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Groups of components of Néron models of Jacobians*

DINO J. LORENZINI

University of California at Berkeley (Present address: Dept. of Mathematics, Yale University, New Haven, CT 06520, USA)

Introduction

Before stating our main results, we list some notations used in this work.

K	a complete field with respect to a discrete valuation v and $\eta = \operatorname{Spec} K$.
$\mathcal{O}_K$	the ring of integers in K and $S = \operatorname{Spec} \mathcal{O}_K$.
k	the residue field, assumed to be algebraically closed and $s = \operatorname{Spec} k$.
A/K	an abelian variety of dimension g.
Φ, ϕ	the group of components of the Néron model of A/K and its order.
L	the minimal Galois extension of K such that A_L/L has semi-stable reduction.
Φ_L	the group of components of the Néron model of A_L/L.
Ψ, ψ	the kernel of the canonical map $\alpha: \Phi \to \Phi_L$ and its order.
X/η	a proper smooth geometrically connected curve of genus g.

We present in this paper some bounds for the group of components of a Néron model associated to the jacobian of a curve. In order to use Raynaud's description [16] of this group in terms of a regular model of the curve, we need to assume that the gcd of the multiplicities of the irreducible components of the special fiber obtained from the minimal model of X/η is equal to one (this happens for instance if the curve has an η-rational point). We shall call such a curve X/η an *S-curve.*

THEOREM 2.6. *Let X/η be an S-curve and $A = \operatorname{Jac}(X)$. Let Δ_{EP} be the difference between the Euler-Poincaré characteristics of the special fiber and the generic fiber of the minimal model of X/η. Then $\phi \leqslant 2^{\Delta_{EP}-1}$.*

It is worth noting that when E/K is an elliptic curve and $p \geqslant 5$, the integer Δ_{EP} is equal to the valuation of the minimal discriminant of E (Ogg's Formula). The following result is an improvement for jacobians of a theorem of Lenstra and Oort [7].

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THEOREM 2.4. *Let X/η be a smooth proper geometrically connected S-curve of genus g and $A = \mathrm{Jac}(X)$. If the toric dimension t of this abelian variety equals zero, then the order ϕ of its group of components can be bounded in terms of the unipotent rank u of A/K in the following way: $\Sigma_{l\,\mathrm{prime}}\, \mathrm{ord}_l(\phi)(l-1) \leqslant 2u$. In particular, $\phi \leqslant 2^{2u}$ and if l divides ϕ, then $l \leqslant 2u + 1$.*

The primes that divide ϕ were first studied by Oort in [15] and a bound for $\phi^{(p)}$ depending only on the dimension of A (when A has potential good reduction) was first found by Silverman [20]. However, these authors do not treat the case where l equals the characteristic $p > 0$ of the residue field. The above bound was shown to hold by Lenstra and Oort in [7] when $u = \dim A$ and $l \neq p$. It should undoubtly hold for any abelian variety with $t = 0$.

Serre and Tate [19] have proven that if a prime q divides $\varepsilon = \exp(\mathrm{Gal}(L/K))$, then $q \leqslant 2g + 1$ and Oort [15] showed that if a prime l divides ϕ then $l \leqslant 2g + 1$ (when $p = 0$ and $u = g$). This is not a coincidence; in [11], McCallum explains why ε kills the group Ψ. In section 3, we give a proof of a weaker version of his theorem and we extend his bounds for ε to the case of potentially toroidal reduction. In particular, we show that $q \leqslant 2u + t - t_{ss} + 1$ if q is a prime dividing ε, and t_{ss} is the toric rank of the semi-stable Néron model of A_L/L. We conjecture (3.7) that the group Ψ satisfies the Oort/Lenstra bound of $2u + t - t_{ss}$, i.e. that the following inequality holds: $\Sigma_{l\,\mathrm{prime}}\, \mathrm{ord}_l(\psi)(l-1) \leqslant 2u + t - t_{ss}$.

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1. Curves, Types and Picard Schemes

(1.1) Let X/η be a smooth proper geometrically connected curve of genus g. A proper and flat morphism $\mathscr{X}/S$ is a *regular model* for X/η if $\mathscr{X}$ is a connected regular scheme and $\mathscr{X}_\eta$ is isomorphic to X/η over η. As an effective Cartier divisor, the special fiber $\mathscr{X}_s = \Sigma_{i=1}^n r_i C_i$, where r_i is the multiplicity of the irreducible component C_i. The integer $s = gcd(r_1, \ldots, r_n)$ satisfies the following properties:

- s divides $g - 1$ (see 2.1).
- If $\mathscr{X}/S$ has a section (for instance if X/η has a rational point) then $r_i = 1$ for some i and $s = 1$.
- s is independent of the choice of a regular model for X. We shall say that X/η is an *S-curve* if its minimal model has $s = 1$.

We have on $\mathscr{X}$ an intersection theory (see [2], Chap. XIII or [6], page 58–61) with the following properties:

(a) $(C_i \cdot C_j) = (C_j \cdot C_i) \geqslant 0$ for all $i \neq j$,
(b) $(C_i \cdot \mathscr{X}_s) = 0$ for all $i = 1, \ldots, n$. In particular, $(C_i \cdot C_i) < 0$ for all $i = 1, \ldots, n$,
(c) $2p(C_i) - 2 = (C_i \cdot C_i) + (C_i \cdot K)$ where $p(C_i) \geqslant 0$ is the arithmetical genus of C_i and K is the relative canonical divisor.
(d) $2g - 2 = (\mathscr{X}_s \cdot \mathscr{X}_s) + (\mathscr{X}_s \cdot K) = \Sigma_{i=1}^n r_i(C_i \cdot K)$.

Assertion (b) implies that the intersection matrix $M = ((C_i \cdot C_j))$ and the vector ${}^tR = (r_1, \ldots, r_n)$ satisfy to the relation $M \cdot R = 0$. Given such a matrix M, we define what some authors call the dual graph of the curve; this graph is connected because the special fiber $\mathscr{X}_s$ is. The vertices of the *dual graph* G associated to the matrix M are the "curves" C_i. Two vertices C_i and C_j are linked by exactly $c_{ij} = (C_i \cdot C_j)$ edges. $(G, -M, s^{-1}R)$ defines what we called an arithmetical graph in [9].

