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Reduction of points in the group of components of the Néron model of a jacobian

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Introduction

Let K be a complete field with a discrete valuation. Let $\mathcal{O}_K$ denote the ring of integers of K, with maximal ideal (t). Let k be the residue field of $\mathcal{O}_K$, assumed to be algebraically closed of characteristic $p \geqq 0$. We shall call a *curve* in this article a smooth proper geometrically connected variety X/K of dimension 1. Let A/K denote the jacobian of X/K. Let P and Q be two K-rational points of X. The divisor of degree zero $P - Q$ defines a K-rational point of A/K. In this article, we study the reduction of the point $P - Q$ in the Néron model of A/K in terms of the reductions of the points P and Q in a regular model $\mathcal{X}/\mathcal{O}_K$ of X/K.

Let A/K be any abelian variety of dimension g. Denote by $\mathcal{A}/\mathcal{O}_K$ its Néron model. Recall that the special fiber $\mathcal{A}_k/k$ of $\mathcal{A}/\mathcal{O}_K$ is an extension of a finite abelian group $\Phi_K := \Phi_K(A)$, called the group of components, by a smooth connected group scheme $\mathcal{A}_k^0$, the connected component of zero in $\mathcal{A}_k$. We denote by $\pi : A(K) \to \mathcal{A}_k(k)$ the canonical reduction map. We will often abuse notation and also denote by π the composition $A(K) \to \mathcal{A}_k(k) \to \Phi_K$.

In [Lor3], the author introduced two functorial filtrations of the prime-to-p part $\Phi_K^{(p)}$ of the group Φ_K. These filtrations are key in the complete description of all possible groups $\Phi_K^{(p)}$ [Edi]. Filtrations for the full group Φ_K were later introduced by Bosch and Xarles in [B-X]. An example of a functorial subgroup of Φ_K occurring in one of the filtrations is the group $\Psi_{K,L}$ described below, where L/K denotes the minimal extension of K such that A_L/L has semistable reduction (see [Des], 5.15). More generally, let F/K be any separable extension. Let Φ_F denote the group of components of A_F/F. The functoriality property of the Néron models induces a map $\gamma : \Phi_K \to \Phi_F$, whose kernel is denoted by $\Psi_{K,F}$.

Given two points P and Q in $X(K)$, it is natural to wonder whether it is possible to predict when the reduction of $P - Q$ in Φ_K belongs to one of the functorial subgroups mentioned above. This question is not easy since even deciding whether the reduction of $P - Q$ is trivial is not immediate. We give in this paper a sufficient condition on the special fiber of a model $\mathscr{X}$ for the image of the point $P - Q$ in Φ_K to belong to the subgroup $\Psi_{K,L}$. When this condition is satisfied, we are able to provide a formula for the order of this image. To explicitly compute this order, we exploit the fact that a natural pairing attached to Φ is non-degenerate. We also discuss cases where the image of the point $P - Q$ belongs to the subgroup $\Theta_K^{[3]}$ of $\Psi_{K,L}$ (notation recalled in 5.6).

1. The main results

Let X/K be a curve. Let $\mathscr{X}/\mathscr{O}_K$ be a regular model of X/K. Let $\mathscr{X}_k := \sum_{i=1}^{v} r_i C_i$ denote the special fiber of $\mathscr{X}$ and let $M := \big((C_i \cdot C_j)\big)_{1 \leqq i,j \leqq v}$ be the associated *intersection matrix*. The *dual graph* G associated to $\mathscr{X}_k$ is defined as follows. The vertices of G are the curves C_i and, when $j \neq h$, the vertex C_j is linked in G to the vertex C_h by exactly $(C_j \cdot C_h)$ edges. The degree of the vertex C_i in G is the integer $d_i := \sum_{i \neq j} (C_i \cdot C_j)$.

Let ${}^tR := (r_1, \ldots, r_v)$, so that $MR = 0$. We assume in this paper that $\gcd(r_1, \ldots, r_v) = 1$. The triple (G, M, R) is called an *arithmetical graph*. When the coefficients of M are not thought of as intersection numbers, we may denote $(C_i \cdot C_j)$ simply by c_{ij}. As we will recall in section 5, Raynaud has shown that the group of components $\Phi_K\big(\mathrm{Jac}(X)\big)$ is isomorphic to $\mathrm{Ker}({}^tR)/\mathrm{Im}(M)$, where ${}^tR : \mathbb{Z}^v \to \mathbb{Z}$ and $M : \mathbb{Z}^v \to \mathbb{Z}^v$ are the linear maps associated with the matrices M and ${}^tR$. We call the group $\Phi(G) := \mathrm{Ker}({}^tR)/\mathrm{Im}(M)$ the *group of components* of the arithmetical graph (G, M, R).

Let (C, r) and (C', r') be two distinct vertices of G. Let $E(C, C')$ denote the vector of $\mathbb{Z}^v$ with null components everywhere except for $r'/\gcd(r, r')$ in the C-component, and $-r/\gcd(r, r')$ in the C'-component. Clearly, $E(C, C') \in \mathrm{Ker}({}^tR)$. The image of $E(C, C')$ in the quotient $\mathrm{Ker}({}^tR)/\mathrm{Im}(M)$ will be called the element of $\Phi(G)$ associated to the pair of vertices (C, C').

Let $P \in X(K)$. Let $\bar{P} \in \mathscr{X}$ denote the closure of P in $\mathscr{X}$. The Cartier divisor $\bar{P}$ intersects $\mathscr{X}_k$ in a smooth point of $\mathscr{X}_k$ (see, for instance, [L-L], 1.3). Hence, there exists a unique component C_P of $\mathscr{X}_k$, of multiplicity one, such that $\bar{P} \cap \mathscr{X}_k \in C_P$. Let now P and Q be distinct points of $X(K)$. It follows from Raynaud's results recalled in section 5 that if $C_P = C_Q$, then the image of $P - Q$ in $\Phi_K\big(\mathrm{Jac}(X)\big)$ is trivial, and that to determine the image of $P - Q$ in $\Phi_K\big(\mathrm{Jac}(X)\big)$ when $C_P \neq C_Q$, it is sufficient to determine the image of the vector $E(C_P, C_Q)$ in $\mathrm{Ker}({}^tR)/\mathrm{Im}(M)$. Thus, in the remainder of this article, we shall usually assume that $C_P \neq C_Q$.

1.1. Let us recall the following terminology. A *node* of a graph G is a vertex of degree greater than 2. A *terminal vertex* is a vertex of degree 1. The topological space obtained from G by removing all its nodes is the union of connected components. A *chain* of G is a connected subgraph of the closure of such a connected component. In particular, a chain contains at most two nodes of G. If a chain contains a terminal vertex, we call it a *terminal chain*. We define the *weight* of a chain $\mathscr{C}$ to be the integer

$$w(\mathscr{C}) := \gcd(r_j, C_j \text{ a vertex on } \mathscr{C}).$$

Let $(C,r),(C_1,r_1),\dots,(C_n,r_n),(C',r')$ be the vertices on a chain $\mathscr{C}$ of an arithmetical graph, with C and C' nodes: then $(C\cdot C_1)=(C_i\cdot C_{i+1})=(C_n\cdot C')=1$. The reader will check that $\gcd(r,r_1)=\gcd(r_1,r_2)=\cdots=\gcd(r_n,r')$. In particular, $w(\mathscr{C})=\gcd(r,r_1)$. When $(C,r),(C_1,r_1),\dots,(C_n,r_n)$ are the vertices of a terminal chain, with C_n the terminal vertex, then $\gcd(r,r_1)=r_n$. Note that if the set of vertices on a chain consists of exactly two nodes C and C', it may happen that $(C\cdot C')>1$.

Let (C,r) and (C',r') be two distinct vertices of G. We say that the pair (C,C') is *weakly connected* if there exists a path $\mathscr{P}$ in G between C and C' such that, for each edge e on $\mathscr{P}$, the graph $G\backslash\{e\}$ is disconnected. Note that when a pair (C,C') is weakly connected, then the path $\mathscr{P}$ is the unique shortest path between C and C'. If a pair is not weakly connected, we will say that it is *multiply connected*. A graph is a tree if and only if every pair of vertices of G is weakly connected.

Let (C,r) and (C',r') be a weakly connected pair with associated path $\mathscr{P}$. While walking on $\mathscr{P}\backslash\{C,C'\}$ from C to C', label each encountered node consecutively by $(C_1,r_1),(C_2,r_2),\dots,(C_s,r_s)$. (There may be no such nodes, in which case the integer s is set to be 0.) Thus $\mathscr{P}$ is the union of chains: the chain $\mathscr{C}_0$ from C to C_1, then the chain $\mathscr{C}_1$ from C_1 to C_2, and so on. The last chain on $\mathscr{P}$ is the chain $\mathscr{C}_s$ from C_s to C'. If there are no nodes on $\mathscr{P}\backslash\{C,C'\}$, then $\mathscr{P}$ is a chain from C to C', and if there are no vertices on $\mathscr{P}\backslash\{C,C'\}$, then by definition of weakly connected, $(C\cdot C')=1$. Let ℓ be a prime number. We say that the weakly connected pair (C,C') is *ℓ-breakable* if, for all $i=0,\dots,s$, the weight $w(\mathscr{C}_i)$ is not divisible by ℓ. In particular, if the pair (C,C') is ℓ-breakable, then each chain $\mathscr{C}_i$ contains a vertex of multiplicity prime to ℓ. To study the element of $\Phi(G)$ associated to the pair (C,C'), we will break the graph G at each such vertex and study each smaller graph so obtained individually.

Note that there is only one reduction type of curve of genus $g=1$ which contains a weakly connected pair that is not ℓ-breakable: the type I_v^*, with $\ell=2$ and $v>0$. For examples with $g>1$, see 6.6.

1.2. Let (C,C') be a weakly connected and ℓ-breakable pair. Let $\mathscr{P}$ denote the associated path between C and C', with nodes $(C_1,r_1),\dots,(C_s,r_s)$. If $s=0$, set $\lambda(C,C'):=1$. If $s>0$, define $\lambda(C,C')$ as follows. Remove all edges of $\mathscr{P}$ from G to obtain a disconnected graph $\mathscr{G}$. Let $\mathscr{G}_i$, $i=1,\dots,m$, denote the connected components of $\mathscr{G}$. Let us number these connected components in such a way that the node (C_i,r_i) on the path $\mathscr{P}$ belongs to the graph $\mathscr{G}_i$. Let

$$m_i := \gcd\big(r_j, (D_j,r_j) \text{ a vertex of } \mathscr{G}_i\big)$$

and let $\lambda(C,C')$ denote the power of ℓ such that

$$\operatorname{ord}_\ell\big(\lambda(C,C')\big) := \max\{\operatorname{ord}_\ell(r_i/m_i), C_i \text{ a node on } \mathscr{P}\}.$$

1.3. Recall that a finite abelian group H can be written as a product $H \cong \prod_{\ell \text{ prime}} H_\ell$. The group H_ℓ is called the *ℓ-part of H*. Let h be an element of H of order m. We call the *ℓ-part of h* the following element h_ℓ of H. If $\ell \nmid m$, then h_ℓ is trivial. Otherwise, write $1 = \sum_{\ell \text{ prime}} a_\ell m/\ell^{\mathrm{ord}_\ell(m)}$. Then set $h_\ell := h^{a_\ell m/\ell^{\mathrm{ord}_\ell(m)}}$. The reader will check that the element h_ℓ does not depend on the choice of the coefficients a_ℓ. We may now state the main results of this article.

Theorem 5.5. *Let X/K be a curve. Let $\mathscr{X}/\mathcal{O}_K$ be a regular model of X/K with associated arithmetical graph (G, M, R). Let $\ell \neq p$ be a prime. Let $P, Q \in X(K)$ with $C_P \neq C_Q$. If the pair (C_P, C_Q) is weakly connected and ℓ-breakable, then the image of the ℓ-part of $P - Q$ in $\Phi_K(\mathrm{Jac}(X))$ belongs to $\Psi_{K,L}$, and has order $\lambda(C_P, C_Q)$.*

Theorem 6.3/6.4. *Let X/K be a curve. Assume that L/K is tame. Let $\mathscr{X}/\mathcal{O}_K$ be a regular model of X/K with associated arithmetical graph (G, M, R). Let $P, Q \in X(K)$ with $C_P \neq C_Q$. If the pair (C_P, C_Q) is weakly connected but not ℓ-breakable for some prime $\ell \neq p$, then the image of $P - Q$ in $\Phi_K(\mathrm{Jac}(X))$ does not belong to $\Psi_{K,L}$.*

Note that Theorem 6.3 is only a partial converse to Theorem 5.5 since 6.3 provides information only on the image of $P - Q$ and not on the image of the ℓ-part of $P - Q$.

1.4. Recall that the connected component $\mathscr{A}_k^0$ of the Néron model $\mathscr{A}/\mathcal{O}_K$ is the extension of an abelian variety of dimension a_K by the product of a torus and a unipotent group of dimension t_K and u_K respectively. The integers a_K, t_K, and u_K are called the abelian, toric, and unipotent ranks of A/K, respectively. For each prime ℓ dividing $[L:K]$, $\ell \neq p$, let K_ℓ/K denote the unique subfield of L with the property that $[K_\ell : K] = \ell^{\mathrm{ord}_\ell([L:K])}$. An abelian variety has *potentially good reduction* if $t_L = 0$. It is said to have *potentially good ℓ-reduction* if $t_{K_\ell} = 0$. An abelian variety with potentially good reduction has potentially good ℓ-reduction for all primes $\ell \neq p$, but the converse is false, even when $p = 0$.

We shall say that an element h of a group H is *divisible by ℓ*, or is *ℓ-divisible* if there exists $g \in H$ such that $\ell g = h$. Note that the ℓ-part h_ℓ of h is ℓ-divisible if and only if h is ℓ-divisible.

Theorem 7.2. *Let X/K be a curve. Let $\ell \neq p$ be a prime. Let $P, Q \in X(K)$. Assume that $\mathrm{Jac}(X)/K$ has potentially good ℓ-reduction. Then $\Psi_{K,L,\ell} = \Phi_K(\mathrm{Jac}(X))_\ell$. If the ℓ-part of the image of $P - Q$ in $\Phi_K(\mathrm{Jac}(X))$ is not trivial, then $P - Q$ is not divisible by ℓ in $\mathrm{Jac}(X)(K)$.*

This article will proceed as follows. In the next three sections, we prove several propositions on arithmetical graphs needed to compute the order in Φ_K of elements of the form $\pi(P - Q)$. In particular, we introduce in the third section a very useful pairing on $\Phi_K \times \Phi_K$ that is non-degenerate. These first three sections are linear algebraic in essence and can be read independently of the rest of the paper. In the fifth section, we prove the first theorem stated above. In section six, we discuss a partial converse to this theorem. In the last section, we study the case where the jacobian has potentially good ℓ-reduction and prove Theorem 7.2.

2. Terminal chains

Let (G, M, R) be an arithmetical graph. As the reader may have noted, it is not easy in general to compute the order of the group $\Phi(G)$, or the order in $\Phi(G)$ of a given pair of vertices of G. There is no easy criterion to determine in terms of G whether, for instance, $|\Phi(G)| = 1$ (see, however, 6.5 and 3.3). When the arithmetical graph is reduced, that is, when all its multiplicities are equal to 1, such a criterion exists: $\Phi(G)$ is trivial if and only if G is a tree. We provide in this section a sufficient condition on a pair (C, C') for the image of $E(C, C')$ to have order 1 in $\Phi(G)$. When the arithmetical graph is reduced, a necessary and sufficient criterion already exists. Indeed, it is shown in [Lor4], 2.3, or [Edi2], 9.2, that:

Proposition 2.1. *When G is reduced, the image of $E(C, C')$ has order 1 if and only if (C, C') is weakly connected.*

We shall see below that even in the general case, it is possible to show that certain weakly connected pairs have order 1. After a series of preliminary lemmas on chains, we prove in 2.7 the main result of this section, that $E(C, C')$ is trivial if C and C' both belong to the same terminal chain. The case where C and C' are consecutive vertices on a chain is easy and is treated in the following lemma.

