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Grothendieck's pairing on component groups of Jacobians

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Let $\mathfrak{R}$ be a discrete valuation ring with field of fractions K. Let A_K be an abelian variety over K with dual A'_K. Denote by A and A' the corresponding Néron models and by Φ_A and $\Phi_{A'}$ their component groups. In [Gr], Exp. VII–IX, Grothendieck used the notion of biextension invented by Mumford to investigate how the duality between A_K and A'_K is reflected on the level of Néron models. In fact, the essence of the relationship between A and A' is captured by a bilinear pairing

$$\langle\ ,\ \rangle \colon \Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

introduced in [Gr], Exp. IX, 1.2, and which represents the obstruction to extending the Poincaré bundle $\mathcal{P}_K$ on $A_K \times A'_K$ to a biextension of $A \times A'$ by $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$.

Grothendieck conjectured in [Gr], Exp. IX, 1.3, that the pairing $\langle\ ,\ \rangle$ is perfect and gave some indications on how to prove this in certain cases, namely on ℓ-parts with ℓ prime to the residue characteristic of $\mathfrak{R}$, as well as in the semi-stable reduction case; see [Gr], Exp. IX, 11.3 and 11.4, see also [Ber], [We] for full proofs. The conjecture has been established in various other cases, notably by Bégueri [Beg] for valuation rings $\mathfrak{R}$ of mixed characteristic with perfect residue fields, by McCallum [McC] for finite residue fields and by Bosch [B] for abelian varieties with potentially multiplicative reduction, again for perfect residue fields. Grothendieck also mentions in [Gr], Exp. IX, 1.3.1, that for Jacobians, the conjecture follows from unpublished work of Artin and Mazur on the autoduality of relative Jacobians (for algebraically closed residue fields). On the other hand, using previous work of Edixhoven [E] on the behavior of component groups under the process of Weil restriction, Bertapelle and Bosch [B-B] have recently given a series of counter-examples to Grothendieck's conjecture when the residue field k of $\mathfrak{R}$ is not perfect.

Our main result in this paper is an explicit formula for the pairing $\langle\ ,\ \rangle$ in the case of the Jacobian J_K of a smooth proper curve X_K having a K-rational point. More precisely, fixing a flat proper $\mathfrak{R}$-model X of X_K, which is regular, we show that the pairing $\langle\ ,\ \rangle$ is completely determined by the intersection matrix M of the special fiber X_k of X and by the geometric multiplicities of the irreducible components of X_k. This explicit formula allows us, on one hand, to prove Grothendieck's conjecture for Jacobians as above in the case where all the geometric multiplicities are equal to 1 and, in particular, for perfect residue fields; see Corollary 4.7. On the other hand, working over an imperfect residue field, we use the formula to provide examples of Jacobians where Grothendieck's conjecture fails to hold; see 6.2.

Our method of proof is purely geometric. In Sect. 1, we attach to any symmetric matrix M a "component group" Φ_M and a symmetric pairing

$$\langle\ ,\ \rangle_M : \Phi_M \times \Phi_M \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which we show to be always perfect. In Sect. 2, we recall Raynaud's description of the component group Φ_J of a Jacobian J_K in terms of the intersection matrix M associated to a regular model of X_K. This description allows us to interpret Φ_J as a subgroup of Φ_M. The group Φ_J coincides with Φ_M when the residue field k is perfect or, more generally, when all the geometric multiplicities of the irreducible components of X_k are equal to 1. We thus obtain a canonical pairing on $\Phi_J \times \Phi_J$ by restricting $\langle\ ,\ \rangle_M$ to $\Phi_J \times \Phi_J$. Note that even though the pairing on $\Phi_M \times \Phi_M$ is always perfect, the restricted pairing on $\Phi_J \times \Phi_J$ may not be perfect. Our main result, Theorem 4.6, states that Grothendieck's pairing $\langle\ ,\ \rangle$ coincides with this restricted pairing once we identify J_K with its dual J'_K in a canonical way.