Let P denote the vector $(p(C_1), \ldots, p(C_n))$. A *type* T is a set (n, M, R, P) as above, satisfying in particular the relation $M \cdot R = 0$ and such that the graph G associated to M is connected. The *group of components* of the type T is defined as $\Phi(T) = \mathrm{Ker}({}^tR)/\mathrm{Im}(M)$, where $M\colon \mathbb{Z}^n \to \mathbb{Z}^n$ and ${}^tR\colon \mathbb{Z}^n \to \mathbb{Z}$. The notion of type has been introduced by Artin and Winters in [1]. The problem of associating a curve to a given type is discussed in [23] and [24].

(1.2) Let A/η be an abelian variety of dimension g and $\mathscr{A}/S$ its Néron model. We have the following exact sequences of group schemes over s:

$$0 \to \mathscr{A}_s^0 \to \mathscr{A}_s \to \pi_0(\mathscr{A}_s) \to 0$$

$\mathscr{A}_s^0$ is the connected component of 0 in $\mathscr{A}_s$.
$\pi_0(\mathscr{A}_s)$ is the (finite etale) group scheme of components.
We let Φ denote the finite group $\pi_0(\mathscr{A}_s)(k)$.

$$0 \to U \times \mathscr{T} \to \mathscr{A}_s^0 \to B \to 0$$

U is a unipotent group scheme of dimension $u(A)$.
$\mathscr{T}$ is a torus of dimension $t(A)$.
B is an abelian variety of dimension $a(A)$.

In particular, $g = u + t + a$. The following key theorem is due to Raynaud [16] (see also [5], IX, Section 12).

THEOREM 1.3. *Let $\mathscr{X}/S$ be a regular model of the S-curve X_η and let $T = (n, M, R, P)$ be its associated type.*

1. *The connected component $\mathscr{A}_s^0$ is isomorphic over s to the group scheme* $\mathrm{Pic}^0_{\mathscr{X}_s/s}$.
2. *The finite abelian group $\Phi := \pi_0(\mathscr{A}_s)(k)$ is isomorphic to $\Phi(T)$.*

We review some standard results on the Picard scheme of a curve. Let k be an algebraically closed field and C/k a proper connected curve purely of dimension 1 whose irreducible components C_i, $i = 1, \ldots, n$ have multiplicity r_i. Let $\bar{C}_i$ be the normalization of C_i.

- $\mathrm{Pic}_{C/k}$ is a smooth group scheme over k and $\mathrm{Pic}^0_{C/k}$ denotes the connected component of the identity in $\mathrm{Pic}_{C/k}$.
- The map $i: C_{\mathrm{red}} \to C$ induces a morphism of group schemes $i^*: \mathrm{Pic}_{C/k} \to \mathrm{Pic}_{C_{\mathrm{red}}/k}$ which is surjective with connected unipotent kernel (see [14]).
- The map $p: \tilde{C} = \coprod C_i \to C_{\mathrm{red}}$ induces a morphism of group schemes $p^*: \mathrm{Pic}^0_{C_{\mathrm{red}}/k} \to \mathrm{Pic}^0_{\tilde{C}/k}$ which is surjective. Its kernel is a smooth and affine group scheme; when C lies on a regular surface, the kernel has dimension equal to the first Betti number of the dual graph $G(C)$ of C ([1], Lemma 2.8).
- The map $\pi: \bar{C} = \coprod \bar{C}_i \to \tilde{C} = \coprod C_i$ induces a morphism of group schemes $\pi^*: \mathrm{Pic}^0_{\tilde{C}/k} \to \mathrm{Pic}^0_{\bar{C}/k}$ which is surjective, and $\mathrm{Pic}^0_{\bar{C}/k}$ is the largest quotient of $\mathrm{Pic}^0_{C/k}$ which is an abelian variety. In particular, if $C = \mathscr{X}_s$ is the special fiber of a regular model of an S-curve, $a = \Sigma g(\bar{C}_i)$.

A regular model $\mathscr{X}/S$ is called an *SNC-model* when its special fiber is isomorphic over k to a curve C whose irreducible components are smooth and such that the singularities of C_{red} are formally isomorphic to the one of the union of the coordinates axis in an affine space $\mathbb{A}^2$.

- In the case of an SNC-model of an S-curve, the kernel of p^* is a torus ([5], IX, 12.3 or [12], page 47); the toric rank of A equals the first Betti number of the graph associated to the special fiber.

The "Embedded Resolution of Singularities" [8] shows the existence, for any curve X/η, of a regular SNC-model $\mathscr{X}/S$. It follows easily from what has just been reviewed that:

COROLLARY 1.4. *Let X/η be an S-curve. The dimension of the maximal torus in the special fiber of the connected component of the Néron model of $A = \mathrm{Jac}(X)$ is equal to the first Betti number of the graph associated to the special fiber of the regular SNC-model of X/η over S.*

We summarize in the next corollary some results that follow from 1.3 and 1.4, 2.3, 2.5 of [9].

COROLLARY 1.5. *Let $\mathscr{X}_s = \sum_{i=1}^n r_i C_i$ be the special fiber of an SNC-model of an S-curve X/η. Let $d_i = \sum_{i \neq j}(C_i \cdot C_j)$. Suppose that the Jacobian $A = \mathrm{Jac}(X)$ has toric dimension equal to zero. Then the group of components of A has order equal to $\phi = \prod_{i=1}^n r_i^{d_i - 2}$ and is killed by* $\mathrm{lcm}(r_i r_j, (C_i \cdot C_j) \neq 0)$.

2. Bounds for the Order of Φ

DEFINITION 2.1. Let $T = (n, M, R, P, G)$ be a type and d_i the degree of the vertex C_i in the graph G. The linear rank $g_o(T)$ is defined by the formula:

$$2g_o(T) - 2 = \sum_{i=1}^{n} (|(C_i \cdot C_i)| - 2) \cdot r_i = \sum_{i=1}^{n} (d_i - 2) r_i.$$

Let $\beta(T)$ denote the first Betti number of the graph G. It is not hard to check that $2\beta(T) - 2 = \sum(d_i - 2)$. The linear rank g_o could be thought as a generalization for types of the Betti number. Note that $g_o(T)$ is an integer: see for instance 3.6 in [9].