Lemma 2.2. *Let (C, r) and (C', r') be two vertices of an arithmetical graph (G, M, R) joined by a single edge e. Assume that $G \backslash \{e\}$ is disconnected. Let G_C denote the connected component of $G \backslash \{e\}$ that contains C. Let $s := \gcd\big(d, (D, d) \text{ vertex on } G_C\big)$. Then the image of $E(C, C')$ in $\Phi(G)$ is killed by $\gcd(r, r')/s$. In particular, if C and C' belong to the same terminal chain, then the image of $E(C, C')$ is trivial.*

Proof. Multiply each column of M corresponding to a vertex (D, d) of G_C by d/s. Add all these columns to the C-column multiplied by r/s. The new matrix has the vector $\big(-\gcd(r, r')/s\big)E(C, C')$ in the C-column. Hence, $\big(-\gcd(r, r')/s\big)E(C, C')$ belongs to $\mathrm{Im}(M)$ and is thus trivial in $\Phi(G)$. If C and C' belong to the same terminal chain, we may without loss of generality assume that G_C contains the terminal vertex of the chain. The terminal vertex has then multiplicity s, which equals $\gcd(r, r')$.

2.3. Let $n \geqq 1$. Let $(C, r), (C_1, r_1), \ldots, (C_n, r_n), (C', r')$ be the vertices on a chain of an arithmetical graph. Letting $-c_i$ denote the self-intersection of C_i, we obtain an $(n \times n)$ matrix N and a relation:

$$N := \begin{pmatrix} -c_1 & 1 & 0 & \ldots & 0 \\ 1 & -c_2 & 1 & & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & -c_{n-1} & 1 \\ 0 & \ldots & 0 & 1 & -c_n \end{pmatrix} \quad \text{and} \quad N \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} -r \\ 0 \\ \vdots \\ 0 \\ -r' \end{pmatrix}.$$

If C, $C_1, \ldots, C_n$ is a terminal chain with terminal vertex C_n, then

$$(r_1, \ldots, r_n)N = (-r, 0, \ldots, 0).$$

It is possible to find a sequence of integers $b_1 = 1, b_2, \dots, b_n$ such that

$$(b_1, \dots, b_n) \cdot N = (0, \dots, 0, -b)$$

for some $b \in \mathbb{Z}$. Indeed, set $b_1 = 1$ and solve for b_2 in the above equation. Once b_1 and b_2 are known, then it is possible to solve for b_3, and so on.

Lemma 2.4. *We have $br_n = r + b_n r'$. When the chain $(C, r), \dots, (C_n, r_n)$ is terminal with terminal vertex C_n, then $br_n = r$.*

Proof. Compute $(b_1, \dots, b_n) \cdot N \cdot {}^t(r_1, \dots, r_n)$ in two different ways.

Let us note there that the integers $b_1 = 1, b_2, \dots, b_n, b$ are all positive. Indeed, if $b \leqq 0$, then $br_n = r + b_n r' \leqq 0$ implies $b_n < 0$. If $b_i < 0$ for some i, then the equality $b_i r_{i-1} = r + b_{i-1} r_i$ implies that $b_{i-1} < 0$, which is a contradiction since $b_1 > 0$.

2.5. The sequence $(C_1, r_1), (C_2, r_2), \dots, (C_n, r_n), (C', r')$ is also a chain, with associated matrix N^{11}, the principal minor of N obtained by removing the first row and first column of N. Let $d_1 = 1, d_2, \dots, d_{n-1}$, d denote the integers associated to N^{11} such that

$$(d_1, d_2, \dots, d_{n-1}) N^{11} = (0, \dots, 0, -d).$$

Let

$$A := \begin{pmatrix} -1 & 0 & d_1 & d_2 & \dots & d_{n-1} \\ 0 & b_1 & b_2 & b_3 & \dots & b_n \\ \vdots & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 0 & 1 & & \vdots \\ & \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad N' := \begin{pmatrix} 1 \quad 0 \quad 0 \quad \dots \quad 0 \\ \hline \\ \mathrm{N} \\ \\ \end{pmatrix}.$$

The matrix AN' is an $(n+1) \times n$ matrix. Using operations involving only the columns of AN', it is easy to see that AN' is equivalent over $\mathbb{Z}$ to the following matrix (we shall say that AN' is 'column equivalent' to):

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -d \\ 0 & 0 & \dots & 0 & -b \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Set $d_0 = 0$.

Lemma 2.6. *Let $(C, r), (C_1, r_1), \dots, (C_n, r_n)$ be a terminal chain of an arithmetical graph. Then $\det(N) = (-1)^n r / r_n$. Moreover, r divides $r_i b_j - b_i r_j$, for all $i \neq j$, and*

$$\frac{r}{r_1} = \frac{r_i b_j - b_i r_j}{r_i d_{j-1} - d_{i-1} r_j}, \quad \textit{for all } i \neq j,\ 1 \leqq i, j \leqq n.$$

In particular, $b_n r_1/r_n$ *is congruent to* 1 *modulo* r/r_n *and* $\gcd(b_n, r/r_n) = 1$.

Proof. Recall that, with the notation introduced above, we have

$$(r_1, \dots, r_n)N = (-r, 0, \dots, 0),$$

$$(b_1, \dots, b_n)N = (0, \dots, 0, -b),$$

$$(0, d_1, \dots, d_{n-1})N = (d_1, 0, \dots, 0, -d).$$

Recall also that $b_1 = d_1 = 1$, and that since the vertices form a terminal chain, Lemma 2.4 shows that $b = r/r_n$ and $d = r_1/r_n$. It is easy to check that $r_n = \gcd(r, r_1)$ and that r_n divides all r_i.

Let $N^* := \big((a_{ij})\big)_{1 \leqq i, j \leqq n}$ denote the comatrix of N: $N^*N = NN^* = \det(N)I_n$. Multiply both sides of the three equalities above by $\big((a_{ij})\big)$. We find that

$$\det(N)r_i = -a_{i,1}r \quad \text{for all } i = 1, \dots, n,$$

$$\det(N)b_i = -a_{i,n}b \quad \text{for all } i = 1, \dots, n,$$

$$\det(N)d_i = a_{i+1,1} - a_{i+1,n}d \quad \text{for all } i = 0, \dots, n-1.$$

In particular, $\det(N)r_n = -a_{n,1}r = (-1)^n r$. It follows from the three equalities above that

$$r_i b_j - r_j b_i = r_n(a_{i,1}a_{j,n} - a_{j,1}a_{i,n}),$$

$$(r_i d_{j-1} - r_j d_{i-1})r = r_n(a_{i,1}a_{j,n} - a_{j,1}a_{i,n})r_1.$$

From the equality $(r_i b_j - r_j b_i)r_1 = (r_i d_{j-1} - r_j d_{i-1})r$, we find that $r | r_i b_j - r_j b_i$. This concludes the proof of Lemma 2.6.

As a corollary to our study of the properties of the matrix N, we may now prove the following result.

Proposition 2.7. *Let* (G, M, R) *be an arithmetical graph, and let*

$$(C, r), (C_1, r_1), \dots, (C_n, r_n)$$

be a terminal chain of G. *Then* $E(C_i, C_j)$ *is trivial in* $\Phi(G)$, *for all* $i, j \in \{1, \dots, n\}$, $i \neq j$.

Proof. The matrix M has the form

$$M = \left(\begin{array}{ccc|c} & & * & \\ & & \vdots & \\ * & \cdots & * & 1 \\ \hline & & 1 & \\ & & & N \\ & & & \end{array}\right). \quad \text{Let} \quad A' := \left(\begin{array}{c|c} \mathrm{Id}_s & \\ \hline & \\ & A \\ & \end{array}\right),$$

where A is as in 2.5 and, if v denotes the number of vertices of G, then $s := v - n - 1$. Then, using 2.5 and the facts that $d = r_1/r_n$ and $b = r/r_n$, the reader will check that $A'M$ is column equivalent to a matrix of the form

$$\left(\begin{array}{cc|cccc} * & * & & & & \\ * & * & 0 & \dots & 0 & -r_1/r_n \\ \hline & 1 & 0 & \dots & 0 & -r/r_n \\ & & 1 & \ddots & & 0 \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & 0 \end{array}\right).$$

The transpose of the vector $A'E(C_i, C_j)$ has the form (if $i < j$):

$$\frac{1}{r_n}(0, \dots, 0, d_{i-1}r_j - d_{j-1}r_i, b_ir_j - r_ib_j, 0, \dots, 0, +r_j, 0, \dots, 0, -r_i, 0, \dots, 0)$$

(where the first s coefficients are 0). We claim that $A'E(C_i, C_j)$ is in the span of the last n columns of the matrix $A'M$. To prove this claim, it is sufficient to show that $\frac{1}{r_n}(d_{i-1}r_j - d_{j-1}r_i, b_ir_j - r_ib_j)$ is an integer multiple of $(-r_1/r_n, -r/r_n)$, which follows immediately from Lemma 2.6. Since A' is invertible over $\mathbb{Z}$, $A'E(C_i, C_j)$ is in the span of $A'M$ if and only if $E(C_i, C_j)$ is in the span of M. Hence, $E(C_i, C_j)$ is trivial in $\Phi(G)$.

We conclude this section with a key lemma used in the next sections.

Lemma 2.8. *Let $(C, r), (C_1, r_1), \dots, (C_n, r_n)$ be a terminal chain. Then*

$$\frac{1}{rr_1} + \frac{1}{r_1r_2} + \dots + \frac{1}{r_{n-1}r_n} = \frac{b_n}{rr_n}.$$

Proof. We proceed by induction on n. If $n = 1$, Lemma 2.8 holds since $b_1 = 1$. By induction hypothesis applied to $C_1, \dots, C_n$,

$$\frac{1}{r_1r_2} + \dots + \frac{1}{r_{n-1}r_n} = \frac{d_{n-1}}{r_1r_n}.$$

Lemma 2.6 shows that $r/r_1 = (r_1b_n - r_nb_1)/(r_1d_{n-1} - d_0r_n)$. In other words, $d_{n-1}r = r_1b_n - r_n$. Dividing both sides by rr_1r_n shows that

$$\frac{d_{n-1}}{r_1r_n} = \frac{b_n}{rr_n} - \frac{1}{rr_1}.$$

3. Computations using a pairing attached to Φ

3.1. Let us introduce in this section a pairing associated to $\Phi(G)$. Let (G, M, R) be any arithmetical graph. Let $\tau, \tau' \in \Phi$ and let $T, T' \in \mathrm{Ker}({}^tR)$ be vectors whose images in Φ

are τ and τ', respectively. Let $S, S' \in \mathbb{Z}^v$ be such that $MS = nT$ and $MS' = n'T'$. Note that n and n' are divisible by the order of τ and τ', respectively. Define

$$\langle\, ;\, \rangle : \Phi \times \Phi \to \mathbb{Q}/\mathbb{Z},$$
$$(\tau, \tau') \mapsto ({}^tS/n)M(S'/n') \pmod{\mathbb{Z}}.$$

It is shown in [B-L], 2.1, that this pairing is well-defined and perfect (i.e., that if $\langle \tau; \mu \rangle = 0$ for all $\mu \in \Phi$, then $\tau = 0$).

Let (C, r) and (C', r') be a weakly connected pair with associated path $\mathscr{P}$. While walking on $\mathscr{P} \setminus \{C, C'\}$ from C to C', label each encountered vertex consecutively by $(C_1, r_1), (C_2, r_2), \ldots, (C_n, r_n)$. The following proposition is proved in [B-L], 2.4.

Proposition 3.2. *Keep the notation introduced above. Assume that (C, C') is a weakly connected pair of G. Let γ denote the image of the element $E(C, C')$ in $\Phi(G)$. If (D, s) and (D', s') are any two distinct vertices on G, then let δ denote the image of $E(D, D')$ in $\Phi(G)$. Let C_α denote the vertex of $\mathscr{P}$ closest to D in G, and let C_β denote the vertex of $\mathscr{P}$ closest to D'. Assume that $\alpha \leqq \beta$. (Note that we may have $\alpha = \beta$, and we may have $D = C_\alpha$ or $D' = C_\beta$.) Then*

$$\langle \gamma; \delta \rangle = \operatorname{lcm}(r, r') \operatorname{lcm}(s, s')(1/r_\alpha r_{\alpha+1} + 1/r_{\alpha+1} r_{\alpha+2} + \cdots + 1/r_{\beta-1} r_\beta).$$

In particular, if $C_\alpha = C_\beta$, then $\langle \gamma; \delta \rangle = 0$. Moreover,

$$\langle \gamma; \gamma \rangle = \operatorname{lcm}(r, r')^2 (1/rr_1 + 1/r_1 r_2 + \cdots + 1/r_n r').$$

The existence of this explicit perfect pairing has the following interesting consequences.

Proposition 3.3. *Let (G, M, R) be any arithmetical graph. Let ℓ be any prime. Let (C, C') be a weakly connected pair such that $\ell \nmid rr'$. Let τ denote the image in Φ of $E(C, C')$. Then the order of the ℓ-part of τ is greater than or equal to the maximum of the ℓ-parts of the weights $w(\mathscr{C})$, where $\mathscr{C}$ is a chain on the path from C to C'. In particular, if (C, C') is not ℓ-breakable, then the ℓ-part of τ is not trivial in Φ.*

Proof. Let $\mathscr{C}$ be any chain on the path $\mathscr{P}$ in G linking C and C' (see 1.1). Let (D, d) and (D', d') be two consecutive vertices on $\mathscr{C}$. Recall that $w(\mathscr{C}) = \gcd(d, d')$. Let τ' denote the image in Φ of $E(D, D')$. Then Proposition 3.2 implies that

$$\langle \tau; \tau' \rangle = \operatorname{lcm}(r, r') \operatorname{lcm}(d, d')(1/dd').$$

Since $\ell \nmid rr'$, $\langle \tau; \tau' \rangle$ has order $\gcd(d, d')$ in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. Thus, the ℓ-parts of τ and τ' have orders at least equal to the ℓ-part of $\gcd(d, d')$.

By definition, when (C, C') is weakly connected and not ℓ-breakable, the path $\mathscr{P}$ contains a chain $\mathscr{C}$ with consecutive vertices (D, d) and (D', d') such that $\gcd(d, d')$ is divisible by ℓ. Thus, in this case, the ℓ-parts of τ and τ' are not trivial in Φ. Note that it follows from Lemma 2.2 that $\operatorname{ord}_\ell(\tau') = \operatorname{ord}_\ell(\gcd(d, d'))$.

The arithmetical graph I_{2n}^*, occurring as a type of reduction of elliptic curves, provides an example of a pair (C, C') where the ℓ-part of τ is equal to the maximum of the ℓ-parts of the weights $w(\mathscr{C})$.

Proposition 3.4. *Let (G, M, R) be any arithmetical graph. Let*

$$(C, r), (C_1, r_1), \ldots, (C_n, r_n)$$

be a terminal chain T of G, with node C and terminal vertex C_n. Then the image τ of $E(C, C_j)$ in $\Phi(G)$ is trivial for all $j = 1, \ldots, n$.