To prove Theorem 4.6, we first give in Sect. 3 a more practical description of Grothendieck's pairing $\langle\ ,\ \rangle$. We view it as a homomorphism

$$\Phi_{A'} \longrightarrow \mathrm{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Ext}^1(\Phi_A, \mathbb{Z})$$

and, starting with an element $x \in \Phi_{A'}$, we describe its image in $\mathrm{Ext}^1(\Phi_A, \mathbb{Z})$ as a cocycle in $H^2(\Phi_A, \mathbb{Z})_s$; see 3.3. This description is the key ingredient for the proof of 4.6 and involves the vanishing orders on certain divisors of suitable functions in $K(A_K)$. We then express in 3.7 the value $\langle a, x\rangle$ for $a \in \Phi_A$ and $x \in \Phi_{A'}$ in terms of such vanishing orders. Finally, we express in 4.4 the value $\langle a, x\rangle$ in terms of Néron's local symbol j, introduced in [Nér] and recalled in Sect. 4. In the case of a Jacobian J_K, it is then the functoriality property of Néron's local height pairing in conjunction with its characterization in terms of intersection theory on regular models of curves that allows us to show that $\langle a, x\rangle$ can be computed in terms of data pertaining only to the underlying curve X_K and to identify $\langle\ ,\ \rangle$ with the pairing $\langle\ ,\ \rangle_M$ given by the intersection matrix of the special fiber X_k of X.

In Sects. 5 and 6, we provide explicit computations of the pairing $\langle\ ,\ \rangle$, including the case of elliptic curves and the case where the Jacobian J_K has potentially good reduction.

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1. A pairing attached to a symmetric matrix

We attach in this section a pairing to any symmetric matrix. The content of this section is of a purely linear algebraic nature.

Let Z be any domain with field of fractions Q. Let $M \in M_v(Z)$ be any matrix, considered as a linear map $M : Z^v \to Z^v$. The group $Z^v/\mathrm{Im}(M)$ is a finitely generated Z-module. We denote by Φ_M, or simply by Φ, its torsion submodule. If $Y \subset Z^v$ is any submodule, let $Y^\perp$ denote the orthogonal of Y with respect to the standard scalar product on Z^v. Assume now that M is symmetric, so that $\mathrm{Im}(M) \subseteq \mathrm{Ker}(M)^\perp$. Then

$$\Phi = \mathrm{Ker}(M)^\perp/\mathrm{Im}(M).$$

Indeed $\mathrm{rk}(\mathrm{Im}(M)) = \mathrm{rk}(\mathrm{Ker}(M)^\perp)$, and $\mathrm{Ker}(M)^\perp$ is a saturated submodule of Z^v (i.e., if $zu \in \mathrm{Ker}(M)^\perp$ for some $z \in Z$, $u \in Z^v$, then $u \in \mathrm{Ker}(M)^\perp$).

Let $\tau, \tau' \in \Phi$, and let $T, T' \in \mathrm{Ker}(M)^\perp$ be vectors whose image in Φ are τ and τ', respectively. Let $S, S' \in Z^v$ be such that $MS = nT$ and $MS' = n'T'$ for some non-zero $n, n' \in Z$. Define

$$\begin{aligned} \langle\ ,\ \rangle_M : \Phi \times \Phi &\longrightarrow Q/Z \\ (\tau, \tau') &\longmapsto ({}^tS/n)M(S'/n') \mod Z. \end{aligned}$$

When M is invertible over Q, this construction is classical (see, e.g., [Dur], Sect. 2). The pairing $\langle\ ,\ \rangle_M$ can then be written as

$$\langle \tau, \tau'\rangle_M = {}^tTM^{-1}T' \mod Z.$$

Lemma 1.1. *The pairing $\langle\ ,\ \rangle_M$ is well-defined, bilinear, and symmetric.*