- *The genus $g(T)$ is defined to be: $g(T) = g_o(T) + \sum_{i=1}^{n} r_i p(C_i)$. Let $\alpha(T) = \sum_{i=1}^{n} r_i p(C_i)$ and $\gamma(T) = g_o(T) - \beta(G(T))$. With these notations, $g(T) = \gamma + \beta + \alpha$.*

These numbers could be thought as analogue of the unipotent, toric and abelian ranks u, t, a of the special fiber of the Néron model of the jacobian of a curve X having type T. It is clear from the definitions that $\alpha \geqslant a$. One shows easily, using 2.10, that $\gamma \leqslant u$ for any S-curve.

- *For any curve X/η we let $g_o(X) = g(X) - \sum_{i=1}^{n} r_i g(C_i)$ where $g(C_i)$ is the geometrical genus of the irreducible component (C_i, r_i).*

Since any regular model of X/η is obtained from the minimal model by a sequence of blow ups, $g_o(X)$ depends only on the minimal model of X, and hence only on X itself. We note that when T is associated to a curve X, $g(T)$ equals $g(X)$, the genus of X. Since $g(C_i) \leqslant p(C_i)$, the linear rank $g_o(T)$ of a type T associated to any regular model $\mathscr{X}/S$ of X satisfies $g_o(T) \leqslant g_o(X)$; moreover $g_o(T) = g_o(X)$ iff all the irreducible components of the special fiber of $\mathscr{X}$ are smooth. This is the case for any regular SNC-model.

To any type $T = (n, M, R, P, s)$ we can associate a new type $s^{-1}T = (n, M, s^{-1}R, P, 1)$ with the relation: $g_o(T) - 1 = s \cdot (g_o(s^{-1}T) - 1)$. Since $g_o(T)$ and $g_o(s^{-1}T)$ are integers, s divides $g_o(T) - 1$. If $g_o(T) \geqslant 0$, then $g_o(T) \geqslant g_o(s^{-1}T) \geqslant \beta(G)$, as quoted below.

LEMMA 2.2. *Let $T = (n, M, R, P)$ be a type. If $s = 1$ or $g_o(T) \geqslant 1$, then $g_o(T) \geqslant \beta(G)$.*

Proof. We proved this fact in an elementary way in [9], Theorem 4.7. Note that it is not clear a priori that the integer $2g_o - 2\beta = \sum(r_i - 1)(d_i - 2)$ is non negative since some of the degrees $d_i s$ might be equal to one.

LEMMA 2.3. *Let X/η be an S-curve, and $A = Jac(X)$. Then $t(A) \leqslant g_o(X) \leqslant u(A) + t(A)$.*

Proof. Let $\mathscr{X}/S$ be a regular SNC-model of X/η and T its associated type. By definition, $g_o(X) = g_o(T)$ and by 1.4, $t(A) = \beta(T)$. Using the same type T_o as in the next theorem, we find a curve $\mathscr{Y}_t$ such that $\beta(T) = \beta(T_o) = t(\mathscr{Y})$ and $g_o(T) = g_o(T_o) = g(\mathscr{Y}_t)$. Since $t(\mathscr{Y}_t) \leqslant g(\mathscr{Y}_t)$, the first inequality follows. The second inequality is immediate because $g(X) = u(A) + t(A) + a(A)$ and $a(A) = \sum_{i=1}^{n} g(C_i)$.

We can now generalize a result of Oort and Lenstra [7] in the case where the abelian variety A is the jacobian of an S-curve (see also the introduction). Let ϕ denote the order of the group $\pi_o(\mathscr{A}_s)(k)$ and $\phi^{(p)}$ the order of its prime-to-p part. Let also $l(x) := \sum_{l \text{ prime}} \text{ord}_l(x)(l - 1)$, for any integer x.

THEOREM 2.4. *Let X/η be an S-curve and $A = \text{Jac}(X)$. If the toric dimension $t(A, K) = 0$, then $l(\phi) \leqslant 2g_o(X) \leqslant 2u(A, K)$.*

Proof. Let $\mathscr{X}/S$ be a regular SNC-model for X/η and $T = (n, M, R, P, \phi, G(T))$ the associated type. $G(T)$ is a tree because $t(A) = 0$ (1.4). The integers $\phi, g_o(X)$ and the graph $G(T)$ depend only on n, M, R and not on P. Let $P_o = (0, \ldots, 0)$ and $T_o = (n, M, R, P_o, \phi, G(T))$. By Winters' Existence Theorem [24], one can find a discrete valuation ring Λ of equicharacteristic 0, a regular SNC-model $\mathscr{Y}/\text{Spec } \Lambda$ of its generic fiber $\mathscr{Y}_t/t$ such that the type associated to $\mathscr{Y}$ is the given type T_o.

Since $G(T_o)$ is a tree and $P_o = (0, \ldots, 0)$, the jacobian of $\mathscr{Y}_t$ is an abelian variety of dimension $g_o(X)$ with purely additive reduction over Λ. Moreover, since Λ is of equicharacteristic 0, $\phi^{(p)} = \phi$. We can then apply Lenstra/Oort's Theorem to get $l(\phi) \leqslant 2g_o(X)$.

REMARK 2.5. The bound $2u(A)$ undoubtly holds for abelian varieties in general. McCallum [11] has improved Lenstra/Oort's proof to obtain $l(\phi^{(p)}) \leqslant 2u(A)$ when $t(A) = 0$.

Note that Raynaud's Theorem 1.3 let us translate the problem of bounding ϕ in terms of the intersection matrix M only. We provide a more elementary proof that $l(\phi) \leqslant 2g_o$ in [9], Theorem 4.8, where we do not use Winters' and Lenstra/Oort's Theorems.

Winters' Theorem can be used to show the existence of all kinds of jacobians with specified group of components. It is sufficient to exhibit the right graph.