Proof. Since the pairing $\langle\,;\,\rangle$ is perfect, it is sufficient, to show that $\tau = 0$, to show that $\langle \tau; \sigma \rangle = 0$ for all $\sigma \in \Phi(G)$. Let σ denote the image in Φ of $E(D, D')$, where D, D' are any vertices of G, of multiplicity r_D and $r_{D'}$. If neither D nor D' belong to the terminal chain T, or if $D = C$ and $D' \notin T$, then Proposition 3.2 implies that $\langle \tau; \sigma \rangle = 0$. Assume now that $D = C_i$ and $D' \neq C_s$, for all $s = 1, \ldots, n$. Let $m = \min(i, j)$. Then, using 3.2 and 2.8, we find that there exist two integers b and c such that

$$\begin{aligned}\langle \tau; \sigma \rangle &= \operatorname{lcm}(r, r_j) \operatorname{lcm}(r_i, r_{D'}) \left(\frac{1}{r_m r_{m-1}} + \cdots + \frac{1}{r_1 r} \right) \\ &= \operatorname{lcm}(r, r_j) \operatorname{lcm}(r_i, r_{D'}) \left(\frac{b}{r r_n} - \frac{c}{r_m r_n} \right).\end{aligned}$$

Since $r_n | r_i$, we have $\operatorname{lcm}(r, r_j) \operatorname{lcm}(r_i, r_{D'}) b / r r_n = 0$ in $\mathbb{Q}/\mathbb{Z}$. If $m = i$, then we use the fact that $r_n | r$ to find that $\operatorname{lcm}(r, r_j) \operatorname{lcm}(r_i, r_{D'}) c / r_i r_n = 0$ in $\mathbb{Q}/\mathbb{Z}$. If $m = j$, we use again the fact that $r_n | r_i$ to find that $\operatorname{lcm}(r, r_j) \operatorname{lcm}(r_i, r_{D'}) c / r_j r_n = 0$ in $\mathbb{Q}/\mathbb{Z}$. Thus, in all cases, $\langle \tau; \sigma \rangle = 0$. If $D = C_i$ and $D' = C_s$, Proposition 2.7 shows that $\sigma = 0$. This concludes the proof of Proposition 3.4. The reader may use the techniques developed in the above proof to give a different proof of Proposition 2.7.

Remark 3.5. Let (C_1, r_1), (C_2, r_2), and (D, r), be three vertices on an arithmetical graph (G, M, R). Then

$$rE(C_1, C_2) = \frac{r_2 \gcd(r, r_1)}{\gcd(r_2, r_1)} E(C_1, D) + \frac{r_1 \gcd(r, r_2)}{\gcd(r_2, r_1)} E(D, C_2).$$

If $\ell \nmid r r_1 r_2$, we find that the order of the ℓ-part of $E(C_1, C_2)$ divides the maximum of the orders of the ℓ-parts of $E(C_1, D)$ and $E(D, C_2)$.

If C_1 and C_2 belong to the same terminal chain of G and if $\ell \nmid r_1 r_2$, we find, using 2.7 and 3.4, that the orders of the ℓ-parts of $E(C_1, D)$ and $E(C_2, D)$ are equal.

3.6. If (D, r) is a node and (C_n, r_n) and $(C'_{n'}, r'_{n'})$ are terminal vertices on two terminal chains attached to D, then we shall call $(C_n, C'_{n'})$ an *elementary pair*. In the case of an elementary pair, both r_n and $r'_{n'}$ divide r and we find that as vectors in $\mathbb{Z}^v$,

$$\frac{r}{\operatorname{lcm}(r_n, r'_{n'})} E(C_n, C'_{n'}) = E(C_n, D) + E(D, C'_{n'}).$$

Using Proposition 3.4, we see that $E(C_n, C'_{n'})$ has order dividing $r/\mathrm{lcm}(r_n, r'_{n'})$. We shall compute below the precise order of such a pair of vertices.

Let (C, C') be any pair in a graph G, and let τ denote the image of $E(C, C')$ in Φ. When the pair (C, C') is weakly connected, the perfectness of the pairing $\langle\, ; \rangle$ reduces the sometimes difficult computation of the order of τ in Φ to a series of easier computations. Indeed, the perfectness of the pairing implies that the order of τ is equal to the maximum of the orders of the elements of $\mathbb{Q}/\mathbb{Z}$ of the form $\langle \tau; \mu \rangle$, where μ ranges over all elements of Φ. Since $\langle \tau; \mu \rangle$ is very easy to compute when (C, C') is weakly connected, the order of τ can be easily computed. We illustrate this technique in our next proposition.

Consider the following elementary pair $(C_n, C'_{n'})$. Let (D, r) be a node of the graph G. Let (D, r), $(C_1, r_1), \dots, (C_n, r_n)$ be a terminal chain T on G with terminal vertex C_n. Let (D, r), $(C'_1, r'_1), \dots, (C'_{n'}, r'_{n'})$ be a terminal chain T' on G with terminal vertex $C'_{n'}$. Let G_D denote the connected component of D in $G \backslash \{\text{edges of } T \cup T'\}$. Let

$$m := \gcd\big(r_j, (D_j, r_j) \text{ any vertex of } G_D\big).$$

Note that $m|r$. We know from the relation $MR = 0$ that $|D \cdot D|r = r_1 + r'_1 + z$, for some integer z divisible by m. Let $\tau \in \Phi(G)$ denote the image of $E(C_n, C'_{n'})$. It follows from Proposition 3.2 that

$$\langle \tau; \tau \rangle = \mathrm{lcm}(r_n, r'_{n'})^2 (1/r_n r_{n-1} + \cdots + 1/r_1 r + 1/r r'_1 + \cdots + 1/r'_{n'-1} r'_{n'}).$$

Let $\{b_1, \dots, b_n\}$ denote the sequence of integers associated in 2.3 to the terminal chain T. Lemma 2.8 shows that

$$(1/r_n r_{n-1} + 1/r_{n-1} r_{n-2} + \cdots + 1/r_1 r) = b_n / r r_n.$$

Lemma 2.6 shows that b_n and r/r_n are coprime. The same arguments show that

$$(1/r'_{n'} r_{n'-1} + 1/r'_{n'-1} r'_{n'-2} + \cdots + 1/r'_1 r) = b'_{n'} / r r'_{n'},$$

and for any truncated sum,

$$(1/r'_{n'} r'_{n'-1} + \cdots + 1/r'_{i+1} r'_i) = c_i / r'_i r'_{n'},$$

for some integer c_i with $\gcd(c_i, r'_i/r'_{n'}) = 1$. It follows that

$$\langle \tau; \tau \rangle = \mathrm{lcm}(r_n, r'_{n'})^2 (b_n / r r_n + b'_{n'} / r r'_{n'}).$$

Proposition 3.7. *Keep the notation introduced above.*

(a) *If $\ell \nmid r_n r'_{n'}$, then the ℓ-part of the order of $\langle \tau; \tau \rangle$ in $\mathbb{Q}/\mathbb{Z}$ is equal to $\ell^{\mathrm{ord}_\ell(r/z)}$. The ℓ-part of the order of τ in $\Phi(G)$ is equal to $\ell^{\mathrm{ord}_\ell(r/m)}$.*

(b) *If $\mathrm{ord}_\ell(r_n) = \mathrm{ord}_\ell(r'_{n'}) > 0$, then the ℓ-part of the order of $\langle \tau; \tau \rangle$ in $\mathbb{Q}/\mathbb{Z}$ is equal to $\ell^{\mathrm{ord}_\ell(r/z)}$. The ℓ-part of the order of τ in $\Phi(G)$ is equal to $\ell^{\mathrm{ord}_\ell(r/\mathrm{lcm}(r_n, r'_{n'}))}$.*

(c) *If* $\mathrm{ord}_\ell(r_n) < \mathrm{ord}_\ell(r'_{n'})$, *then the* ℓ*-part of the order of* $\langle\tau;\tau\rangle$ *in* $\mathbb{Q}/\mathbb{Z}$ *and the* ℓ*-part of the order of* τ *in* $\Phi(G)$ *are both equal to* $\ell^{\mathrm{ord}_\ell(r/\mathrm{lcm}(r_n,r'_{n'}))}$.

Proof. Using 3.6, we find that, if $\mathrm{ord}_\ell(r/\mathrm{lcm}(r_n,r'_{n'}))=0$, then the ℓ-part of τ is trivial. Thus, Proposition 3.7 holds. Assume now that $\mathrm{ord}_\ell(r/\mathrm{lcm}(r_n,r'_{n'}))>0$. Recall that

$$\langle\tau;\tau\rangle = \frac{\mathrm{lcm}(r_n,r'_{n'})^2}{r_n r'_{n'}}\left(\frac{b_n r'_{n'}+b'_{n'}r_n}{r}\right).$$

Lemma 2.6 shows that $b_n r_1 - r_n = er$ and $b'_{n'}r'_1 - r'_{n'} = fr$ for some integers e,f. Thus

$$\begin{aligned} r'_1(b_n r'_{n'}+b'_{n'}r_n) &= r'_1 b_n r'_{n'} + (fr+r'_{n'})r_n \\ &= (|D\cdot D|r - z - r_1)b_n r'_{n'} + (fr+r'_{n'})r_n \\ &= (|D\cdot D|r - z)b_n r'_{n'} - (er+r_n)r'_{n'} + (fr+r'_{n'})r_n. \end{aligned}$$

It follows that modulo $\mathbb{Z}$,

$$r'_1\frac{\gcd(r_n,r'_{n'})}{\mathrm{lcm}(r_n,r'_{n'})}\langle\tau;\tau\rangle = r'_1(b_n r'_{n'}+b'_{n'}r_n)/r \equiv -zb_n r'_{n'}/r.$$

Since $r_n=\gcd(r_1,r)$ and $b_n r_1 - r_n = er$, we find that $\gcd(\ell,b_n)=1$. Since $r'_{n'}=\gcd(r'_1,r)$, $\mathrm{ord}_\ell(r'_{n'})=\mathrm{ord}_\ell(r'_1)$. If $\mathrm{ord}_\ell(r'_{n'})>\mathrm{ord}_\ell(r_n)$, then the relation $r_1+r'_1+z=|D\cdot D|r$ shows that in this case $\mathrm{ord}_\ell(z)=\mathrm{ord}_\ell(r_n)$. Part (c) of Proposition 3.7 follows in this case from 3.6 since the order of τ is at least equal to the order of $\langle\tau;\tau\rangle$.

Assume now that $\mathrm{ord}_\ell(r'_{n'})=\mathrm{ord}_\ell(r_n)$. Then, clearly, the ℓ-part of the order of $\langle\tau;\tau\rangle$ in $\mathbb{Q}/\mathbb{Z}$ is equal to $\ell^{\mathrm{ord}_\ell(r/z)}$. Let (C,r_C) be a vertex of G. Let μ denote the image in Φ of $E(C_n,C)$. Assume first that C does not belong to the shortest path $\mathscr{P}$ in G from C_n to $C'_{n'}$. Then

$$\begin{aligned} \langle\tau;\mu\rangle &= \mathrm{lcm}(r_n,r'_{n'})\,\mathrm{lcm}(r_n,r_C)(1/r_n r_{n-1}+\cdots+1/r_1 r) \\ &= \mathrm{lcm}(r_n,r'_{n'})\,\mathrm{lcm}(r_n,r_C)b_n/rr_n. \end{aligned}$$

If $\mathrm{ord}_\ell(r_n)>0$, then there exists a vertex C outside of $\mathscr{P}$ whose multiplicity is not divisible by ℓ. For such C, we find that the ℓ-part of the order of $\langle\tau;\mu\rangle$ is divisible by the ℓ-part of $r/\mathrm{lcm}(r_n,r'_{n'})$. Thus, 3.7 (b) follows again in this case from 3.6.

Let us then assume that $\ell\nmid r_n r'_{n'}$. Then, for any C not on $\mathscr{P}$, the ℓ-part of the order of $\langle\tau;\mu\rangle$ equals $\ell^{\mathrm{ord}_\ell(r/r_C)}$. We are going to show that the ℓ-part of the order of τ in $\Phi(G)$ is equal to $\ell^{\mathrm{ord}_\ell(r/m)}$ by showing that $\ell^{\mathrm{ord}_\ell(r/m)}$ is the maximum of the orders of the ℓ-parts of the elements $\langle\tau;\mu\rangle$, with $\mu\in\Phi$. If C belongs to the chain from C_n to D, then it follows from 2.7 and 3.4 that $\mu=0$, so that $\langle\tau;\mu\rangle=0$. If $C=C'_i$ for some i, write

$$\begin{aligned} 1/r'_i r'_{i-1}+\cdots+1/r'_1 r &= (1/r'_{n'}r'_{n'-1}+\cdots+1/r'_1 r) - (1/r'_{n'}r'_{n'-1}+\cdots+1/r'_{i+1}r'_i) \\ &= b'_{n'}/rr'_{n'} - c_i/r'_i r'_{n'}. \end{aligned}$$

It follows that

$$\begin{aligned}\langle\tau;\mu\rangle &= \mathrm{lcm}(r_n, r'_{n'})\,\mathrm{lcm}(r_n, r'_i)(b_n/rr_n + b'_{n'}/rr'_{n'} - c_i/r'_i r'_{n'})\\ &= \mathrm{lcm}(r_n, r'_{n'})\,\mathrm{lcm}(r_n, r'_i)(b_n/rr_n + b'_{n'}/rr'_{n'}).\end{aligned}$$

Using the computation of $\langle\tau;\tau\rangle$ done above, we find that when $C = C'_i$, the ℓ-part of the order of $\langle\tau;\mu\rangle$ is at most $\ell^{\mathrm{ord}_\ell(r/z)}$, and $\mathrm{ord}_\ell(r/z) \leqq \mathrm{ord}_\ell(r/m)$. It follows that the ℓ-part of the order of $\langle\tau;\mu\rangle$ is at most $\ell^{\mathrm{ord}_\ell(r/m)}$ for all $\mu \in \Phi$ of the form image of $E(C_n, C)$. By definition of m, there exists a vertex (C, r_C) outside of the path $\mathscr{P}$ whose multiplicity is such that $\mathrm{ord}_\ell(r_C) = \mathrm{ord}_\ell(m)$. Thus for this vertex C, the ℓ-part of the order of $\langle\tau;\mu\rangle$ equals $\ell^{\mathrm{ord}_\ell(r/m)}$. To conclude the proof of 3.7 (a), it remains to show that when $\ell \nmid r_n$, the set of elements of Φ of the form μ as above generates the ℓ-part of the group Φ. By construction, the ℓ-part of Φ is isomorphic to $\mathrm{Ker}({}^tR) \otimes \mathbb{Z}_\ell/\mathrm{Im}(M) \otimes \mathbb{Z}_\ell$. Since the multiplicity of C_n is coprime to ℓ, it is invertible in $\mathbb{Z}_\ell$, and our claim follows.

Example 3.8. Consider the following graphs.

The pair $E(C, C')$ has order 4 in $\Phi(G_1)$, and order 2 in $\Phi(G_2)$.

Corollary 3.9. *Keep the notation introduced above. If either* (a) $\ell \nmid r_n r'_{n'}$ *and* $\mathrm{ord}_\ell(z) = \mathrm{ord}_\ell(m)$, *or* (b) $\mathrm{ord}_\ell(z) = \mathrm{ord}_\ell(r_n) = \mathrm{ord}_\ell(r'_{n'}) > 0$ *or* (c) $\mathrm{ord}_\ell(r_n) < \mathrm{ord}_\ell(r'_{n'})$, *then τ is not divisible by ℓ in Φ.*

Proof. Given any two elements τ and σ of Φ of orders t and s, respectively, it is easy to check that the order of the element $\langle\tau;\sigma\rangle$ divides $\gcd(t,s)$. Hence, if $\tau = \ell\xi$ in Φ, then $\langle\tau;\tau\rangle = \ell\langle\tau;\xi\rangle$ is killed by t/ℓ. Now let τ be the image of an elementary pair, as in 3.7. Under our hypotheses, Proposition 3.7 shows that $\ell^{\mathrm{ord}_\ell(t)}$ divides the order of $\langle\tau;\tau\rangle$. Thus, τ is not divisible by ℓ.

The reader will find in 6.6 an explicit example where $\ell \nmid r_n r'_{n'}$ and $\mathrm{ord}_\ell(z) > \mathrm{ord}_\ell(m)$, and where τ is divisible by ℓ in Φ. Note that it is not true in general that if $\ell \nmid r_n r'_{n'}$ and τ is not divisible by ℓ in Φ, then $\mathrm{ord}_\ell(z) = \mathrm{ord}_\ell(m)$.

As a last example of the usefulness of the pairing in providing information on the order of elements in Φ, let us consider the following situation. Let (G, M, R) be an arithmetical graph with a weakly connected pair of terminal vertices (C_n, C'_m) such that the unique path $\mathscr{P}$ of G that connects C_n and C'_m has the following vertices:

$$\{(C_n, r_n), (C_{n-1}, r_{n-1}), \ldots, (C_1, r_1), (D, r)\}$$

is a terminal chain with node D, $\{(D, r), (D_1, t_1), \ldots, (D_k, t_k), (D', s)\}$ is a chain with exactly two nodes D and D', and $\{(D', s), (C_1', s_1), \ldots, (C_{m-1}', s_{m-1}), (C_m', s_m)\}$ is a terminal chain with node D' and terminal vertex C_m'. The reader will find an explicit example of such a graph in 6.6.