Proof. Assume that $MS_1 = n_1T$ and $MS_2 = n_2T$. Then $M(n_2S_1 - n_1S_2) = 0$, so $S_1/n_1 - S_2/n_2 \in \mathrm{Ker}(M) \otimes_Z Q$. Since $T' \in \mathrm{Ker}(M)^\perp$, we find that

$$0 = ({}^tS_1/n_1 - {}^tS_2/n_2)T' = ({}^tS_1/n_1)M(S'/n') - ({}^tS_2/n_2)M(S'/n').$$

Thus, the value $\langle \tau, \tau'\rangle$ does not depend on the choices of S and n in the relation $MS = nT$. Assume now that T_1 and T_2 both have image τ in Φ.

Then $T_1 - T_2 = MV$ for some vector $V \in Z^v$. We may always find $n \in Z$ and $S_1, S_2 \in Z^v$ such that $MS_1 = nT_1$ and $MS_2 = nT_2$. Then

$$\begin{aligned}({}^tS_1/n)M(S'/n') &= {}^tT_1(S'/n') = ({}^tT_2 + {}^t(MV))(S'/n') \\ &= {}^tT_2(S'/n') + {}^tVM(S'/n') \\ &= {}^tT_2(S'/n') + {}^tVT' \\ &\equiv ({}^tS_2/n)M(S'/n') \mod Z.\end{aligned}$$

Hence, the value $\langle \tau, \tau' \rangle$ does not depend on the choice of a representative $T \in \text{Ker}(M)^{\perp}$. It is clear that the pairing is bilinear and symmetric. □

Remark 1.2. For $A \in \text{GL}_v(Z)$ consider the symmetric matrix $M' = {}^t(A^{-1})M(A^{-1})$. Then the natural map $Z^v \to Z^v$, which sends V to AV, induces an isomorphism $\alpha : \Phi_M \to \Phi_{M'}$ such that we have $\langle x, y \rangle_M = \langle \alpha(x), \alpha(y) \rangle_{M'}$ for all $x, y \in \Phi_M$.

Recall that a bilinear pairing $\langle\ ,\ \rangle : \Phi \times \Phi' \longrightarrow Q/Z$ is called *perfect*, if the associated Z-morphisms

$$\Phi \longrightarrow \text{Hom}_Z(\Phi', Q/Z), \qquad \Phi' \longrightarrow \text{Hom}_Z(\Phi, Q/Z),$$

are isomorphisms. Of course, if $\Phi = \Phi'$ and the pairing is symmetric (as in our case), the two maps coincide.

Theorem 1.3. *Let $M \in M_v(Z)$ be any symmetric matrix. The pairing $\langle\ ,\ \rangle_M$ is perfect if either*

a) $\det(M) \neq 0$, *or*
b) $\text{Ker}(M)$ *is a free Z-module and Z^v is the direct sum of* $\text{Ker}(M)$ *and a free complement, or*
c) *Z is a principal ideal domain or, more generally,*
g) *Z is a Dedekind domain.*

Proof. Assume that $\det(M) \neq 0$. To show that the map $\Phi_M \longrightarrow \text{Hom}_Z(\Phi_M, Q/Z)$ is injective, choose x in its kernel and let $T \in Z^v$ be a representative of x. Then ${}^tTM^{-1}T' \in Z$ for all $T' \in Z^v$ and, hence, ${}^tTM^{-1} = {}^tS$ for some $S \in Z^v$. Thus, $T = MS$ and, as x is the image of T in $\Phi_M = Z^v/\text{Im}(M)$, it is trivial.