THEOREM 2.6. *Let $\mathscr{X}/S$ be the minimal regular model of an S-curve X/η. Let Δ_{EP} be the difference between the l-adic Euler-Poincaré characteristics of $\mathscr{X}_s$ and X, where $l \neq p = \text{char}(k)$. The order of the group $\Phi(\text{Jac}(X))$ is bounded by $2^{\Delta_{EP} - 1}$.*

Proof. Following Dolgačev [4], we define the Betti numbers of a proper connected curve $C = \sum_{i=1}^{n} r_i C_i$ as follows. Let $(\pi \circ p)\colon \amalg \bar{C}_i \to \amalg C_i \to \bigcup C_i$ with $\bar{C}_i$ denoting the normalization of C_i and $\delta_C := \sum_{P \in C^{\text{red}}} |(\pi \circ p)^{-1}(P)| - 1$.

$$\beta_i(C) = \beta_i(C_{\text{red}}) = \begin{cases} 1 & i = 0 \\ 2\sum_{i=1}^{n} g(C_i) + \delta_C - (n-1) & i = 1 \\ n & i = 2 \\ 0 & i > 2. \end{cases}$$

The étale cohomology group with coefficients in the constant sheaf $(\mathbb{Z}/l\mathbb{Z})_C$, $(\text{char}(k) \nmid l)$, can be computed by $H^i(C_{et}, \mathbb{Z}/l\mathbb{Z}_C) \cong (\mathbb{Z}/l\mathbb{Z})^{\beta_i(C)}$. It follows from the definitions that

$$\begin{aligned} EP(X) &= 2 - 2g = 2 - 2(u + t + a), \\ EP(\mathscr{X}_s) &= 1 - (2a + \delta_{\mathscr{X}_s} - (n-1)) + n, \\ \Delta_{EP} &= 2u + 2t - \delta_{\mathscr{X}_s} + 2(n-1). \end{aligned}$$

We showed in [9] that the order ϕ of the group of components is bounded by $\phi \leqslant v \cdot \kappa(G)$, where $\kappa(G)$ is the number of spanning trees of G and v is an integer satisfying the bound $l(v) \leqslant 2g_o - 2\beta$ (3.5, 4.7). In particular, $v \leqslant 2^{2g_o - 2\beta}$. We claim that $\kappa(G) \leqslant 2^{m-1}$ (see Lemma 2.7 below), so that $\phi \leqslant 2^{2g_o - 2\beta + m - 1}$. In the last two lemmas of this section, we shall show that $2g_o - \beta \leqslant 2u + t$ (2.10) and that $\delta_{\mathscr{X}_s} - (n-1) \leqslant t$ (2.8); we can then conclude the proof of the theorem:

$$\begin{aligned} 2g_o - 2\beta + m &= 2g_o - \beta + (n-1) \\ &\leqslant 2u + t + (n-1) + (t - \delta_{\mathscr{X}_s} + (n-1)) \\ &= \Delta_{EP}. \end{aligned}$$

LEMMA 2.7. *Let $T = (n, M, R, P, G)$ be a type. Then $\kappa(G) \leqslant \binom{m}{\beta} \leqslant 2^{m-1}$, where $\kappa(G)$ denotes the number of spanning trees of G.*

Proof. In order to obtain a spanning tree of G, one has to remove exactly β edges from the m edges of G. Hence the bound $\kappa \leqslant \binom{m}{\beta}$ follows. In order to prove the second inequality, we show:

$$\binom{m}{\beta} \leqslant \begin{cases} 2^{m-2} & \text{if } m \geqslant 9 \\ 2^{m-1} & \text{if } 1 \leqslant m \leqslant 8. \end{cases}$$

It is well known that $\binom{m}{\beta} \leqslant \binom{m}{\frac{1}{2}(m-1)}$ or $\binom{m}{\frac{1}{2}m}$ depending on whether m is odd or even. It is also known that $\binom{m}{\frac{1}{2}m} < 2^{m-2}$ when $m \geqslant 10$ is even. This proves our lemma in this case. In case $m \geqslant 9$ is odd, $\binom{m}{\frac{1}{2}(m-1)} = \frac{1}{2}\binom{m+1}{\frac{1}{2}(m+1)}$ and since $m+1$ is even, $\binom{m}{\frac{1}{2}(m-1)} < 2^{m-2}$. The reader will check the remaining cases by direct computations.

LEMMA 2.8. *Let $\mathscr{X}/S$ be a regular model of an S-curve X/η. Then $\beta_1(\mathscr{X}_s) \leqslant 2a + t$ with equality if $\mathscr{X}/S$ is an SNC-model.*

Proof. Let $\mathscr{X}_s = \sum_{i=1}^{n} r_i C_i$ be the special fiber of the given model of X/K. Let $\mathscr{Z} = \mathscr{X}_b \xrightarrow{\phi_b} \mathscr{X}_{b-1} \to \cdots \to \mathscr{X}_1 \xrightarrow{\phi_1} \mathscr{X}$ be a sequence of blow ups of points such that $\mathscr{Z}$ is a regular SNC-model of X/η. Then $\beta(G(\mathscr{Z}_s)) = t$ by 1.4. Let m_z be the number of edges of $G(\mathscr{Z}_s)$, so that, by definition, $\beta(G(\mathscr{Z}_s)) = m_z - (n + b - 1)$. Since $\beta_1(\mathscr{X}_s) = 2a + \delta_{\mathscr{X}_s} - (n-1)$, we only need to show that $m_z \geqslant b + \delta_{\mathscr{X}_s}$.

Let $\mathscr{Z}_s^{\text{red}} = (\bigcup_{i=1}^{n} \bar{C}_i) \cup (\bigcup_{j=1}^{b} E_j)$ where $\bar{C}_i$ is the normalization of $C_i \subset \mathscr{X}_s$ and E_j is the exceptional fiber of ϕ_j. The map $\psi := \phi_1 \circ \cdots \circ \phi_b \colon \mathscr{X}_s^{\text{red}} \to \mathscr{X}_s^{\text{red}}$ induces by restriction the normalization map $\bar{C}_i \to C_i$. We can assume without loss of generality that $\bar{C}_i \bigcap \bar{C}_j = \emptyset$ for all $i \neq j$. It is not hard to check that by restriction the map $\psi \colon \amalg \bar{C}_i \to \mathscr{X}_s^{\text{red}}$ is the map $(\pi \circ p) \colon \amalg \bar{C}_i \to \bigcup C_i$.