Proposition 3.10. *Let (G, M, R) be an arithmetical graph with a weakly connected pair of terminal vertices (C_n, C_m') as above. Assume that $\mathrm{ord}_\ell(t_i) \geqq \mathrm{ord}_\ell(r)$ for all $i = 1, \ldots, k$, and that $\mathrm{ord}_\ell(s) > \mathrm{ord}_\ell(r)$. Assume also for simplicity that*

$$\mathrm{ord}_\ell(r_n) = \mathrm{ord}_\ell(s_m) = 0.$$

Let τ denote the image of $E(C_n, C_m')$ in $\Phi(G)$. Then the order of τ is divisible by $\ell^{\mathrm{ord}_\ell(rs)}$.

Proof. Let $a := \mathrm{ord}_\ell(r)$ and $a + b := \mathrm{ord}_\ell(s)$. The relations t_1 divides $r + t_2$, t_2 divides $t_1 + t_3, \ldots, t_q$ divides $t_{q-1} + s$, show that the sequence

$$s\ell^{-a}, t_k\ell^{-a}, \ldots, t_1\ell^{-a}, r\ell^{-a}$$

can be continued using Euclid's algorithm with $t_1\ell^{-a}$ and $r\ell^{-a}$ into a sequence

$$S := \{s\ell^{-a}, t_k\ell^{-a}, \ldots, t_1\ell^{-a}, r\ell^{-a}, u_1, \ldots, u_w\},$$

as in [Lor2], 2.4, and that this sequence S can be considered as the sequence of multiplicities of a terminal chain of some arithmetical graph. In particular, Lemma 2.8 shows that

$$\frac{1}{u_w u_{w-1}} + \cdots + \frac{1}{u_2 u_1} + \frac{1}{u_1 r\ell^{-a}} + \frac{1}{r\ell^{-a} t_1 \ell^{-a}} + \cdots + \frac{1}{t_k\ell^{-a} s\ell^{-a}} = \frac{\beta}{s\ell^{-a} u_w}$$

for some integer β coprime to $s\ell^{-a}/u_w$, and

$$\frac{1}{u_w u_{w-1}} + \cdots + \frac{1}{u_2 u_1} + \frac{1}{u_1 r\ell^{-a}} = \frac{\gamma}{r\ell^{-a} u_w}$$

for some integer γ coprime to $r\ell^{-a}/u_w$. Using Proposition 3.2, we find that

$$\begin{aligned} \mathrm{lcm}(r_n, s_m)^{-2}\langle\tau;\tau\rangle = {} & \left(\frac{1}{r_n r_{n-1}} + \cdots + \frac{1}{r_1 r}\right) + \left(\frac{1}{rt_1} + \cdots + \frac{1}{t_{k-1}t_k} + \frac{1}{t_k s}\right) \\ & + \left(\frac{1}{ss_1} + \cdots + \frac{1}{s_{m-1}s_m}\right). \end{aligned}$$

Lemma 2.8 implies that there exist integers c_n and d_m coprime to ℓ such that

$$\mathrm{lcm}(r_n, s_m)^{-2}\langle\tau;\tau\rangle = \frac{c_n}{rr_n} + \left(\frac{\beta\ell^{-a}}{su_w} - \frac{\gamma\ell^{-a}}{ru_w}\right) + \frac{d_m}{ss_m}.$$

Regarded as elements in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, c_n/rr_n has order ℓ^a, d_m/ss_m has order ℓ^{a+b}, $\gamma/\ell^a r u_w$ has order at most ℓ^{2a}. Since $b > 0$, $\ell \nmid \beta$ and, thus, $\beta/\ell^a s u_w$ has order ℓ^{2a+b}. Since $a > 0$, we find that $\langle\tau;\tau\rangle$ has order ℓ^{2a+b} in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, which concludes the proof of Proposition 3.10.

3.11. For later use in 6.6, let us note the following fact. The details are left to the reader. Let (D, r) and (D', s) be two nodes on a chain $\mathscr{C}$ of a graph G. Let τ denote the image of $E(D, D')$ in Φ. Assume that ℓ divides the weight $w(\mathscr{C})$. Then the ℓ-part of the order of τ in Φ may be non-trivial, but the ℓ-part of the order of $\langle \tau; \tau \rangle$ is always trivial.

4. A splitting of the group of components

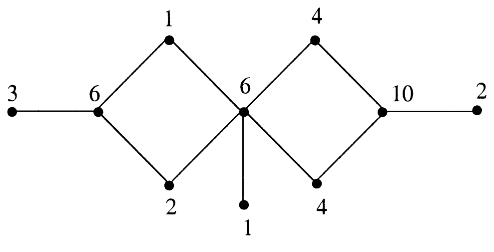
Let (G, M, R) be an arithmetical graph. Fix a prime ℓ. Let $\Phi_\ell(G)$ denote the ℓ-part of the group of components $\Phi(G)$. Let (D, r) be a vertex of G such that $G\backslash\{D\}$ is not connected. Our aim in this section is to establish an isomorphism between $\Phi_\ell(G)$ and the product of the ℓ-parts of the groups of components of arithmetical graphs associated to the connected components of $G\backslash\{D\}$.

Construction 4.1. Label the connected components of $G\backslash\{D\}$ by $\mathscr{G}_1, \dots, \mathscr{G}_t$. Label the vertices of $\mathscr{G}_i$ adjacent in G to D by $(C_{i,1}, r_{i,1}), \dots, (C_{i,s_i}, r_{i,s_i})$. Assume that $t > 1$. For $i = 1, \dots, t$, let g_i denote the greatest common divisor of r and the multiplicities of all vertices of $\mathscr{G}_i$.

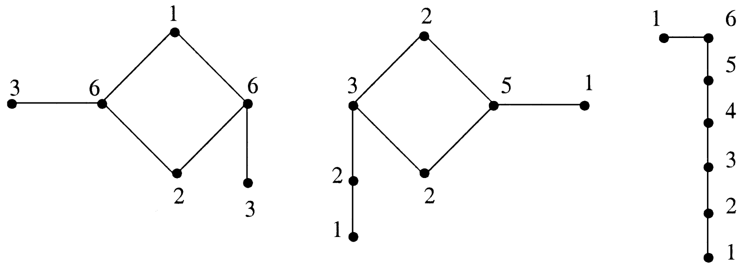
Construct a new connected arithmetical graph G_i associated to $\mathscr{G}_i$ as follows. Start with $\mathscr{G}_i \cup \{D\}$. Give to D the multiplicity r/g_i. Give to a vertex in $\mathscr{G}_i$ its multiplicity in G divided by g_i. Let c_i denote the least integer such that $c_i r - \sum_{j=1}^{s_i} (C_{ij} \cdot D) r_{ij} \geqq 0$. The integer c_i will be the self-intersection of D in G_i.

If r divides $\sum_{j=1}^{s_i} (C_{ij} \cdot D) r_{ij}$, then the graph $G_i := \mathscr{G}_i \cup \{D\}$ with multiplicities as above is an arithmetical graph. If r does not divide $\sum_{i=1}^{s_i} (C_{ij} \cdot D) r_{ij}$, then let $\hat{r}_i := \left(c_i r - \sum_{j=1}^{s_i} (C_{ij} \cdot D) r_{ij} \right) / g_i$. Construct a terminal chain T using $(r/g_i, \hat{r}_i)$ and Euclid's algorithm as in [Lor2], 2.4. The graph G_i consists then in the graph $\mathscr{G}_i \cup \{D\}$, with the chain T attached to D. We shall say that the graph G is *ℓ-breakable at* (D, r) if $\ell \nmid r$ and $t > 1$.

Example 4.2. Let G be the following graph.



Let D denote the central vertex of multiplicity 6. Then $G\backslash\{D\}$ has 3 components $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$, and the above procedure produces 3 new arithmetical graphs:



G_1 G_2 G_3

Note that $|\Phi(G_1)| = 2$, $|\Phi(G_2)| = 8$, and $|\Phi(G_3)| = 1$. Our next proposition shows then that the only primes that can divide $|\Phi(G)|$ are 2 or 3. One may compute that the full group $\Phi(G)$ has order 144.

Proposition 4.3. *Let (G, M, R) be any arithmetical graph. Let ℓ be a prime. Assume that G is ℓ-breakable at a vertex (D, r). Let $G_1, \ldots, G_t$ denote the arithmetical graphs associated as in* 4.1 *to the components of $G \backslash \{D\}$. Then there exists an isomorphism*

$$\alpha : \Phi_\ell(G) \to \prod_{i=1}^{t} \Phi_\ell(G_i).$$

Let (C_1, r_1) and (C_2, r_2) be any two vertices of G. If C_1 and C_2 belong to the same component of $G \backslash \{D\}$, say to $\mathcal{G}_j$, or if $C_1 \in \mathcal{G}_j$ and $C_2 = D$, then we denote by $E(C_1, C_2)$ and $E(C_1, D)$ both the elements of $\Phi(G)$ and the corresponding elements of $\Phi(G_j)$. Then the ℓ-part of $E(C_1, C_2)$ is mapped under α to the element of $\prod_{i=1}^{t} \Phi_\ell(G_i)$ having the ℓ-part of $E(C_1, C_2)$ in the j-th coordinate, and 0 *everywhere else.*

Proof. Let N_i, $i = 1, \ldots, t$, denote the square submatrix of the intersection matrix M corresponding to the vertices of G that belong to $\mathcal{G}_i$. The matrix M has the following form:

$$\begin{pmatrix} (D \cdot D) & * & \ldots & \ldots & * \\ * & N_1 & 0 & \ldots & 0 \\ \vdots & 0 & N_2 & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ * & 0 & \ldots & 0 & N_t \end{pmatrix}$$

(in particular, the first column is the 'D-column'). Multiply the first row by r, and add to it the sum of all other rows, each multiplied by its corresponding multiplicity. Perform a similar operation on the first column of M, to obtain a matrix M' of the form

$$\begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 \\ 0 & N_1 & & & \\ \vdots & & N_2 & & \\ \vdots & & & \ddots & \\ 0 & & & & N_t \end{pmatrix}.$$

Since $\ell \nmid r$, the row and column operations described above are permissible over $\mathbb{Z}_\ell$. The module $\mathrm{Ker}({}^tR) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is thus the direct sum of t $\mathbb{Z}_\ell$-submodules V_i, $i = 1, \dots, t$, where

$$V_i := \bigoplus_{C \in \mathscr{G}_i} \mathbb{Z}_\ell r^{-1} E(C, D).$$

Let W_i denote the $\mathbb{Z}_\ell$-span of the column vectors of N_i, so that $\mathrm{Im}(M) \otimes \mathbb{Z}_\ell \cong \bigoplus_{i=1}^{t} W_i$. Then

$$\Phi_\ell(G) \cong \bigoplus_{i=1}^{t} V_i / W_i.$$

We claim that $\Phi_\ell(G_i) \cong V_i / W_i$. Indeed, the intersection matrix M_i associated to G_i has the following form:

$$\left(\begin{array}{cccc|cc} * & 1 & & & & \\ 1 & \ddots & \ddots & & & \\ & \ddots & * & 1 & & \\ & & 1 & (D \cdot D) & * & \cdots \\ \hline & & & * & & \\ & & & \vdots & & N_i \end{array}\right) \quad \text{or} \quad \begin{pmatrix} (D \cdot D) & \cdots & * \\ \vdots & & \\ & & \boxed{N_i} \\ * & & \end{pmatrix},$$

where the case on the right occurs if $G_i = \mathscr{G}_i \sqcup \{D\}$ (see 4.1). Multiply the D-row by r/g_i, and add to it all other rows multiplied by their corresponding multiplicities. Perform a similar operation on the columns of M_i to get a matrix M_i' of the form

$$\begin{pmatrix} * & 1 & & & 0 & & \\ 1 & \ddots & \ddots & & \vdots & & \\ & \ddots & \ddots & 1 & \vdots & & \\ & & 1 & * & 0 & & \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ & & & & \vdots & \multicolumn{2}{|c}{N_i} \\ & & & & 0 & \multicolumn{2}{|c}{} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ & & \boxed{N_i} \\ 0 & & \end{pmatrix}.$$

In the case of the first matrix M_i', the top left corner can be further reduced to:

$$\begin{pmatrix} 0 & & & -\beta & 0 & & \\ 1 & \ddots & & 0 & \vdots & & \\ & \ddots & \ddots & \vdots & \vdots & & \\ & & 1 & 0 & 0 & & \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ & & & & \vdots & \multicolumn{2}{|c}{N_i} \\ & & & & 0 & \multicolumn{2}{|c}{} \end{pmatrix},$$

where $\beta = -r/\gcd(r, \hat{r}_i)$ (see the proof of 2.7; note that $\gcd(r, \hat{r}_i)$ is the terminal multiplicity of the terminal chain attached to D in G_i). Since $\ell \nmid r$, we find that in both cases,

$$\Phi_\ell(G_i) \cong \bigoplus_{C \in \mathscr{G}_i} \mathbb{Z}_\ell r^{-1} E(C, D)/\mathrm{Im}(N_i).$$

Therefore, we have an isomorphism between $\Phi_\ell(G_i)$ and V_i/W_i, where the ℓ-part of $E(C, D)$ in $\Phi_\ell(G_i)$ is mapped to the ℓ-part of the element $E(C, D)$ in $V_i/W_i \subset \Phi_\ell(G)$. We leave it to the reader to compute the image of $E(C_1, C_2)$ under the above isomorphism. This concludes the proof of Proposition 4.3.

We now use Proposition 4.3 to prove the following important theorem.

Theorem 4.4. *Let (G, M, R) be any arithmetical graph. Let ℓ be any prime. Let (C, r) and (C', r') be a weakly connected and ℓ-breakable pair of vertices of G with $\ell \nmid rr'$ and associated integer $\lambda(C, C')$ as in* 1.2. *Then the ℓ-part of $E(C, C')$ has order $\lambda(C, C')$ in $\Phi(G)$.*

Proof. Suppose that both C and C' are not terminal vertices of G. Then by hypothesis the graph is ℓ-breakable at C into two or more arithmetical graphs. Denote by G' the new arithmetical graph that contains C' (graph constructed while breaking G). Then the ℓ-part of $E(C, C')$ in G has the same order as the ℓ-part of $E(C, C')$ in G'. Since (C, C') is weakly connected, C lies on a terminal chain T of G', and its multiplicity is still coprime to ℓ. Moreover, (C, C') is a weakly connected ℓ-breakable pair of G'. Denote by D the terminal vertex of T. If $C \neq D$, we find using Remark 3.5 that the ℓ-part of $E(C, C')$ has the same order as the ℓ-part of $E(D, C')$. The pair (D, C') is clearly a weakly connected ℓ-breakable pair of G'. Thus, to prove Theorem 4.4 for (C, C') in G, it is sufficient to prove it for pairs where one vertex is a terminal vertex, say the vertex C.

Let $\mathscr{P}$ denote the path associated to the weakly connected pair (C, C'). (Note that if $\mathscr{P}\backslash\{C, C'\}$ contains no vertices, the theorem follows from 2.2.) Let $(C_1, r_1), (C_2, r_2), \ldots, (C_s, r_s)$ be the nodes on $\mathscr{P}\backslash\{C, C'\}$, as discussed in 1.1. If $s = 0$, then C and C' belong to the same terminal chain of G, and Theorem 4.4 follows from 2.7. If $s = 1$ and C' is not a terminal vertex, we may apply the reduction step described at the beginning of the proof and assume without loss of generality that C' is a terminal vertex. Then we can apply 3.7 to show that our statement holds in this case.