To verify that $\Phi_M \longrightarrow \text{Hom}_Z(\Phi_M, Q/Z)$ is surjective, start out from an element $\overline{\varphi} \in \text{Hom}_Z(\Phi_M, Q/Z)$, i. e., from a Z-linear map $\overline{\varphi}: \Phi_M \longrightarrow Q/Z$, and lift it to a Z-linear map $\varphi: Z^v \longrightarrow Q$. Then φ is of type $T' \longmapsto {}^tST'$ for some $S \in Q^v$ and satisfies $\varphi(\text{Im}(M)) \subset Z$. The latter means ${}^tSMS' \in Z$ for all $S' \in Z^v$. But then $T := MS \in Z^v$ and the residue class $x \in \Phi_M$ of T satisfies $\langle x, y \rangle_M = \overline{\varphi}(y)$ for all $y \in \Phi_M$. Thus, x is an inverse image of $\overline{\varphi}$.

Assume now that $\text{Ker}(M)$ is a free Z-module, say of rank r. Let $\{V_1, \ldots, V_r\}$ be a basis for $\text{Ker}(M)$. By hypothesis, there exists a set $\{V_{r+1}, \ldots, V_v\}$ of vectors in Z^v such that $Z^v = (\oplus_{i=1}^r ZV_i) \oplus (\oplus_{j=r+1}^v ZV_j)$.

Let $B \in \mathrm{GL}_v(Z)$ be the matrix whose j-th column is V_j. Then the matrix ${}^tBMB$ is symmetric, and is the null matrix except for a matrix $A \in GL_{v-r}(Z)$ in the bottom right corner. Since B is invertible, we find using 1.2 that $(Z^v/\mathrm{Im}(M))_{\mathrm{tors}}$ endowed with the pairing defined by M is isomorphic to $(Z^v/(\mathrm{Im}({}^tBMB))_{\mathrm{tors}}$ endowed with the pairing defined by ${}^tBMB$. Since $(Z^v/\mathrm{Im}({}^tBMB))_{\mathrm{tors}} \cong Z^{v-r}/\mathrm{Im}(A)$ and A is invertible, Part b) follows from Part a).

To prove Part c) suppose that Z is a principal ideal domain. Then $\mathrm{Ker}(M) \subset Z^v$ is certainly free. Since $\mathrm{Ker}(M)$ is saturated, $Z^v/\mathrm{Ker}(M)$ is also free and, thus, $Z^v = \mathrm{Ker}(M) \oplus Z^v/\mathrm{Ker}(M)$. Hence, we may apply Part b) in this case. Finally, Part d) is reduced to Part c) via localization. □

We study below the case of 2 × 2-matrices and produce examples over $Z = k[x, y]$ and $Z = k[x, y]/(y^2 - x^3)$ where the pairing on $\Phi_M \times \Phi_M$ is not perfect. But for now, let Z be arbitrary, and let $x, y, z \in Z$ with $x \neq 0$. Let $M = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$ with $\det(M) = 0 = xy - z^2$. Given $c \in Q$, let

$$I_c := \{a \in Z \mid ac \in Z\}.$$

The set I_c is an ideal of Z. Let $e := {}^t(1, z/x)$. It is clear that $\mathrm{Im}(M)$ is generated by xe and ze.

Lemma 1.4. *The map $I_{z/x}/(x, z) \to \Phi_M$, with $a \mapsto ae$, is an isomorphism, and Φ_M is killed by the ideal (x, y, z).*

Proof. Assume that ${}^t(n, m) \in Z^2$ is such that $a\,{}^t(n, m) \in \mathrm{Im}(M)$ for some $a \in Z, a \neq 0$. Then $a\,{}^t(n, m) = (bx + cz)e$ for some $b, c \in Z$. It follows that $m = nz/x$ and ${}^t(n, m) = ne$. Since ${}^t(n, m) \in Z^2$, we find that $n \in I_{z/x}$. To prove the last statement, let $a \in I_{z/x}$. Then $yae = \frac{az}{x} \cdot {}^t(yx/z, y) = \frac{az}{x} \cdot {}^t(z, y)$ and $az/x \in Z$ with ${}^t(z, y) \in \mathrm{Im}(M)$. □