The map ψ determines a partition $\coprod_{\alpha=1}^{c} A_\alpha = \{1, \ldots, b\}$ where $i, j \in A_\alpha$ iff $\psi(E_i) = \psi(E_j)$. For any point $P \in \mathscr{X}_s^{\text{red}}$, we have $|\psi^{-1}(P)| \leqslant \sum_i (\sum_{j \in A_\alpha} (\bar{C}_i \cdot E_j))$. Hence

$$\sum_P (|\psi^{-1}(P)| - 1) \leqslant \sum_{\alpha=1}^{c} \left(\sum_i \sum_{j \in A_\alpha} (\bar{C}_i \cdot E_j) - 1 \right)$$

$$= \sum_{i,j} (\bar{C}_i \cdot E_j) - c$$

$$= m_z - \left(\sum_{i<j} (E_i \cdot E_j) + c \right).$$

We claim that $\sum (E_i \cdot E_j) + c \geqslant b$. This will follow if we show that $\sum_{i<j \in A_\alpha} (E_i \cdot E_j) \geqslant |A_\alpha| - 1$. In fact, given A_α, we can construct the following graph: the vertices are the E_is and E_i is linked to E_j iff $(E_i \cdot E_j) - 1 \geqslant 0$. Since the E_is are exceptional fibers mapping to the same point, this graph is connected. Hence the number of edges is at least equal to the number of vertices minus one.

(2.9) We defined the blow up of a graph G with respect to a vector Q in 1.8 of [9]. The following straightforward lemma discusses the effect of a blow up on the integers $g_o(T)$ and $\beta(T)$. Let E_k denote the kth column vector of the identity matrix. We say that a blow up is *trivial* if $Q = E_k$, *singular* if $Q = qE_k$ with $q > 1$ and *elementary* if $Q = E_i + E_j$.

A trivial blow up corresponds to blowing up a regular point of the special fiber, a singular blow up could correspond to blowing up a singular point of the special fiber belonging to exactly one irreducible component and an elementary blow up corresponds to blowing up the intersection of two curves that intersect normally.

LEMMA 2.10. *Let (G, M, R) be an arithmetical graph with invariants β and g_o, $Q = (q_1, \ldots, q_n)$ an integer vector and (G_Q, M_Q, R_Q) the blow-up of G with respect to Q, with invariants $\beta_Q := \beta(G_Q)$ and $g_Q := g_o(G_Q)$. Then $\beta_Q = \beta + (q - 1)$ if the blow up is singular, $\beta = \beta_Q$ if the blow up is trivial or elementary and $\beta_Q < \beta$ otherwise. Moreover, $g_o \leqslant g_Q$, $2g_o - 2\beta \leqslant 2g_Q - 2\beta_Q$ and $2g_o - \beta \leqslant 2g_Q - \beta_Q$.*

3. Base Change to Semi-Stability

Let A/K be an abelian variety. Since we assume that K is complete with respect to a discrete valuation and that the residue field is algebraically closed, we have $\mathrm{Gal}(\bar{K}/K) = I(\bar{K}/K) := I_K$. Fix an odd prime $l \geqslant 3$, $l \neq p$ and let $\rho: I_K \to \mathrm{Aut}(T_l A)$ be the action of I_K on the Tate module $T_l A$ of A. For any finite extension M/K of K, we let $u(M)$, $t(M)$, $a(M)$ be respectively the unipotent, toric and abelian rank of A_M/M.

- There exists (see [5], IX) an integer $e \geqslant 1$ minimal with the property that

  $(\sigma^e - id)^2 = 0$ for all $\sigma \in \rho(I_K)$.

  The characteristic polynomial of any matrix $\sigma \in \rho(I_K) \subset \mathrm{Aut}(T_l A)$ has rational integer coefficients. When $e = 1$, the abelian variety A_K/K is said to have *semi-stable* reduction and A_K/K has semi-stable reduction if and only if $u(K) = 0$.
- There exists (see [3], 5.15) a finite Galois extension L/K such that $(\sigma - id)^2 = 0$ for all $\sigma \in \rho(I_L)$ and such that L is minimal with this property:

  A_M/M has semi-stable reduction iff $L \subseteq M$.

  Let $a_{ss} = a(L)$ and $t_{ss} = t(L)$.

We generalize now to the potential toroidal case (i.e. $t_{ss} \neq 0$) some bounds for $\varepsilon = \exp(\mathrm{Gal}(L/K))$ given in [11] by McCallum. For any integer $x = p_1^{a_1} \ldots p_k^{a_k}$, with $p_1, \ldots, p_k$ distinct primes, we let

$$L(x) = \begin{cases} \sum_{i=1}^{k} \varphi(p_i^{a_i}) = \sum_{i=1}^{k} p_i^{a_i - 1}(p_i - 1) & \text{if } \mathrm{ord}_2(x) \neq 1, \\ L(x/2) & \text{if } \mathrm{ord}_2(x) = 1, \\ 0 & \text{if } x = 1, 2. \end{cases}$$

Note that $L(x)$ is always an even integer, so that $\{x \mid L(x) \leqslant 2n\} = \{x \mid L(x) \leqslant 2n + 1\}$.

PROPOSITION 3.1. *Let A/K be an abelian variety which does not have semi-stable reduction over K (i.e. $u = u(K) \neq 0$). Let ε denote the exponent of the finite group $I(L/K)$ and write $\varepsilon = p^w \cdot \varepsilon^{(p)}$ with $p \nmid \varepsilon^{(p)}$. Then* $\max(L(p^w), L(\varepsilon^{(p)})) \leqslant 2u + t - t_{ss}$. *If $t_{ss} > t$, ε is divisible by a prime at most equal to $(t_{ss} - t + 1) \leqslant g + 1$ and if $a_{ss} > a$, ε is divisible by a prime at most equal to $(2a_{ss} - 2a + 1)$.*

REMARK 3.2. For elliptic curves with potential good reduction, $\max(L(p^w), L(\varepsilon^{(p)})) \leqslant 2$. This bound is achieved in the case of $y^2 = x^3 - 2x^2 - x$ over $\mathbb{Q}_{2,nr}$ (Example 5.9.1 in [18]). The extension L/K needed has its inertia group isomorphic to $SL_2(\mathbb{F}_3)$; this group has order 24 and exponent 12. In particular, we see that a bound of the form $L(\varepsilon) \leqslant 2u$ does not hold for elliptic curves.

For elliptic curves with potential multiplicative reduction, it is known that the inertia group has order 2 (the curve is a quadratic twist of the Tate curve, see [21], 14.1). Our bound is then also achieved.