We proceed by induction on the number s of nodes on $\mathscr{P}$. Let $m > 1$ and assume that Theorem 4.4 holds for $s \leqq m - 1$. Let (C, C') be a pair whose associated path $\mathscr{P}$ contains m nodes. Since the pair is ℓ-breakable, there exists a vertex (D, r_D) on $\mathscr{P}$ with $\ell \nmid r_D$ and such that both components of $\mathscr{P}\backslash\{D\}$ contain at most $m - 1$ nodes C_i. (Note that one component may contain no nodes such as, for instance, when $r_D = C_1$.) Break the graph G at D. Call G_1 the arithmetical graph associated to the connected component of $G\backslash\{D\}$ which contains C. Call G_2 the arithmetical graph that contains C'. The pairs (C, D) and (D, C') are weakly connected and ℓ-breakable pairs of G_1 and G_2, respectively. We may thus apply the induction hypothesis to both pairs. To conclude the proof of Theorem 4.4, we need only to show that the order of the ℓ-part of $E(C, C')$ is equal to the maximum of the orders of the ℓ-parts of $E(C, D)$ and $E(D, C')$. Note that 3.5 only shows that the order of $E(C, C')$ divides the maximum of the orders of $E(C, D)$ and $E(D, C')$. To prove our claim, we need to use the fact that breaking the graph G at D produces a splitting of $\Phi(G)_\ell$, with $\Phi(G_1)_\ell$ and $\Phi(G_2)_\ell$ as direct summands (4.3).

5. The subgroups $\Psi_{K,L}$ and $\Theta^{[3]}_{K,\ell}$

Let X/K be a curve. We recall below Raynaud's description of the group Φ_K and of the map π in terms of a regular model $\mathscr{X}/\mathcal{O}_K$ of X/K. Let $\mathscr{X}_k = \sum_{i=1}^{v} r_i C_i$, and assume that $\gcd(r_1, \dots, r_v) = 1$. Let $\mathscr{L} := \bigoplus_{i=1}^{v} \mathbb{Z} C_i$ denote the free abelian group generated by the components C_i, $i = 1, \dots, v$. Let $\mathscr{L}^* := \mathrm{Hom}_{\mathbb{Z}}(\mathscr{L}, \mathbb{Z})$, and let $\{x_1, \dots, x_v\}$ denote the dual basis of $\mathscr{L}$, so that $x_i(C_j) = \delta_{ij}$. Let ${}^tR : \mathscr{L}^* \to \mathbb{Z}$ be the map $\sum_{i=1}^{v} a_i x_i \mapsto \sum_{i=1}^{v} a_i r_i$. Consider the following diagram:

$$\begin{array}{ccccccc}
\mathscr{L} & \xrightarrow{i} & \mathrm{Pic}(\mathscr{X}) & \xrightarrow{\mathrm{res}} & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} \\
\| & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \| \\
\mathscr{L} & \xrightarrow{\mu} & \mathscr{L}^* & \longrightarrow & \mathscr{L}^*/\mu(\mathscr{L}) & \xrightarrow{{}^tR} & \mathbb{Z}.
\end{array}$$

The map i is defined as follows: $i(C_j) :=$ curve C_j in $\mathscr{X}$, where the curve C_j is viewed as an element of $\mathrm{Pic}(\mathscr{X})$. The map res restricts a divisor of $\mathscr{X}$ to the open set X of $\mathscr{X}$. The map res is surjective because the scheme $\mathscr{X}$ is regular. The map deg is defined as follows: $\deg\left(\sum_{i=1}^{s} a_i P_i\right) := \sum_{i=1}^{s} a_i [K(P_i) : K]$, where $K(P_i)$ is the residue field of P_i in X. We denote by $\mathrm{Pic}^0(X)$ the kernel of the map deg. The intersection matrix M of $\mathscr{X}_k$ can be thought of as a bilinear map on $\mathscr{L} \times \mathscr{L}$ and, therefore, induces a map $\mu : \mathscr{L} \to \mathscr{L}^*$ defined by $\mu(C_i) := \sum_{j=1}^{v} (C_i \cdot C_j) x_j$. Then ${}^tR \circ \mu = 0$. Let D be an irreducible divisor on $\mathscr{X}$, and define $\phi(D) := \sum_{j=1}^{v} (C_j \cdot D) x_j$. The map ψ is the natural map induced by ϕ. It is well-known that the diagram above is commutative.

One easily checks that $\mathrm{Ker}({}^tR)/\mu(\mathscr{L})$ is the torsion subgroup of $\mathscr{L}^*/\mu(\mathscr{L})$. Raynaud [BLR], 9.6, showed that the group of components Φ_K of the jacobian A/K of the curve X/K is isomorphic to the group $\mathrm{Ker}({}^tR)/\mu(\mathscr{L})$. It follows from this description that the group Φ_K can be explicitly computed using a row and column reduction of the intersection matrix M (see [Lor1], 1.4). Since the residue field k is algebraically closed, $A(K) = \mathrm{Pic}^0(X)$. Raynaud ([BLR], 9.5/9 and 9.6/1) has shown that the reduction map $\pi : A(K) \to \Phi$ corresponds to the restricted map $\psi : \mathrm{Pic}^0(X) \to \mathrm{Ker}({}^tR)/\mu(\mathscr{L})$. Thus, given two points P and Q in $X(K)$, the image of $P - Q$ in the group Φ_K is trivial if $C_P = C_Q$, and can be identified with the image of $E(C_P, C_Q)$ in $\Phi(G)$ when $C_P \neq C_Q$. To prove Theorem 5.5 below, we need to recall the following facts.

5.1. Let $\mathscr{X}/\mathcal{O}_K$ be any regular model of X/K. Let F/K be a finite extension. Let $\mathscr{Y}/\mathcal{O}_F$ denote the normalization of the scheme $\mathscr{X} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_F)$. Let $b : \mathscr{Y} \to \mathscr{X}$

denote the composition of the natural maps

$$\mathscr{Y} \to \mathscr{X} \times_{\mathrm{Spec}\,(\mathcal{O}_K)} \mathrm{Spec}\,(\mathcal{O}_F) \to \mathscr{X}.$$

Let $\rho : \mathscr{Z} \to \mathscr{Y}$ denote the minimal desingularization of $\mathscr{Y}$. To recall the descriptions of the maps ρ and b, we need the following definition. Let $C_1, \ldots, C_m$ be smooth irreducible components of the special fiber $\mathscr{X}_k$. The divisor $C := \sum_{i=1}^{m} C_i$ is said to be a *Hirzebruch-Young string* if the following four conditions hold: 1) $g(C_i) = 0$, for all $i = 1, \ldots, m$, and 2) $(C_i \cdot C_i) \leqq -2$, for all $i = 1, \ldots, m$, and 3) $(C_i \cdot C_j) = 1$ if $|i - j| = 1$, and 4) $(C_i \cdot C_j) = 0$ if $|i - j| > 1$.

Let D and D' be two reduced effective divisors on $\mathscr{X}$ with no irreducible component in common. Recall that D and D' meet at a point P with normal crossings if the local intersection number $(D \cdot D')_P$ is equal to 1. In particular, P is a smooth point on both D and D'. We say that two effective divisors meet with normal crossings if they meet with normal crossings at each intersection point. We say that $\mathscr{X}_k$ has normal crossings of every singular point of $\mathscr{X}_k^{\mathrm{red}}$ is an ordinary double point.

Given any integer m prime to p, let us denote by F_m/K the unique Galois extension of K of degree m. We shall call F/K an ℓ-extension of K if $[F : K]$ is a power of ℓ.

The following facts are well known; we state them without proof (see for instance [BPV], Theorem 5.2, when $\mathscr{X}/\mathbb{C}$ is a surface).

Facts 5.2. Let q be a prime, $q \neq p$. Let $F := F_q$. Let (C, r) be a component of $\mathscr{X}_k$.

- The map $b : \mathscr{Y} \to \mathscr{X}$ is ramified only over the divisor $R := \sum_{\gcd(q, r_i) = 1} C_i$.

- If $q \nmid r$, then $b^{-1}(C) =: D$ is irreducible and the restricted map $b_{|D} : D \to C$ is an isomorphism. The curve D has multiplicity r in $\mathscr{Y}_k$.

- Let $P \in \mathscr{Y}$ be a point such that $b(P) \in R$. If $b(P)$ is a smooth point of $\mathscr{X}_k^{\mathrm{red}}$, then P is regular on $\mathscr{Y}$.

- Let $P \in \mathscr{Y}$ be such that $b(P)$ is the intersection point of exactly two components C and C' of R. If C and C' meet with normal crossings at $b(P)$, then the divisor $\rho^{-1}(P) := \sum_{i=1}^{m(P)} E_i$ is a Hirzebruch-Young string. Let $P \in D \cap D'$, where D and D' are the preimages of C and C' in $\mathscr{Y}_k$. Write $\tilde{D}$ for the strict transform of D in $\mathscr{Z}$. Then:

$$\big(\rho^{-1}(P) \cdot \tilde{D}\big) = (E_1 \cdot \tilde{D}) = 1 = (E_{m(P)} \cdot \tilde{D}') = \big(\rho^{-1}(P) \cdot \tilde{D}'\big).$$

Moreover, $\big(\rho^{-1}(P) \cdot E\big) = 0$ if E is an irreducible component of $\mathscr{Z}_k$ with $E \neq \tilde{D}, \tilde{D}'$.

- If $q | r$ and $C \cap R \neq \emptyset$, then the restricted map $b_{|b^{-1}(C)} : b^{-1}(C) \to C$ is a morphism of degree q ramified over $|C \cap R|$ points of C. If C intersects R with normal crossings in at least one point, then $D := b^{-1}(C)$ is irreducible and the curve D has multiplicity r/q in $\mathscr{Y}_k$. When C is smooth and meets R with normal crossings, then D is smooth and its genus is computed using the Riemann-Hurwitz formula.

- If $q|r$ and $C \cap R = \emptyset$, then $b : b^{-1}(C) \to C$ is an etale map and each irreducible component of $b^{-1}(C)$ has multiplicity r/q in $\mathscr{Y}_k$. If $b^{-1}(C)$ is not irreducible, then it is equal to the disjoint union $D_1 \sqcup \cdots \sqcup D_q$ of q irreducible curves, and each restricted map $b_{|D_j} : D_j \to C$ is an isomorphism.

- If $\mathscr{X}_k$ has smooth components and normal crossings, then $\mathscr{Z}_k$ has smooth components and normal crossings.

Lemma 5.3. *Let X/K be a curve with a regular model $\mathscr{X}/\mathcal{O}_K$ and associated arithmetical graph (G, M, R). Let $\ell \neq p$ be a prime. Let (C, C') be a weakly connected pair with $\ell \nmid rr'$. Let $F := F_\ell$ and consider the associated map $b \circ \rho : \mathscr{Z} \to \mathscr{X}$. Since $\ell \nmid rr'$, the preimages of C and C' in $\mathscr{Y}$ are irreducible, and we also denote them by C and C'. We will also denote by C and C' the strict transforms of C and C' in $\mathscr{Z}$.*

(a) *Assume that the pair (C, C') is ℓ-breakable in G. Then (C, C') is also weakly connected and ℓ-breakable in the graph of $\mathscr{Z}$.*

(b) *Assume that the pair (C, C') is not ℓ-breakable in G. Then (C, C') is a multiply connected pair in the graph of $\mathscr{Z}$.*

(c) *Let (G', M', R') denote the arithmetical graph associated with the regular model $\mathscr{Z}$. Let τ_K and τ_F denote respectively the elements of Φ_K and Φ_F corresponding to the images of $E(C, C')$ in $\mathrm{Ker}({}^tR)/\mu(\mathscr{L})$ and $\mathrm{Ker}({}^tR')/\mu'(\mathscr{L}')$. Then τ_F is the image of τ_K under the natural map $\gamma : \Phi_K \to \Phi_F$.*

Proof. Note that, by definition of weakly connected, two curves of the path $\mathscr{P}$ between C and C' that intersect do intersect with normal crossings. (Note on the other hand that our hypothesis allows other singularities on each component.) The weakly connected pair (C, C') is ℓ-breakable if and only if no two consecutive vertices on the path $\mathscr{P}$ have multiplicity divisible by ℓ. The first two parts of the lemma follow immediately from 5.2.

To prove (c), let us pick a point P of degree r and a point Q of degree r' on the scheme X such that the closures $\bar{P}$ and $\bar{Q}$ of P and Q in $\mathscr{X}$ each intersect $\mathscr{X}_k$ in a single point, on C and C', respectively. Let $c := \gcd(r, r')$. Then $(r'P - rQ)/c$ belongs to $\mathrm{Pic}^0(X)$ and its image in Φ_K is identified with the image of $E(C, C')$ in $\mathrm{Ker}({}^tR)/\mu(\mathscr{L})$. Since F/K is Galois of degree coprime to r and r', the points P and Q in X define two points P' and Q' in X_F, also of degree r and r', respectively. Thus $(r'P' - rQ')/c$ belongs to $\mathrm{Pic}^0(X_F)$ and its image in Φ_F is identified with the image of $E(C, C')$ in $\mathrm{Ker}({}^tR')/\mu'(\mathscr{L}')$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}^0(X_F) \\ \downarrow & & \downarrow \\ \Phi_K & \longrightarrow & \Phi_F, \end{array}$$

where the top horizontal map is the natural map β induced by $X_F \to X$, and the bottom horizontal map is the natural map γ. Then this diagram is commutative. It is easy to check that $\beta(r'P - rQ) = r'P' - rQ'$, thus proving (c).

Given any curve X/K (resp., any abelian variety A/K), we let L/K denote the extension minimal with the property that X_L/L (resp., A_L/L) has semistable reduction.

Lemma 5.4. *Let A/K be an abelian variety. Let $\tau \in \Phi_K$. Then $\tau \in \Psi_{K,L}$ if and only if there exists a finite extension F/K such that $\tau \in \Psi_{K,F}$.*

Proof. It is clear that $\Psi_{K,F} \subseteqq \Psi_{K,FL}$. It follows from the fact that A_L/L has semistable reduction that the canonical map $\Phi_L \to \Phi_{FL}$ is injective. Hence, $\Psi_{K,F} \subseteqq \Psi_{K,L}$.

Theorem 5.5. *Let X/K be a curve with a regular model $\mathscr{X}/\mathcal{O}_K$ and associated arithmetical graph (G, M, R). Let $\ell \neq p$ be a prime. Let (C, C') be a weakly connected ℓ-breakable pair with $\ell \nmid rr'$. Then the ℓ-part of the image of $E(C, C')$ in $\Phi_K(\mathrm{Jac}(X))$ belongs to $\Psi_{K,L}$ and has order $\lambda(C, C')$.*

Proof. Let $\mathscr{P}$ denote the path linking C and C'. Let F/K be any ℓ-extension. Let $b \circ \rho : \mathscr{Z} \to \mathscr{X}$ be the associated base change and desingularization map as in 5.1. Denote again by C and C' the preimages in $\mathscr{Y}$ of the components C and C' in $\mathscr{X}$ as well as their strict transform in $\mathscr{Z}$. It follows from 5.3 that the pair (C, C') is also weakly connected and ℓ-breakable in $\mathscr{Z}$. Thus, the order of the ℓ-part of the image of $E(C, C')$ can be computed using Theorem 4.4.

Let us now consider the nodes on the path $\mathscr{P}'$ linking C and C' in $\mathscr{Z}$. If D is a vertex of $\mathscr{P}'$ such that $(b \circ \rho)(D)$ is a node of $\mathscr{P}$, then D is a node on $\mathscr{P}'$. If D is a node on $\mathscr{P}'$ such that $(b \circ \rho)(D)$ is not a node of $\mathscr{P}$, then the component $(b \circ \rho)(D)$ is not smooth. The reader will note that after an extension of degree ℓ^d, the multiplicity of the preimage in $\mathscr{Y}$ of a component (C, r) on the path $\mathscr{P}$ is equal to $r\ell^{-\min(d, \mathrm{ord}_\ell(r))}$. Define μ to be the power of ℓ such that

$$\mathrm{ord}_\ell(\mu) := \max\{\mathrm{ord}_\ell(r), (C, r) \text{ a component on } \mathscr{P}\}.$$

It follows that over F_μ, all the nodes of the path $\mathscr{P}'$ linking C and C' in $\mathscr{Z}$ have multiplicity prime to ℓ. Thus, Theorem 4.4 shows that the image of $E(C, C')$ has trivial ℓ-part in Φ_{F_μ}. Therefore, the ℓ-part of the image of $E(C, C')$ in Φ_K belongs to Ψ_{K,F_μ}. Thus, Lemma 5.4 implies that the ℓ-part of the image of $E(C, C')$ in Φ_K belongs to $\Psi_{K,L}$. Note that it is not always true that $F_\mu \subseteqq L$. This concludes the proof of Theorem 5.5.

