\ ,\ \rangle_F$ is not perfect for any extension F/K with $[F : K] = p = e_{F/K}$. It does not seem possible to prove by elementary means only that $\langle\ ,\ \rangle_K$ is perfect. Thus, we will use the results of [B-L], where $\langle\ ,\ \rangle_K$ is computed explicitly for jacobians. We proceed as follows. Pick the points P and Q on $X(K)$ as in 3.8 to get a generator τ of Φ_K.

Lemma 3.11. *Assume that X/K has reduction K_m. Then $\langle\tau, \tau\rangle_K = \frac{-m+1}{p}$ in $\mathbb{Q}/\mathbb{Z}$.*

Proof: Let us consider the intersection matrix M associated with K_m, as well as the two vectors S, T with the relation $MS = pT$. The main result of [B-L] shows that

$$\begin{aligned}
\langle\tau, \tau\rangle_K &= ({}^tS/p)M(S/p) \mod \mathbb{Z} \\
&= \frac{-m}{p} - \frac{p-1}{p} \\
&= \frac{-m+1}{p}.
\end{aligned}$$

It follows in particular that when $p \nmid m - 1$, $\langle\ ,\ \rangle_K$ is perfect. ∎

Thus, returning to the example of the curve $y^p = x^{p+1} + ax^p + \pi^p$ (so $r = 1$), we find that the reduction is of type K_m with $m = 1 + (p^s - 1)/(p - 1)$. Hence,

$$\begin{aligned} \langle \tau, \tau \rangle_K &= \frac{-1 - (p^{s-1} + p^{s-2} + \cdots + p + 1) + 1}{p} \mod \mathbb{Z} \\ &= \frac{-1}{p}, \end{aligned}$$

so $\langle \ , \ \rangle_K$ is perfect. We note that when $r > 1$, the reduction is of type K_m with $m = p^{r-1} + \frac{p^s - 1}{p-1}$, and in this case 3.11 shows that $\langle \tau, \tau \rangle_K = 0$, as we found already in 3.9.

References

[Beg] L. Bégueri, *Dualité sur un corps local à corps résiduel algébriquement clos*, Mém. Soc. Math. Fr. **108** (1980), fasc. 4.

[Ber] A. Bertapelle, *On perfectness of Grothendieck's pairing for ℓ-parts of component groups*, J. reine angew. Math. **538** (2001), 223-236.

[B-B] A. Bertapelle and S. Bosch, *Weil restriction and Grothendieck's duality conjecture*, J. Alg. Geom. **9** (2000), 155-164.

[Bos] S. Bosch, *Component groups of abelian varieties and Grothendieck's duality conjecture*, Ann. Inst. Fourier **47** (1997), no. 5, 1257-1287.

[B-L] S. Bosch and D. Lorenzini, *Grothendieck's pairing on groups of components of jacobians*, Invent. Math. **148** (2002), 353-396.

[BLR] S. Bosch, W. Lüktebohmert, and M. Raynaud, *Néron Models*, Springer Verlag, 1990.

[B-X] S. Bosch and X. Xarles, *Component groups of Néron models via rigid uniformization*, Math. Ann. **306** (1996), 459-486.

[Chi] T. Chinburg, *Minimal models for curves over Dedekind rings*, in Arithmetic Geometry, G. Cornell and J. Silverman, Eds, Springer Verlag, 1986.

[CES] B. Conrad, B. Edixhoven, and W. Stein, *$J_1(p)$ has connected fibers*, Preprint 2002.

[Des] M. Deschamps, *Reduction semi-stable*, in Séminaire sur les pinceaux de courbes de genre au moins deux, L. Szpiro Editor, Astérisque **86** (1981), 1-34.

[ELL] B. Edixhoven, Q. Liu, and D. Lorenzini, *The p-part of the group of components of a Néron model*, J. of Alg. Geom. **5** (1996), 801-813.

[Gro] A. Grothendieck, *Groupe de monodromie en Géométrie Algébrique SGA 7, I*, Springer Verlag LNM **288** (1972).

[Gro2] A. Grothendieck, *Revêtements étales et groupe fondamental SGA 1*, Springer Verlag LNM **224** (1971).

[J-M] B. Jordan and D. Morrison, *On the Néron model of abelian surfaces with quaternionic multiplication*, J. reine angew. Math. **447** (1994), 1-22.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Math., 2002.

[L-L] Q. Liu and D. Lorenzini, *Special fibers of Néron models and wild ramification*, J. reine angew. Math. **532** (2001), 179-222.

[LLR] Q. Liu, D. Lorenzini, and M. Raynaud, *Néron models, Lie algebras, and reduction of curves of genus one*, In preparation.

[Lip] J. Lipman, *Introduction to resolution of singularities*, Proc. Symp. in Pure Math. **29** (1975), 187-230.

[Lo1] D. Lorenzini, *Groups of components of Néron models of Jacobians*, Comp. Math. **73** (1990), 145–160.

[Lo2] D. Lorenzini, *On the group of components of a Néron model*, J. reine angew. Math. **445** (1993), 109–160.

[Lo3] D. Lorenzini, *Reduction of points in the group of components of the Néron model of a jacobian*, J. reine angew. Math. **527** (2001), 117–150.

[McC] W. McCallum, *Duality Theorems for Néron models*, Duke Math. J. **53** (1986), 1093-1124.

[Mil] J. Milne, *On the arithmetic of abelian varieties*, Inv. Math. **17** (1972), 177-190.

[Pin] H. Pinkham, *Singularités rationelles de surfaces*, in Séminaire sur les singularités des surfaces, Springer LNM **777** (1980).

[Ser] J.-P. Serre, *Propriétés galoisiennes des points d'ordre finis des courbes elliptiques*, Invent. Math. **15** (1972), 259-331.

[Sil] J. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Springer-Verlag, 1994.

[S-Z] A. Silverberg and Y. Zarhin, *Semistable reduction and torsion subgroups of abelian varieties*, Ann. Inst. Fourier **45** (1995), 403-420.

[Sz] M. Szydlo, *Elliptic pencils with non-perfect residue fields*, Preprint, 2001, available at http://www.szydlo.net.

[Sz2] M. Szydlo, *Flat regular models of elliptic schemes*, Harvard Ph.D. Thesis, (2000), available at http://www.szydlo.net.

[Tat] J. Tate, *Genus change in inseparable extensions of function fields*, Proc. AMS **3** (1952), 400-406.

[Vie] E. Viehweg, *Invarianten der degenerierten Fasern in lokalen Familien von Kurven*, J. reine angew. Math. **293** (1977), 284-308.

[Wer] A. Werner, *On Grothendieck's pairing of component groups in the semistable reduction case*, J. reine angew. Math. **486** (1997), 205–215

Dino Lorenzini, University of Georgia, Athens, GA 30602, USA
E-mail address: `lorenzini@math.uga.edu`