We show in [10], Proposition 2.7, that if A is the jacobian of an S-curve having tame potential good reduction then $\varepsilon = [L:K] \leqslant 2(2u + 1)$.

Proof of 3.1. Let $T = T_l A$ and consider the filtration $T \supset T_1 \supset T_2$, where $T_1 = T^{I_L}$ and T_2 is the orthogonal of T_1 under the Weil pairing on T (see [7], Proof of 1.3). I_K acts on each graded piece of $T/T_1 \oplus T_1/T_2 \oplus T_2$ as a finite group of automorphisms. By minimality of L/K, I_L is the kernel of this action. The actions of I_K on T/T_1 and T_2 are isomorphic. We have ([5], IX, Section 2):

$$\mathrm{rank}_{\mathbb{Z}_l} T_1/T_2 = 2a_{ss} \quad \text{and} \quad \mathrm{rank}_{\mathbb{Z}_l}(T_1/T_2)^{I_K} = 2a,$$

$$\mathrm{rank}_{\mathbb{Z}_l} T_2 = t_{ss} \quad \text{and} \quad \mathrm{rank}_{\mathbb{Z}_l}(T_2)^{I_K} = t.$$

Each element of $\rho(I_K)$ acting on any of the graded pieces has a characteristic polynomial with rational integers coefficients ([5], IX, Proof of 4.3). We can then bound $\max(L(p^w), L(\varepsilon^{(p)}))$ by using the lemma below; the multiplicity of one as eigenvalue of any element of I_K acting on T_1/T_2 is at least equal to $2a$. Similarly for T_2, where this multiplicity is at least equal to t. Note also that $(2a_{ss} - 2a) + (t_{ss} - t) = 2u + t - t_{ss}$.

If $t_{ss} > t$, I_K acts non trivially on T_2 and hence some element of I_K satisfies $(x^e - 1)^2 = 0$, e minimal with this property and $e > 1$; the bound for $L(e)$ given in the next lemma shows that a prime dividing e equals at most $t_{ss} - t + 1$. The argument in case $a_{ss} > a$ is similar.

LEMMA 3.3. *Let $S \in GL_n(\mathbb{Z}_l)$ be such that its characteristic polynomial has integer coefficients and its minimal polynomial divides $(x^e - 1)^q$. Assume that e is minimal with this property and let t_1 denote the multiplicity of the eigenvalue one. Then $L(e) \leqslant n - t_1$.*

Proof. Recall the following factorization of the polynomial $x^e - 1$ over $\mathbb{Z}$:

$$x^e - 1 = \prod_{d|e} \theta_d(x)$$

where $\theta_d(x)$ is the minimal irreducible polynomial of a primitive dth root of one. This cyclotomic polynomial has degree $\varphi(d)$ ($\varphi(x)$ is the Euler function). When $d \equiv 2 \pmod 4$, $\theta_d(x) = \theta_{d/2}(-x)$; in particular if $e \equiv 2 \pmod 4$, we have $x^e - 1 = \prod_{d|e/2} \theta_d(x)\theta_d(-x)$. Since the characteristic polynomial divides a power of the minimal polynomial, we have

$$\mathrm{char}_S(x) = \prod_{d|e} \theta_d(x)^{t_d} \ t_d \geqslant 0$$

Suppose that $e \equiv 0 \pmod 4$. Let $e = p_1^{a_1} \dots p_k^{a_k}$, so that $L(e) = \sum \varphi(p_i^{a_i})$. By minimality, $e = \mathrm{lcm}(d|e$, such that $t_d \neq 0)$. Hence there exists integers $d_1, \dots, d_h$, dividing e such that, for each $1 \leqslant i \leqslant k$, there exists $1 \leqslant j \leqslant h$ with $\mathrm{ord}_{p_i}(d_j) = a_i$ and $t_{d_j} \geqslant 1$. Since $n = t_1 + \sum_{d|\exp(S), d \neq 1} t_d \varphi(d)$, we can write

$$L(e) \leqslant \sum_{j=1}^{h} \varphi(d_j) \leqslant n - t_1 - \sum_{d_j} \varphi(d_j)(t_{d_j} - 1) - \sum_{d|e, d \neq d_j} \varphi(d)(t_d) \leqslant n - t_1.$$

Suppose that $e \equiv 2 \pmod 4$. Then $L(e) = L(e/2)$ and $n = t_1 + t_2 + \sum_{d|e/2, d \neq 1} (t_d + t_{2d})\varphi(d)$. By the same reasoning as in the previous case, we have $L(e) \leqslant n - t_1$.

(3.4) Let A/K be an abelian variety and let L/K be the minimal field extension such that A_L/L has semi-stable reduction over L. For any finite extension M/L, the natural map from the Néron model $(\mathscr{A}_L)_M$ to the Néron model of the abelian variety $(A_L)_M/M$ induces an isomorphism of their connected components (see [5], IX, 3.2). This isomorphism induces an injection

$$\Phi_L \hookrightarrow \Phi_M.$$

We denote by $\Psi = \Psi(A, K)$ the kernel of the canonical map $\alpha: \Phi(A, K) \to \Phi(A_L, L)$.

For any integer $m \geqslant 3$, $\gcd(m, p) = 1$, let A_m be the group scheme of m-torsion points of A and $K_m = K(A_m)$ be the smallest field extension over which the points of A_m are rational. The minimal extension L is always contained in K_m ([3], 5.15) and $L = K_m$ when A/K has potential good reduction (see [19], Cor. 3, page 498). In the following theorem, we prove a weaker version of a result of McCallum in [11], using a method of Silverman [20].

PROPOSITION 3.5. *The prime-to-p part of* Ψ *injects into* $H^1(I_{L/K}, A_m^{I_{\bar{K}/L}})$. *In particular, the order of* $\Psi^{(p)}$ *is bounded by a constant depending only on g and* $\Psi^{(p)}$ *is killed by the order of* $I(L/K)$.