5.6. Let us recall now the description of the first functorial subgroup of $\Phi_{K,\ell}$ appearing in the filtration

$$\Theta_{K,\ell}^{[3]} \subseteqq \Psi_{K,L,\ell} \subseteqq \Theta_{K,\ell} \subseteqq \Phi_{K,\ell}$$

introduced in [Lor3], 3.21. Let A/K be an abelian variety. Let T_ℓ denote the Tate module $T_\ell A$, $\ell \neq p$. Let $\mathbb{D}_\ell := \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Let $I_K := I(\bar{K}/K)$. There is a natural isomorphism

$$\phi_{K,\ell} : \Phi_{K,\ell} \to E := \frac{(T_\ell \otimes \mathbb{D}_\ell)^{I_K}}{T_\ell^{I_K} \otimes \mathbb{D}_\ell}.$$

Given any submodule X of T_ℓ, let $f_X : X \otimes \mathbb{Q}_\ell \to T_\ell \otimes \mathbb{D}_\ell$ denote the natural map. We denote by $t(X)$ the subgroup of E generated by the elements $x \in (T_\ell \otimes \mathbb{D}_\ell)^{I_K}$ such that there exists $\tilde{x} \in X \otimes \mathbb{Q}_\ell$ with $f_X(\tilde{x}) = x$. Consider the submodules $W_{\ell,L} \subseteqq T_\ell^{I_L} \subseteqq T_\ell$, where

$W_{\ell,L}$ is canonically isomorphic to the Tate module of the maximal torus $\mathscr{T}_L$ in the connected component of the Néron model of A_L/L. Then, by definition ([Lor3], 3.8),

$$\phi_{K,\ell}(\Theta^{[3]}_{K,\ell}) = t(W_{\ell,L}) \quad \text{and} \quad \phi_{K,\ell}(\Psi_{K,L,\ell}) = t(T_\ell^{I_L}).$$

As we shall see in 5.12, the description of the elements of $\Theta^{[3]}_{K,\ell}$ seems to be more complicated than the description of the elements of $\Psi_{K,L,\ell}$.

Let F/K be any finite separable extension. Denote by $\mathscr{A}_F/\mathscr{O}_F$ the Néron model of A_F/F. Let $\mathscr{A}_{F,k}/k$ denote its special fiber, with connected component $\mathscr{A}^0_{F,k}$. Let $\mathscr{T}_F \subset \mathscr{A}^0_{F,k}$ denote the maximal torus of $\mathscr{A}^0_{F,k}$. Denote by $\pi_F : A(F) \to \mathscr{A}_{F,k}(k)$ the reduction map.

Lemma 5.7. *Let A/K be an abelian variety with purely additive reduction. Let $\ell \neq p$ be any prime. Let $\tau \in \Phi_{K,\ell}$. Let t denote the unique element of $A(K)_{\text{tors},\ell}$ whose image in Φ_K is τ. The element τ belongs to the subgroup $\Theta^{[3]}_{K,\ell}$ if and only if there exists a finite separable extension F/K such that $\pi_F(t)$ belongs to $\mathscr{T}_F$.*

Proof. When A has purely additive reduction, $T_\ell^{I_K} = (0)$ and the canonical reduction map $A(K)_{\text{tors},\ell} \to \Phi_{K,\ell}$ is an isomorphism. Consider the map

$$g : A(K)_{\text{tors},\ell} \to (T_\ell \otimes \mathbb{D}_\ell)^{I_K}$$

defined as follows. If $x \in A(K)_{\text{tors},\ell}$, pick $\{x_i\}_{i=1}^\infty \in T_\ell$ such that $x = x_j$ for some $j \in \mathbb{N}$. Then set $g(x) :=$ class of $(\{x_i\}_{i=1}^\infty \otimes \ell^{-j})$. That the map g is well defined and an isomorphism is proved in [Lor3], 3.4. When A has purely additive reduction, the canonical reduction map $A(K)_{\text{tors},\ell} \to \Phi_{K,\ell}$ factors through $(T_\ell \otimes \mathbb{D}_\ell)^{I_K}$ as follows:

$$A(K)_{\text{tors},\ell} \xrightarrow{g} (T_\ell \otimes \mathbb{D}_\ell)^{I_K} \xrightarrow{\phi^{-1}_{K,\ell}} \Phi_{K,\ell}.$$

Let $\tau \in \Phi_{K,\ell}$. Let $\tilde{\tau} := \{t_i\}_{i=1}^\infty \otimes \ell^{-r} \in T_\ell \otimes \mathbb{Q}_\ell$ be such that its image in $(T_\ell \otimes \mathbb{D}_\ell)^{I_K}$ is $\phi^{-1}_{K,\ell}(\tau)$. Then $t_r \in A(K)$ is the preimage of τ under the reduction map. Thus, if $\tau \in \Theta^{[3]}_{K,\ell}$, then by hypothesis we may choose $\tilde{\tau}$ in $W_{\ell,L} \otimes \mathbb{Q}_\ell$, so that $\pi_L(t_r) \in \mathscr{T}_L$.

Let us now assume that there exists a finite separable extension F/K such that $\pi_F(t)$ belongs to $\mathscr{T}_F$. Then $\pi_L(t)$ belongs to $\mathscr{T}_L$. Indeed, it follows from the properties of smooth connected commutative groups that the natural map $\mathscr{A}_F \to \mathscr{A}_{FL}$ restricts to a map $\mathscr{T}_F \to \mathscr{T}_{FL}$. Thus $\pi_{FL}(t)$ belongs to $\mathscr{T}_{FL}$. In particular, the image of τ under the natural map $\Phi_K \to \Phi_{FL}$ is trivial. Since the map $\Phi_L \to \Phi_{FL}$ is injective, we conclude that $\pi_L(t) \in \mathscr{A}^0_{L,k}(k)$. Since $\mathscr{A}^0_{L,k} = \mathscr{A}^0_{FL,k}$ by semistability, we find that $\pi_L(t) \in \mathscr{T}_L$.

Choose now $y := \{y_i\}_{i=1}^\infty \in W_{\ell,L}$ such that $t = y_r$ for some r. Then the image of $y \otimes \ell^{-r}$ in $T_\ell \otimes \mathbb{D}_\ell$ belongs to $(T_\ell \otimes \mathbb{D}_\ell)^{I_K}$. Thus $\phi^{-1}_{K,\ell}(\tau)$ is in the image of $W_{\ell,L} \otimes \mathbb{Q}_\ell$, and τ belongs to $\Theta^{[3]}_{K,\ell}$.

5.8. Our next theorem describes an element $\tau \in \Phi(G)$ whose ℓ-part belongs to the subgroup $\Theta^{[3]}_{K,\ell}$. To describe this element, we need to introduce the following notation. Let (G, M, R) be any arithmetical graph. Let (D, r) be a node of G. Let (D_i, r_i), $i = 1, \dots, d$, denote the vertices of G linked to D. Assume that $(D_i \cdot D) = 1$ for all $i = 1, \dots, d$, and that the numbering of the vertices D_i is such that for $i = 1, \dots, s$, the vertex D_i belongs to a terminal chain T_i attached at D, and for $i = s+1, \ldots, d$, the vertex D_i is not on a terminal chain at D. We assume that $s \geqq 2$. For simplicity, we will assume that $\gcd(r, r_i) = 1$, for all $i = 1, \ldots, s$. Thus the terminal vertex C_i on T_i has multiplicity 1. Let τ_i denote the image of $E(C_i, C_s)$ in $\Phi(G)$, $i = 1, \ldots, s-1$. Let

$$\tau := \sum_{i=1}^{s-1} r_i \tau_i.$$

To motivate this definition of τ, let us first note the following.

Lemma 5.9. *Let ℓ be any prime. If $\operatorname{ord}_\ell(r) \leqq \operatorname{ord}_\ell\left(\sum_{i=1}^{s} r_i\right)$, then $\langle \tau; \tau_i \rangle = 0$ in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, for all $i = 1, \ldots, s-1$.*

Proof. If C is any vertex of G, let $r(C)$ denote its multiplicity. Then Lemma 2.8 shows the existence of integers b_i, $i = 1, \ldots, s$, such that

$$\sum_{\substack{C, C' \in T_i \\ (C \cdot C') = 1}} \frac{1}{r(C) r(C')} = \frac{b_i}{r \gcd(r, r_i)}.$$

Proposition 3.2 shows that

$$\langle \tau_i; \tau_j \rangle = \begin{cases} \dfrac{b_s}{r} & \text{if } i \neq j, \\[2ex] \dfrac{b_i}{r} + \dfrac{b_s}{r} & \text{if } i = j. \end{cases}$$

Thus, for $k = 1, \ldots, s-1$, we find that

$$\begin{aligned} \langle \tau; \tau_k \rangle = \sum_{i=1}^{s-1} r_i \langle \tau_i; \tau_k \rangle &= \left(\sum_{i=1}^{s-1} r_i \right) b_s/r + b_k r_k / r \\ &= \left(\sum_{i=1}^{s} r_i \right) b_s/r - b_s r_s/r + b_k r_k/r. \end{aligned}$$

Lemma 2.6 shows that $b_s r_s \equiv 1 \equiv b_k r_k \bmod r$, and our hypothesis is that

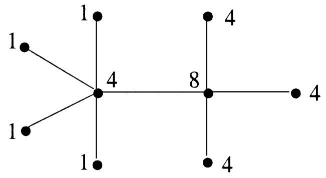
$$\operatorname{ord}_\ell(r) \leqq \operatorname{ord}_\ell\left(\sum_{i=1}^{s} r_i \right).$$

Hence, $\langle \tau; \tau_k \rangle = 0$ in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$.

5.10. Assume that (G, M, R) is associated to a curve X/K whose jacobian has purely additive reduction. We have established in [Lor3], 3.13, that $\Theta_{K,\ell}^{[3]} = \Psi_{K,L,\ell} \cap \Psi_{K,L,\ell}^{\perp}$, where the orthogonal subgroup is computed with respect to the pairing 3.12 in [Lor3]. While no relationship between the pairing 3.12 and the pairing $\langle \, ; \rangle$ described in section 3 is fully established as of yet, one may certainly anticipate a relationship and, thus, we may expect that an element τ of $\Psi_{K,L,\ell}$ that is orthogonal to $\Psi_{K,L,\ell}$ under the pairing $\langle \, ; \rangle$

belongs to $\Theta^{[3]}_{K,\ell}$. Theorem 5.5 shows that the ℓ-parts of τ_i, $i = 1, \dots, s-1$ and, thus, the ℓ-part of τ, belong to $\Psi_{K,L}$. Lemma 5.9 and Theorem 5.5 show that the ℓ-part of τ is orthogonal to any element of $\Psi_{K,L}$ image of $E(C, C')$ with $\ell \nmid rr'$. Thus the ℓ-part of τ is a 'good candidate' to be an element of $\Theta^{[3]}_{K,\ell}$, and Theorem 5.12 below describes some instances where the ℓ-part of τ belongs to $\Theta^{[3]}_{K,\ell}$.

5.11. Note that if $s > 2$, then Theorem 4.4 shows that $\tau_i \neq 0$ for all $i = 1, \dots, s-1$. But it may happen that τ is trivial in $\Phi(G)$, in which case the ℓ-part of τ certainly belongs to $\Theta^{[3]}_{K,\ell}$, as in the following example (with D being the node of multiplicity 4, and $\ell = 2$).



On the other hand, if G contains a vertex C with $\gcd(r, r(C)) = 1$ and $C \notin T_i$, for all $i = 1, \dots, s$, then τ has order r in $\Phi(G)$. Indeed, each τ_i has order r, thus the order of τ divides r. Let τ_C denote the image of $E(C_1, C)$ in $\Phi(G)$. Then $\langle \tau; \tau_C \rangle = b_1 r_1 r_C / r$. Thus, r divides the order of τ (and of τ_C).

Theorem 5.12. *Let X/K be a curve with a regular model $\mathscr{X}/\mathcal{O}_K$ and associated arithmetical graph (G, M, R). Assume that the jacobian A/K of X/K has purely additive reduction over $\mathcal{O}_K$ and that the graph G contains a node (D, r) as in* 5.8. *Assume that D and all components on the terminal chains attached to D are smooth* (*rational*) *curves. Let $\ell \neq p$ be prime. Suppose that $r = \ell^{\operatorname{ord}_\ell(r)}$ and that* $\operatorname{ord}_\ell(r_i) \geqq \operatorname{ord}_\ell(r)$ *for all $i = s+1, \dots, d$. Then τ belongs to $\Theta^{[3]}_{K,\ell}$.*

Proof. Let $t_i \in A(K)_{\mathrm{tors},\ell}$ denote the unique torsion point in $A(K)$ whose image in $\Phi_{K,\ell}$ is equal to τ_i. Let $P_i \in X(K)$ be such that $C_{P_i} = C_i$. Then $\pi_K(P_i - P_s) = \tau_i$ in $\Phi_{K,\ell}$. By hypothesis, the special fiber $\mathscr{A}_{K,k}$ is an extension of Φ_K by a unipotent group. Let U_i denote the connected component of $\mathscr{A}_{K,k}$ such that $\pi_K(P_i - P_s) \in U_i$. Consider the natural map

$$\gamma_{K,F} : \mathscr{A}_K \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_F) \to \mathscr{A}_F.$$

If $\mathscr{A}^0_{F,k}$ does not contain any unipotent group, the image of U_i under $\gamma_{K,F}$ is a single point of $\mathscr{A}^0_{F,k}$. It follows that $\pi_F(P_i - P_s) = \pi_F(t_i)$ in $\mathscr{A}_{F,k}$. To prove Theorem 5.12, it is sufficient to exhibit an extension F/K such that $\pi_F\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right) \in \mathscr{T}_F$. Indeed, if $\pi_F\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right) \in \mathscr{T}_F$, then $\pi_{FL}\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right) \in \mathscr{T}_{FL}$. Since A_{FL}/FL has semistable reduction, we find that $\pi_{FL}\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right) = \pi_{FL}\left(\sum_{i=1}^{s-1} r_i t_i\right)$. Hence, it follows from Lemma 5.7 that $\tau := \sum_{i=1}^{s-1} r_i \tau_i$ belongs to $\Theta^{[3]}_{K,\ell}$.

Let $F := F_r$. Consider the model $\mathscr{Y}/\mathcal{O}_F$ of X_F/F associated as in 5.1 to F/K and the model $\mathscr{X}/\mathcal{O}_K$. Let $E/k \subset \mathscr{Y}_k$ denote the strict transform of D in $\mathscr{Y}$. It follows from 5.2 and our hypotheses on D that if $P \in E$, then P is a regular point on $\mathscr{Y}$. In particular, E is a smooth curve. Let $b_{|_E} : E \to D$ be the map obtained by restriction from $\mathscr{Y} \to \mathscr{X}$. Let $k(D)$ be the function field of D. Choose a coordinate function x in $k(D)$ so that when D is identified with $\mathbb{A}^1/k \sqcup \{\infty\}$, then $\infty \notin D \cap D_i$, for all $i = 1, \dots, s$. Let $a_i \in \mathbb{A}^1(k)$ denote the point $D \cap D_i$, $i = 1, \dots, s$. The map $b_{|_E} : E \to D$ is a cyclic Galois cover of degree r ramified only above the points a_i, $i = 1, \dots, s$ (we use here that $\mathrm{ord}_\ell(r_i) \geqq \mathrm{ord}_\ell(r)$ for all $i = s+1, \dots, d$). Thus

$$k(E) \cong k(x)[y] \Big/ \left(y^r - \prod_{i=1}^{s} (x - a_i)^{q_i} \right)$$

for some positive integers $q_1, \dots, q_s$ such that $r \mid \sum_{i=1}^{s} q_i$. Moreover, $b_{|_E}$ is totally ramified above the points a_i, $i = 1, \dots, s$, so that we may assume that $\gcd(r, q_i) = 1$ for all $i = 1, \dots, s$.