Proof. Consider the following commutative diagram with exact columns:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{E} & \cong & \mathscr{F} & \hookrightarrow & H^1(I_{L/K}, A_m^{I_{\bar{K}/L}}) \\
\downarrow & & \downarrow & & \downarrow \text{inf} \\
\Phi_K/m\Phi_K & \cong & A(K)/mA(K) & \hookrightarrow & H^1(I_{\bar{K}/K}, A_m) \\
\downarrow & & \downarrow & & \downarrow \text{res} \\
\Phi_L/m\Phi_L & \cong & A(L)/mA(L) & \hookrightarrow & H^1(I_{\bar{K}/L}, A_m)
\end{array}$$

The first column is isomorphic to the second for any integer $m \geqslant 3$ such that $\gcd(m, p) = 1$ (see for instance [20]). The second column injects in the third by standard Kummer theory: the injection $A(K)/mA(K) \hookrightarrow H^1(I_{\bar{K}/K}, A_m)$ is induced by the connecting homomorphism of the long exact sequence of cohomology obtained from the short exact sequence $0 \to A_m \to A \to A \to 0$, where the second map is multiplication by m (see [21], page 197, in case of elliptic curves, but the general case is similar).

Choose $m = \phi^{(p)}$, in which case $\Phi_K^{(p)}/m\Phi_K^{(p)} \cong \Phi_K^{(p)}$. In particular, $\Psi^{(p)}$ injects in $\Phi_K/m\Phi_K$; hence $\Psi^{(p)} \subseteq \mathscr{E} \hookrightarrow H^1(I_{L/K}, A_m^{I_{\bar{K}/L}})$ and is then killed by the order of $I_{L/K}$. By a lemma of Silverman [20], the order of $H^1(I_{L/K}, A_m^{I_{\bar{K}/L}})$ is bounded by a constant depending only on g.

REMARK 3.6. Let X/K be an S-curve with $p \nmid [L:K]$. Let $\mathscr{X}_s = \sum r_i C_i$ denote the special fiber of a regular SNC-model of X/K. In [5], I, 3.4, Grothendieck shows that $[L:K]$ divides $\text{lcm}(r_1, \ldots, r_n)$. When $A = \text{Jac}(X)$ has potential good reduction (toric ranks $t_K = t_L = 0$), it follows from the definitions that $\Psi = \Phi$ and hence Φ is killed by $\text{lcm}(r_1, \ldots, r_n)$. If we assume only that the toric rank t_K equals zero, it follows from 1.5 that Φ is killed by a multiple of $\text{lcm}(r_1, \ldots, r_n)$; this bound is sharp in the case of elliptic curves with Kodaira reduction type I_ν^*, ν odd.

3.7 Let A/K be an abelian variety. The canonical exact sequence $0 \to \Psi \to \Phi \to \Phi/\Psi \to 0$ should enjoy the following property:

- $\sum_{l \text{ prime}} \text{ord}_l(\psi)(l - 1) \leqslant 2u + t - t_{ss}$.

The order ψ is divisible only by primes q smaller than or equal to $2u + t - t_{ss} + 1$ (see 3.5, 3.1, or [11], where the case $l = p$ is treated).

- Φ/Ψ is minimally generated by at most t_{ss} elements, where $t_{ss} = t(L)$ is the toric rank of the semi-stable model of A_L/L. The bound is sharp.

To prove this statement, we only need to show that Φ_L is minimally generated by at most $t(L)$ elements, since $\Phi/\Psi \hookrightarrow \Phi_L$. Grothendieck shows in [5], IX, 11.9, 11.11, that the inductive limit $\lim_{M \supset L} \Phi(M)$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{t_{ss}}$. Hence the injection $\Phi_L \hookrightarrow (\mathbb{Q}/\mathbb{Z})^{t_{ss}}$ implies that $\Phi(L)$ is minimally generated by at most t_{ss} elements. Note that if A is the jacobian of an S-curve, we can obtain the desired result by using Raynaud's Theorem 1.3 and our explicit computations in 5.2 of [9].

Let E/K be an elliptic curve with reduction of type I^*_{2k+1} and let $p \geqslant 5$. By Tate's algorithm, we know that $\Phi = \mathbb{Z}/4\mathbb{Z}$ and that $\Phi_L = \mathbb{Z}/2(2k+1)\mathbb{Z}$. Hence the subgroup Ψ is non trivial. We claim that $\Psi = \mathbb{Z}/2\mathbb{Z}$. It is sufficient to show that $\Psi \not\cong \mathbb{Z}/4\mathbb{Z}$. We proved in the above theorem that ε kills Ψ and it is well-known that for such curve, $\varepsilon = 2$ (see 3.2). Hence $\Psi = \mathbb{Z}/2\mathbb{Z}$, Φ/Ψ is non trivial and the bound above is sharp because $t = 0$ and $t_{ss} = 1$. Note that the exact sequence is not always split, as it can be seen on the above example.

4 Elliptic Curves and Wild Ramification

Let E/K be an elliptic curve. Let L/K be the minimal field extension such that E_L/L has semi-stable reduction and $\varepsilon = \exp(\mathrm{Gal}(L/K))$. The exponent of the wild conductor δ, defined in [13] has the following property:

$$\delta = 0 \Longleftrightarrow p \nmid \varepsilon.$$

For elliptic curves with additive reduction, this integer can be computed using Ogg's Formula [13]

$$\delta = v(\Delta) - (n + 1)$$

where $v(\Delta)$ is the valuation of the discriminant of the minimal Weierstrass model of E/K and n is the number of irreducible components of the special fiber of the minimal model of E over S. The following theorem is an easy application of Ogg's Formula and of Tate's Algorithm [22].

THEOREM 4.1 *Let E/K be an elliptic curve with additive reduction.*
1. *If $p = 2$, $\delta = 0 \Longleftrightarrow E/K$ has reduction of type IV or IV^*.*
2. *If $p = 3$, $\delta = 0 \Longleftrightarrow E/K$ has reduction of type $III, III^*, I_0^*, I_\nu^* (\nu \geqslant 1)$.*

where the reductions are described by their Kodaira symbol as in [22].

Proof. We closely follow the notations in Tate's Algorithm. Let

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \quad a_i \in \mathcal{O}$$

be a minimal equation of E/K. By $\pi^k \| a$, we mean that $\pi^k | a$ and $\pi^{k+1} \nmid a$.

Since we assume that E has purely additive reduction, we may start with Case 3: $\pi|a_3, a_4, a_6, b_2$. It is not hard to check that when $p = 2$, $\pi^4|\Delta$ and when $p = 3$, $\pi^3|\Delta$. If the reduction is of type II, $\delta = v(\Delta) - 2$ and hence in both cases $\delta > 0$.

In Case 4, we assume moreover that $\pi^2|a_6$. If $\pi^2\|b_8$, the reduction is of type III and $\delta = v(\Delta) - 3$. Hence in the case $p = 2$, $\delta > 0$. In case $p = 3$, we claim that $\delta = 0$ or equivalently that $v(\Delta) = 3$. In fact, $v(\Delta) = 3$ iff $v(8) + 3v(b_4) = 3$ and it is easy to check that $\pi^2\|b_8$ implies $\pi\|b_4$.

In Case 5, we assume that $\pi^3|b_8$. If $\pi^2\|b_6$, then the reduction is of type IV and $\delta = v(\Delta) - 4$. When $p = 3$, it is easy to check that $\pi^5|\Delta$ and hence $\delta > 0$. When $p = 2$, we claim that $\delta = 0$ is the only possibility. In fact, $\pi|a_1$, $\pi^2|b_2$ and $v(\Delta) = 4$ iff $v(27) + 2v(b_6) = 4$.

In Case 6, it is easy to check that when $p = 2$, $\pi^8|\Delta$. Hence if the reduction is of type I_0^*, $\delta = v(\Delta) - 6 > 0$. When $p = 3$, it is easy to check that $\pi^6|\Delta$ and that $\pi^6\|\Delta$ iff $\pi^6\|(4a_2^3a_6 - a_2^2a_4^2 + 4a_4^3)$. But this last condition is also necessary and sufficient for the discriminant of $P(T) = T^3 + (a_2/\pi)T^2 + (a_4/\pi^2)T + a_6/\pi^3$ to be non zero mod π. This shows that when the reduction is of type I_0^*, $\delta = 0$.

In Case 7, the reduction is of type I_ν^*, $\nu \geqslant 1$ and $\delta = v(\Delta) - 6 - \nu$. It is not hard to check that

$$\pi|a_1, \pi\|a_2, \pi^{\nu+3}|a_6 \quad \text{and} \quad \begin{array}{ll} \nu \text{ odd} & \pi^{(\nu+3)/2}|a_3, \pi^{(\nu+5)/2}|a_4 \\ \nu \text{ even} & \pi^{(\nu+4)/2}|a_3, a_4. \end{array}$$

Moreover, when ν is odd, the reduction is of type I_ν^* iff $\pi^{\nu+3}\|b_6$ and when ν is even, the reduction is of type I_ν^* iff $\pi^{\nu+4}\|(a_4^2 - 4a_2a_6)$. When $p = 2$, one checks that $\pi^{\nu+8}|\Delta$ and hence $\delta > 0$. When $p = 3$, it is easy to check that $\pi^{\nu+6}|\Delta$ and that $\pi^{\nu+6}\|\Delta$ iff $\pi^{\nu+6}\|b_2^2b_8$ iff $\pi^{\nu+4}\|b_8$. When ν is odd, $\pi^{\nu+4}\|b_8$ iff $\pi^{\nu+3}\|b_6$ because $4b_8 = b_2b_6 - b_4^2$ and when ν is even, $\pi^{\nu+4}\|b_8$ iff $\pi^{\nu+4}\|(a_4^2 - 4a_2a_6)$.

In Case 8, if $y^2 + (a_3/\pi^2)y - a_6/\pi^4$ has distinct roots mod π then the reduction is of type IV* and $\delta = v(\Delta) - 8$. When $p = 3$, it is easy to check that $\pi^9|\Delta$ and hence $\delta > 0$. When $p = 2$, we claim that $v(\Delta) = 8$ is the only case occurring. In fact, $v(\Delta) = 8$ iff $v(27) + 2v(b_6) = 8$. This is the case because the condition on the discriminant of the polynomial above is equivalent to $\pi^2\|a_3$ and hence $\pi^4\|b_6$.

In Case 9, the reduction has type III* and $\delta = v(\Delta) - 9$ if $\pi^4 \nmid a_4$. When $p = 2$, it is easy to check that $\pi^6|b_6$ and that $\pi^{10}|\Delta$. Hence $\delta > 0$. When $p = 3$, we claim that $\delta = 0$ is the only case occurring. In fact, $v(\Delta) = 9$ iff $v(8) + 3v(b_4) = 9$. But $b_4 = a_1a_3 + 2a_4$ with $\pi^4|a_1a_3$ and $\pi^3\|a_4$. Hence $\pi^3\|b_4$.

In Case 10, the reduction has type II* and $\delta = v(\Delta) - 10$. In both characteristics, $\pi^7|b_8$ and $\pi^{11}|\Delta$. Hence in both cases, $\delta > 0$.

COROLLARY 4.2. *Let E/K be an elliptic curve with additive reduction and $p = \mathrm{char}(k) = 2$ or 3. Then p divides $|\mathrm{Gal}(L/K)|$ if and only if $\phi = 1$* or *$p|\phi$.*

Proof. We note that IV and IV* are the only types whose group of components has order divisible by 3, and similarly, the types III, III*, I_0^*, I_ν^*, $\nu \geqslant 1$ are the only ones whose group of components has order divisible by 2.

COROLLARY 4.3. *Let E_1/K be an elliptic curve with additive reduction such that $\phi(E_1)$ is divisible by a prime q. If E_2 is an elliptic curve isogenous to E_1 over K, then $\phi(E_2) = 1$ or is also divisible by q.*

Proof. Let L/K be the minimal extension such that E_{1L}/L has semi-stable reduction. It is a very deep fact proven in [5], IX, Cor. 2.2.7, that L/K is invariant under isogeny. Hence when $p = 2, 3$, this corollary follows from the previous one. When $p \geqslant 5 > 2g + 1$, the extension L/K is tame and our claim follows from the following table:

ε	2	3	4	6
ϕ	4	3	2	1

REMARK 4.4. This table is well-known, but we could not find a precise reference in the literature. It follows for instance from a quotient/desingularization construction due to Viehweg [23] (see also [10], Sec. 1).

In [17], T. Saito gives necessary and sufficient conditions for X/K to have tame semi-stable reduction in terms of a regular SNC-model $\mathscr{X}/S$ of X/K.

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