Proposition 5.13. *We may choose $q_i = r_i$, for all $i = 1, \dots, s$.*

Proof. Let ξ denote a primitive r-th root of unity, and let σ denote the automorphism of E which sends y to ξy and x to x. Let e_i denote the point of E totally ramified above the point a_i of D. Write $1 = \alpha_i r + \beta_i q_i$. Then $v_i := y^{\beta_i}(x - a_i)^{\alpha_i}$ is a local uniformizer at e_i with the property that $\sigma(v_i) = \xi^{\beta_i} v_i$. In fact, given any local uniformizer v_i at e_i with the property that $\sigma(v_i) = \xi^{n_i} v_i$ for some integer n_i, we find that $n_i \equiv \beta_i$ modulo r. In other words, n_i is the inverse of q_i modulo r. Indeed, if there exists a unit $u \in \mathcal{O}_{E,e_i}$ such that $\sigma(v_i) = \xi^{n_i} v_i$ and $\sigma(u v_i) = \xi^{m_i} u v_i$, then u^q is a unit in $\mathcal{O}_{D,a_i}$, with $q := r/\gcd(m_i - n_i, r)$. Hence, the extension $k(D)(u)/k(D)$ is unramified at a_i. Thus, $m_i - n_i \equiv 0$ modulo r since $\mathcal{O}_{E,e_i}/\mathcal{O}_{D,a_i}$ is totally ramified.

Let us consider the map $c : \mathscr{X} \to \mathscr{X}'$ over $\mathcal{O}_K$, which contracts all components of $\mathscr{X}_k$ that belong to a terminal chain attached to D. Thus $\mathscr{X}'$ is a normal model of X having exactly s singular points $Q_1, \dots, Q_s$ on the image of D. Let $F := F_r$ and consider the base change maps $\mathscr{Y} \to \mathscr{X}$ and $\mathscr{Y}' \to \mathscr{X}'$ associated to F/K, as well as the minimal desingularization maps $\mathscr{Z} \to \mathscr{Y}$ and $\mathscr{Z}' \to \mathscr{Y}'$. The map c induces a map $c_{\mathscr{Y}} : \mathscr{Y} \to \mathscr{Y}'$. By construction, the multiplicity of E in $\mathscr{Y}$ and $\mathscr{Z}$ is equal to 1. Thus all components C of $\mathscr{Z}$ whose images in $\mathscr{X}$ belong to a terminal chain attached to D can be contracted by a map $\mathscr{Z} \to \mathscr{Z}''$ over $\mathcal{O}_F$ in such a way that the image of C in $\mathscr{Z}''$ is a regular point of $\mathscr{Z}''$ (we use here the fact that all components of the terminal chains attached at D are smooth). In particular, $\mathscr{Z}''$ is regular, and by minimality of the resolution of singularities $\mathscr{Z}' \to \mathscr{Y}'$, we find that we have a map $\mathscr{Z}'' \to \mathscr{Z}'$. Thus, every point of $\mathscr{Y}'$ in the image of E is a regular point of $\mathscr{Y}'$.

Let us consider the action of the group $\mathrm{Gal}(F/K)$ on the scheme $\mathscr{Y}'$. The quotient of this action is the scheme $\mathscr{X}'$. Let R_i denote the preimage of Q_i in $\mathscr{Y}'$. As we mentioned above, R_i is a regular point on $\mathscr{Y}'$. We may thus use the known results on quotient singularities to describe the resolution of singularities at Q_i. Namely, let t_F denote a uniformizer of $\mathcal{O}_F$. Then the completed local ring at R_i is of the form $\mathcal{O}_F[[z]]$, and z can be chosen such that the action of G on $\mathcal{O}_{\mathbf{F}}[[z]]$ is linear: if σ is a generator of G, then there exists a root of unity ξ such that $\sigma(t_F) = \xi t_F$ and $\sigma(z) = \xi^{b_i} z$ for some $b_i \in \mathbb{N}$. (We use here the fact that the extension F_r/K is tame.) Then the resolution of singularities at Q_i is completely determined by the integer b_i. It follows (see for instance [Vie], 6.6) that in order to have a resolution of singularities of the type $\mathscr{X} \to \mathscr{X}'$, the integer b_i must be congruent to the inverse of r_i modulo r. Hence, we find that q_i is congruent to r_i modulo r.

Lemma 5.14. *Let $s \geqq 2$. Let E/k denote the nonsingular complete model of the plane curve given by the equation*

$$y^r - \prod_{i=1}^{s}(x - a_i)^{r_i} = 0,$$

with $a_i \in k$, $i = 1, \dots, s$ and $\prod_{i \neq j}(a_i - a_j) \neq 0$. Assume that $\gcd(r, r_i) = 1$ for all $i = 1, \dots, s$, and $r \mid \sum_{i=1}^{s} r_i$. Let e_i denote the point of E corresponding to the point $(a_i, 0)$. Then the divisor $e_i - e_s$ has order dividing r in $\mathrm{Jac}(E)$, and $\sum_{i=1}^{s-1} r_i(e_i - e_s) = 0$ in $\mathrm{Jac}(E)$.

Proof. The function $(x - a_i)/(x - a_s)$ belongs to the function field of E, and

$$\mathrm{div}\big((x - a_i)/(x - a_s)\big) = r(e_i - e_s).$$

Moreover, let $d := \sum_{i=1}^{s} r_i/r$. Then $\big(y/(x - a_s)^d\big)^r = \prod_{i=1}^{s-1}\left(\dfrac{x - a_i}{x - a_s}\right)^{r_i}$. Thus,

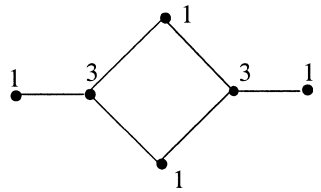
$$\sum_{i=1}^{s-1} r_i(e_i - e_s) = \mathrm{div}\big(y/(x - a_s)^d\big).$$

Let us now conclude the proof of Theorem 5.12. As mentioned at the beginning of the proof, it is sufficient to show that $\pi_{FL}\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right) \in \mathscr{T}_{FL}$. Since the model $\mathscr{Z}''$ is regular, we can use it to describe the special fiber $\mathscr{A}_{F,k}$. The group Φ_F can be computed using $\mathscr{Z}_k''$, and since the points P_i, $i = 1, \dots, s$, all reduce to points in $E \subseteqq \mathscr{Z}_k''$, we find that the image of $\pi_F\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right)$ in Φ_F is trivial, and thus $\pi_F\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right)$ belongs to $\mathscr{A}_{F,k}^0$. The group scheme $\mathscr{A}_{F,k}^0$ is isomorphic to $\mathrm{Pic}^0(\mathscr{Z}_k''/k)$, and $\mathrm{Pic}^0(\mathscr{Z}_k''/k)$ is an extension of the abelian variety $B_F := \prod_{C \subseteqq \mathscr{Z}_k''} \mathrm{Jac}(C)$ by the product of a unipotent group U_F and a torus $\mathscr{T}_F$. Lemma 5.14 implies that the image of $\pi_F\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right)$ in B_F is trivial. Thus, the image of $\pi_{FL}\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right)$ in B_{FL} is also trivial. Since A_{FL}/FL has semi-stable reduction, we find that $\pi_{FL}\left(\sum_{i=1}^{s-1} r_i(P_i - P_s)\right)$ belongs to $\mathscr{T}_{FL}$.

6. Partial converses for Theorem 5.5

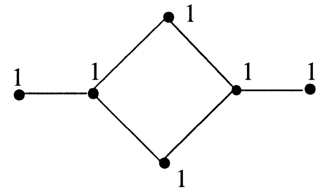
Let X/K be a curve. Let $\mathscr{X}/\mathcal{O}_K$ be a regular model of X/K. Let (C, r) and (C', r') be two distinct components of $\mathscr{X}_k$. Let $\ell \neq p$ be a prime. In view of Theorem 5.5, it is natural to wonder whether it is true that if the ℓ-part of $E(C, C')$ belongs to $\Psi_{K,L}$, then the pair (C, C') is weakly connected and ℓ-breakable. As we shall see in the following example, this question has a negative answer in general.

Example 6.1. Consider the following arithmetical graph (G, M, R):



Using a row and column reduction of M, it is easy to show that $\Phi(G)$ is cyclic of order 12. Let C and C' denote the two vertices of multiplicity 1 in G which are not terminal vertices. Let τ denote the image of $E(C, C')$ in Φ. It is easy to check that $ME(C, C') = -6E(C, C')$ and, thus, τ has order dividing 6. Since $E(C, C') + mR$, $m \in \mathbb{Z}$, is never divisible by 2 or 3, we find that τ has order 6 in Φ.

Winters' Existence Theorem [Win] implies the existence of a field of equicharacteristic zero, say K, and the existence of a curve X/K with a regular model $\mathcal{X}$ whose associated graph is (G, M, R), and such that all components of $\mathcal{X}_k$ are smooth and rational. Consider the base extension F_3/K and the associated model $\mathcal{Z}/\mathcal{O}_{F_3}$. Let (G_3, M_3, R_3) be the graph associated with $\mathcal{Z}$. Then G_3 has the following form:



The two nodes in G_3 correspond to components of genus 1. All other components are rational. Denote again by C and C' the two vertices of multiplicity 1 and degree 2 in G_3. Let τ' denote the image of $E(C, C')$ in $\Phi(G_3)$. Then τ' is the image of τ under the natural map $\Phi(G) \to \Phi(G_3)$. It is easy to check that τ' has order 2. Thus, the 3-part of τ is not trivial and belongs to $\Psi_{K,L}$, even though the pair (C, C') is multiply connected. Let us then ask the following less general question:

Question 6.2. Let $\ell \neq p$. Let (C, C') be a weakly connected pair such that $\ell \nmid rr'$. If (C, C') is not ℓ-breakable, is it true that the ℓ-part of $E(C, C')$ does not belong to $\Psi_{K,L}$?

If $\ell = p$, Question 6.2 has a negative answer, as can be seen on the following example with $p = 2$. Consider an elliptic curve X/K with reduction I_v^*, $v > 1$, and with potentially good reduction. Then the graph of the reduction of X contains pairs that are not p-breakable. On the other hand, since X has potentially good reduction, $\Psi_{K,L} = \Phi_K$.

Clearly, if Question 6.2 has a positive answer, then the ℓ-part of $E(C, C')$ is not trivial in Φ_K. This fact is proved in 3.3. As evidence that Question 6.2 may possibly have a positive answer, we offer the following two theorems.

Theorem 6.3. *Let $\ell \neq p$. Let $\mathcal{X}/\mathcal{O}_K$ be a regular model of a curve X/K, with associated arithmetical graph (G, M, R). Assume that L/K is tame. Let (C, C') be a weakly connected pair with $r = r' = 1$. If (C, C') is not ℓ-breakable in G, then the image τ of $E(C, C')$ does not belong to $\Psi_{K,L}$.*

Proof. Consider the scheme $\mathscr{X}_P$ obtained by blowing-up a single closed point P on $\mathscr{X}_k$. Let C and C' denote again the strict transforms in $\mathscr{X}_P$ of C and C'. It is easy to check that if (C, C') is a weakly connected pair in the graph of $\mathscr{X}$, then (C, C') is also a weakly connected pair in the graph of $\mathscr{X}_P$. Moreover, $E(C, C')$ for $\mathscr{X}$ and $E(C, C')$ for $\mathscr{Z}$ define the same elements in the group of components of the jacobian of X. We may thus assume, without loss of generality, that $\mathscr{X}_k$ has smooth components with normal crossings.

Consider the base change F_ℓ/K and the associated map $\rho \circ b : \mathscr{Z} \to \mathscr{X}$ introduced in 5.1. Since $\mathscr{X}_k$ has smooth components and normal crossings, so does $\mathscr{Z}_k$. We denote again by C and C' the preimages in $\mathscr{Z}$ of C and C'. Lemma 5.3 shows that the pair (C, C') is multiply connected in the graph of the model $\mathscr{Z}$. Moreover, C and C' have multiplicity 1 in $\mathscr{Z}$. Thus, it follows from our next theorem that $\tau \notin \Psi_{K,F_\ell L}$. To conclude the proof of Theorem 6.3, we need only to note that if $\tau \notin \Psi_{K,F_\ell L}$, then $\tau \notin \Psi_{K,L}$ since the map $\Phi_L \to \Phi_{L,F_\ell L}$ is injective.

Theorem 6.4. *Let $\mathscr{X}/\mathcal{O}_K$ be a regular model of a curve X/K, with associated arithmetical graph (G, M, R). Assume that $\mathscr{X}_k$ has smooth components and normal crossings. Assume also that L/K is tame. Let (C, C') be a multiply connected pair with $r = r' = 1$. Then the image of $E(C, C')$ does not belong to $\Psi_{K,L}$.*

Proof. Consider the base change L/K and the associated map $b \circ \rho : \mathscr{Z} \to \mathscr{X}$. Since L/K is assumed to be tame, we can factor $b \circ \rho$ into a sequence of morphisms of prime degree, and apply to each of these morphisms the facts recalled in 5.2. We denote again by C and C' the preimages in $\mathscr{Y}$ of C and C' as well as their strict transforms in $\mathscr{Z}$. Since $\mathscr{X}_k$ has smooth components and normal crossings, so does $\mathscr{Z}_k$. It is easy to check that the pair (C, C') is multiply connected in $\mathscr{Z}$. If $\mathscr{Z}_k$ is reduced, then to conclude the proof of Theorem 6.4, we use Corollary 2.3 in [Lor4] (see 2.1), which states that in a reduced graph G, a pair of vertices gives the trivial element in $\Phi(G)$ if and only if the pair is weakly connected. In general, though, $\mathscr{Z}_k$ is not reduced. On the other hand, since X_L/L has semi-stable reduction, there exists a sequence of elementary blow-downs $\mathscr{Z} := \mathscr{Z}_0 \to \mathscr{Z}_1 \to \cdots \to \mathscr{Z}_s$ such that $(\mathscr{Z}_s)_k$ is reduced. (By an elementary blow-down $\mathscr{Z} \to \mathscr{Z}_1$, we mean that $\mathscr{Z}$ is the blow-up of a single closed point of $\mathscr{Z}_1$.) We may choose this sequence of elementary blow-downs in such a way that for each i, the irreducible curve contracted by the map $\mathscr{Z}_i \to \mathscr{Z}_{i+1}$ has multiplicity greater than 1 in $(\mathscr{Z}_i)_k$. Let C and C' denote again the images of C and C' in $\mathscr{Z}_s$ (these images have dimension 1 by construction). The vectors $E(C, C')$ for $\mathscr{Z}$ and $E(C, C')$ for $\mathscr{Z}_s$ define the same element in the group of components Φ_L of the jacobian of X_L. It is not hard to check that C and C' in $\mathscr{Z}_s$ form a multiply connected pair. We may thus apply 2.1 to the (reduced) graph of $\mathscr{Z}_s$ to find that the image of $E(C, C')$ is not trivial in Φ_L.

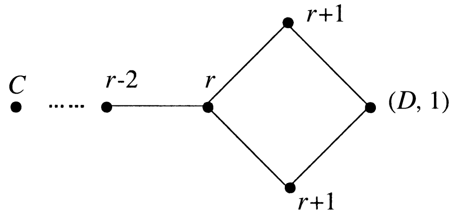
Note that Example 6.1 shows that Theorem 6.4 cannot be sharpened to state that if (C, C') is a multiply connected pair such that $r = r' = 1$ and the ℓ-part of $E(C, C')$ is not trivial in Φ_K, then the ℓ-part of $E(C, C')$ does not belong to $\Psi_{K,L}$.

Let us also note that the hypothesis that $\mathscr{X}_k$ has smooth components and normal crossings cannot be removed from the statement of Theorem 6.4. Consider the reduction of an elliptic curve consisting of three smooth rational lines intersecting in a single point. The divisor $\mathscr{X}_k$ does not have normal crossings, and in its associated graph, every pair of vertices is multiply connected. The group $\Phi(G)$ is cyclic of order 3. When $p \neq 3$, one finds that such an elliptic curve has potentially good reduction. Thus, $\Phi_K = \Psi_{K,L}$ in this case.

Corollary 6.5. *Let (G, M, R) be any arithmetical graph. If G contains a multiply connected pair (C, C') with $r = r' = 1$, then $|\Phi(G)| \neq 1$.*

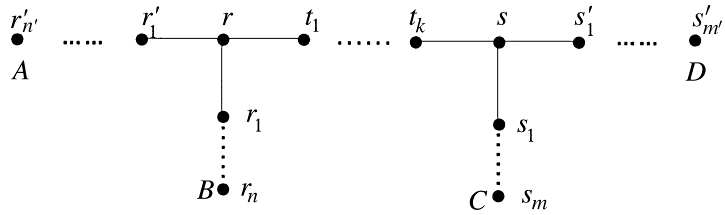
Proof. Winters' Existence Theorem [Win], implies the existence of a field F with a discrete valuation of equicharateristic 0, and a smooth and proper curve Y/F having a model $\mathscr{Y}$ over $\mathcal{O}_F$ whose associated arithmetical graph is the given graph (G, M, R) and such that $\mathscr{Y}_k$ has smooth components and normal crossings. Apply 6.4. It would be interesting to find a direct proof of 6.5 that does not rely on the theory of degenerations of curves.

Consider now the following example:



The order of $\Phi(G)$ equals $2/\gcd(r, r-2)$. Hence, when r is even, $|\Phi(G)| = 1$ even though (C, D) is multiply connected. When r is odd, $|\Phi(G)| = 2$ and $E(C, D)$ is a generator. Let us now show that Question 6.2 has a negative answer in general if $\ell | rr'$.

Example 6.6. Let a and b be two positive integers. Consider the arithmetical graph G given by:



where $\mathrm{ord}_\ell(r) = a$, $\mathrm{ord}_\ell(r_1) = \mathrm{ord}_\ell(r'_1) = 0$, $\mathrm{ord}_\ell(s) = a + b$, and $\mathrm{ord}_\ell(s_1) = \mathrm{ord}_\ell(s'_1) = 0$. The terminal chains are constructed using Euclid's algorithm as in [Lor2], 2.4. We assume that $\mathrm{ord}_\ell(t_i) \geqq a$ for all $i = 1, \ldots, k$.

Lemma 6.7. *The ℓ-part of the group $\Phi(G)$ is cyclic of order ℓ^{2a+b} and is generated by the ℓ-part of the image of $E(B, C)$.*

Proof. Proposition 9.6/6 of [BLR] shows that $|\Phi(G)| = rs/r_n r'_{n'} s_m s'_{m'}$, so its ℓ-part has order ℓ^{2a+b}. Let $\tau \in \Phi(G)$ denote the image of $E(B, C)$. Consider the pairing $\langle\ ;\ \rangle$ introduced in 3.1. To show that the ℓ-part of τ has order ℓ^{2a+b}, it is sufficient to show that the order of $\langle \tau; \tau \rangle$ in $\mathbb{Q}/\mathbb{Z}$ is divisible by ℓ^{2a+b}, and that was done in 3.10.

Let us consider now the case where $r = \ell^a$ and $s = \ell^{a+b}$. Denote by $\tau_{VV'}$ the image in Φ of $E(V, V')$, where (V, V') is any pair of vertices. Proposition 3.7 shows that the order of τ_{AB} is ℓ^a, while the order of τ_{CD} is ℓ^{a+b}. Let C_r and C_s denote the nodes of multiplicity

r and s, respectively. The pair (C_r, C_s) is weakly connected but not ℓ-breakable. We find using Remark 3.5 that $\ell^{a+b}\tau_{BC} = \tau_{C_rC_s}$. Thus, $\tau_{C_rC_s}$ has order ℓ^a and, hence, $\tau_{AB} = \tau_{C_rC_s}$. Winters' Existence Theorem [Win] implies the existence of a field, say K, with a discrete valuation of equicharacteristic 0, and a smooth and proper curve X/K having a model over $\mathcal{O}_K$ whose associated arithmetical graph is the given graph (G, M, R). Since ℓ is not equal to the residue characteristic of K, Theorem 5.5 shows that τ_{AB} belongs to $\Psi_{K,L}(\mathrm{Jac}(X))$. It follows that $\tau_{C_rC_s}$ belongs to $\Psi_{K,L}$ but (C_r, C_s) is not ℓ-breakable, answering negatively Question 6.2 when ℓ divides rr'.

Theorem 5.5 shows that τ_{AB} and τ_{CD} belong to $\Psi_{K,L}$. Hence, since $\Phi(G)$ is cyclic, the element τ_{AB} is a multiple of τ_{CD}, and is thus divisible by ℓ in $\Psi_{K,L}$. We shall see in the next section that this phenomenon cannot occur if $\mathrm{Jac}(X)$ has potentially good ℓ-reduction.

Let L/K denote the extension minimal with the property that X_L/L has semistable reduction. Let t_L and a_L denote the toric and abelian ranks of $\mathrm{Jac}(X_L)/L$, respectively. When ℓ is not the residue characteristic, one can show that $t_L = \ell^a - 1$, and $a_L = (\ell^{a+b} - \ell^a)/2$. It is shown in [Lor3], 1.7 (using the fact that Φ is cyclic), that $|\Psi_{K,L}| - 1 \leqq 2a_L + t_L$. It follows from this bound and the fact that ℓ^{a+b} divides $|\Psi_{K,L}|$ that $|\Psi_{K,L}| = \ell^{a+b}$, and $|\Psi_{K,L}| - 1 = 2a_L + t_L$. It would be very interesting to know what are the possible values of the integers t_L and a_L when ℓ is the residue characteristic of a field K and there exists a curve X/K having a model over $\mathcal{O}_K$ whose associated arithmetical graph is the given graph (G, M, R). We conjecture that in this case $t_L \leqq \ell^a - 1$.

Let us make one final remark about this example. We found that $\tau_{AB} = \tau_{C_rC_s}$. Theorem 5.12 shows that when the graph is associated to the reduction of a curve, τ_{AB} belongs to $\Theta^{[3]}_{K,\ell}$. Let (D, r) and (D', s) be two nodes on a chain $\mathcal{C}$ of a graph G. Let τ denote the image of $E(D, D')$ in Φ. Assume that ℓ divides the weight $w(\mathcal{C})$ of the chain. In view of 5.10 and of the fact that the ℓ-part of the order of $\langle \tau; \tau \rangle$ is always trivial (3.11), it is natural to wonder whether the ℓ-part of such an element τ belongs to the subgroup $\Theta^{[3]}_{K,\ell}$ when (G, M, R) is the graph associated to a regular model of a curve.

Remark 6.8. Let us use a graph G of the type introduced in Example 6.6 to exhibit an example where the ℓ-part of the group $\Phi(G)$ is not generated by the images of the ℓ-parts of the elements of the form $E(C, C')$ with $\gcd(\ell, rr') = 1$. The multiplicities of G are as follows. Let $r_n = r_1 = 1$, $r = 4$, $r'_{n'} = r'_1 = 1$, $t_1 = 8$, $t_2 = 12$, $s = 16$, $s_1 = s'_1 = 10$, $s_2 = s'_2 = 4$, and $s_3 = s'_3 = 2$. The order of $\Phi(G)$ is equal to 16. Let then $\ell = 2$. The image of $E(A, B)$ has order 2 (3.7) and (A, B) is the only pair with $2 \nmid rr'$. Note that the image τ of $E(B, C)$ has order 16, since $\langle \tau; \tau \rangle$ is easily computed to have order at least 16.

Remark 6.9. If (C, C') is any pair of vertices on a graph G, let $\tau_{CC'}$ denote the image of $E(C, C')$ in $\Phi(G)$. Let $\Theta := \Theta(G)$ denote the subgroup of $\Phi(G)$ generated by the set of all $\tau_{CC'}$, with (C, C') a weakly connected pair of G. In this paper, we have described certain elements of the functorial subgroups $\Theta^{[3]}_{K,\ell}$ $(\ell \neq p)$, and $\Psi_{K,L}$, of $\Phi(G)$, when (G, M, R) is associated with the reduction of a curve. It is natural to wonder whether the ℓ-part of the group $\Theta(G)$ is always a subgroup of the first functorial subgroup $\Theta_{K,\ell}$ in the filtration recalled in 5.6.

We will not pursue this question in this article, but we will use the graph introduced above in 6.1 to produce an example of a graph G where $\Theta(G) \subset \Psi_{K,L}$ but $\Theta(G) \neq \Psi_{K,L}$.

Indeed, let (G, M, R) be as in 6.1. In this example, the group $\Theta(G)$ is trivial, since the image of $E(C, C')$ is trivial whenever C and C' belong to the same terminal chain (3.4). On the other hand, we have shown in 6.1 that $|\Psi_{K,L}| \geqq 3$.

7. The case of potentially good ℓ-reduction

Our goal in this section is to prove the following theorem. Recall the definitions introduced in 1.4.

Lemma 7.1. *Let A/K be a principally polarized abelian variety. Let ℓ be a prime, $\ell \neq p$. Assume that A/K has potentially ℓ-good reduction. Then $\Psi_{K,L,\ell} = (\Phi_K)_\ell$.*

Proof. It is shown in [Lor3] (see 3.22, with 3.21 (ii) and 2.15 (ii)), that the kernel of the map $\Phi_{K_\ell} \to \Phi_L$ is killed by $[L : K_\ell]$. Thus $\Psi_{K_\ell,L,\ell} = (0)$. It also follows from [Lor3], using the fact that $t_{K_\ell} = 0$, that $\Psi_{K_\ell,L,\ell} = (\Phi_{K_\ell})_\ell$. Thus, since $(\Phi_{K_\ell})_\ell = (0)$, we find that $\Psi_{K,L,\ell} = (\Phi_K)_\ell$.

Theorem 7.2. *Let X/K be a curve. Let ℓ be a prime, $\ell \neq p$, and assume that* $\mathrm{Jac}(X)/K$ *has potentially good ℓ-reduction. Let $P, Q \in X(K)$ with $C_P \neq C_Q$. Then the ℓ-part τ_ℓ of the image of $P - Q$ in Φ_K belongs to $\Psi_{K,L,\ell}$. If τ_ℓ is not trivial, then it is not ℓ-divisible in Φ_K.*

Proof. Lemma 7.1 shows that τ_ℓ belongs to $\Psi_{K,L}$. Theorem 7.2 is a consequence of Theorem 7.3 below, which pertains only to arithmetical graphs. Indeed, Proposition 1.7 in [Lor2] shows that if $\mathrm{Jac}(X)/K$ has potentially good ℓ-reduction, then there exists a model $\mathscr{X}/\mathscr{O}_K$ of X/K whose associated graph G is a tree satisfying Condition C_ℓ stated in 1.5 of [Lor2].

Theorem 7.3. *Let (G, M, R) be an arithmetical tree. Let ℓ be any prime. Let (C, r) and (C', r') be two vertices of G such that $\ell \nmid rr'$. If G satisfies Condition* C_ℓ, *then the ℓ-part of $E(C, C')$ has order $\lambda(C, C')$. Moreover, if the ℓ-part of $E(C, C')$ is not trivial, then it is not ℓ-divisible.*

Proof. Since G is a tree, every pair (C, C') is weakly connected. Condition C_ℓ implies that any two vertices C and C' with $\ell \nmid rr'$ form an ℓ-breakable weakly connected pair. Thus we may use Theorem 4.4 to compute the order of $E(C, C')$. Let us now show that $E(C, C')$ is not divisible by ℓ if it is not trivial. If the path $\mathscr{P}$ connecting C to C' does not contain any node, then Theorem 4.4 shows that the ℓ-part of the order of $E(C, C')$ is trivial and, thus, in this case the statement of Theorem 7.3 does not apply. Let us now assume that $\mathscr{P}$ contains at least one node.

We claim that Theorem 7.3 holds if it holds in the special case where $\mathscr{P}$ has only one node. Indeed, if the path $\mathscr{P}$ connecting C to C' contains more than one node, use Proposition 4.3 to break the tree G into several trees $G_1, \dots, G_m$, each having a weakly connected ℓ-breakable pair of terminal vertices C_i and C_i' connected by a path having at most one node. Each tree G_j satisfies Condition C_ℓ. The construction of the graphs G_i is such that $\Phi_\ell(G) \cong \prod_{i=1}^{m} \Phi_\ell(G_j)$. Moreover, the image of the ℓ-part of $E(C, C')$ in $\Phi(G_j)$ is the ℓ-part of $E(C_i, C_i')$. Thus, the ℓ-part of $E(C, C')$ is not ℓ-divisible in $\Phi(G)$ if and only if the ℓ-part of $E(C_i, C_i')$ is not ℓ-divisible in $\Phi(G_i)$ for some i.

Consider now the case where (G, M, R) is an arithmetical tree satisfying Condition C_ℓ, with a pair of terminal vertices (C, r) and (C', r') such that $\ell \nmid rr'$, and such that the path $\mathscr{P}$ connecting C to C' in G contains a unique node (D, r_D). Let v denote the total number of nodes of G. We proceed by induction on v. Assume that (G, M, R) is an arithmetical tree with only one node (D, r). Let $(C_1, r_1), \ldots, (C_d, r_d)$ denote the vertices of G adjacent to D. The vertices (D, r) and (C_i, r_i) are on a unique terminal chain T_i, with terminal vertex of multiplicity $s_i := \gcd(r, r_i)$. We may always order the vertices C_i such that

$$\mathrm{ord}_\ell(s_1) \geqq \cdots \geqq \mathrm{ord}_\ell(s_{d-1}) = \mathrm{ord}_\ell(s_d) = 1$$

(see [Lor2], 2.7). In particular, $\ell \nmid s_{d-1}s_d$. Denote by (D_i, s_i) the terminal vertex of the chain T_i. Without loss of generality, we may assume that $C = D_d$ and $C' = D_{d-1}$. It is shown in [Lor2], 2.1, that the group $\Phi_\ell(G)$ is isomorphic to $\prod_{i=1}^{d-2} \mathbb{Z}/\ell^{\mathrm{ord}_\ell(r/s_i)}\mathbb{Z}$. It follows from Proposition 3.7 that the ℓ-part of $E(D_{d-1}, D_d)$ has order $\ell^{\mathrm{ord}_\ell(r/s_{d-2})}$ in $\Phi(G)$. Thus, the ℓ-part of $E(D_{d-1}, D_d)$ is not divisible by ℓ in $\Phi(G)$.

Consider now the case where $v > 1$ and proceed as follows. Pick an edge e of G such that one of the two components of $G \backslash \{e\}$ contains a single node B, with $B \neq D$. The component that does not contain B can be completed into a new arithmetical graph G', as in [Lor2], page 165. The graph G' satisfies Condition C_ℓ, and has $v - 1$ nodes. Thus we may apply the induction hypothesis and obtain that $E(C, C')$ is not divisible by ℓ in $\Phi(G')$. The discussion on page 165 of [Lor2] shows that $\Phi_\ell(G)$ contains $\Phi_\ell(G')$ as a direct summand. Since $E(C, C')$ is not divisible by ℓ in $\Phi(G')$, $E(C, C')$ is not divisible by ℓ in $\Phi(G)$.

Remark 7.4. The fact in Theorem 7.2 that an element of the form $E(C_P, C_Q)$ is not divisible by ℓ does not generalize to a statement pertaining to the group $\Psi_{K,L}$. Indeed, when $\Psi_{K,L} \neq \Phi_K$, Example 6.6 exhibits an element of the form $E(C_P, C_Q)$ in $\Psi_{K,L}$, namely $E(A, B)$, that is divisible by ℓ in $\Psi_{K,L}$.

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