Thus, recalling that $(a, 0)M = ax\,{}^te$, the pairing $\langle\ ,\ \rangle : \Phi_M \times \Phi_M \to Q/Z$ can be described as follows:

Lemma 1.5. *Identifying Φ_M with $I_{z/x}/(x, z)$ via the isomorphism of* 1.4, *we have*

$$\langle a, b\rangle = ab/x \mod Z.$$

Lemma 1.6. *The pairing is perfect if z/x or $x/z \in Z$.*

Proof. If $z/x \in Z$, then $I_{z/x} = Z$, and $\langle a, b\rangle = 0$ for all $b \in Z$ implies that $\langle a, 1\rangle = a/x \equiv 0 \mod Z$. Thus $a/x \in Z$, so a belongs to the ideal (x, z) and $a = 0$ in Φ_M. Furthermore, any Z-linear map $\varphi \in \mathrm{Hom}(\Phi_M, Q/Z)$ is induced from a Z-linear map $Z \longrightarrow Q$. The latter is of type $b \longmapsto cb$ for some $c \in Q$ satisfying $c \cdot (x, z) \subset Z$, and it follows $c = a/x$ for some $a \in Z$. Thus, $\varphi(b) = ab/x \mod Z$ and the pairing is perfect.

If $x/z \in Z$, then $I_{z/x} = (x/z)$, and $\langle a, (x/z) \rangle = ax/zx \equiv 0 \mod Z$ implies that $a/z \in Z$, so a belongs to the ideal (x, z) and $a = 0$ in Φ_M. Similarly as before, any Z-linear map $\varphi\colon \Phi_M \longrightarrow Q/Z$ lifts to a Z-linear map $(x/z) \longrightarrow Q$. The latter is of type $b \longmapsto cb$ for some $c \in Q$, where $c \cdot (x, z) \subset Z$ and we see again that the pairing is perfect. □

Consider now $M = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$, with $\Phi_M \cong I_{xy/x^2}/(x^2, xy)$. When k is any field and $Z := k[x, y]$ is the free polynomial ring in two variables x and y, we find that $I_{xy/x^2} = (x)$. It follows that $\Phi_M \neq 0$, but the pairing on $\Phi_M \times \Phi_M$ is trivial; in fact, $\mathrm{Hom}_Z(\Phi_M, Q/Z)$ is trivial. To obtain a similar example with $\dim(Z) = 1$, consider $Z := k[x, y]/(y^2 - x^3)$. Then, writing $\overline{x}$ and $\overline{y}$ for the residue classes of x and y in Z, we have $I_{\overline{xy}/\overline{x}^2} = (\overline{x}, \overline{y})$ since $(\overline{x}, \overline{y})$ is maximal and $\overline{y}/\overline{x} \notin Z$. It follows that $\Phi_M \neq 0$, but the pairing on $\Phi_M \times \Phi_M$ is trivial.

2. A canonical pairing on component groups of Jacobians

Let us now apply the purely linear algebraic results of the previous section to the case of Jacobians. To do this, fix a strictly henselian discrete valuation ring $\mathfrak{R}$ with field of fractions K and residue field k, of characteristic $p \geq 0$; so k is separably closed. Let X_K be a smooth proper geometrically connected curve over K. Let X be a proper flat $\mathfrak{R}$-model of X_K, which is regular. Such a model always exists and is, in fact, projective (see, for instance, [Art] or [D-M], page 87). Let J_K denote the Jacobian of X_K, let J denote its Néron model over $\mathfrak{R}$, and let Φ_J be the associated component group. The latter is a finite étale k-group scheme and, thus, is constant, as k is separably closed. In order to be able to use Raynaud's results on component groups of Jacobians recalled below, we assume in this article that, in addition, k is perfect or that X admits a section (which amounts to the fact that $X_K(K)$ is not empty). In this situation, Φ_J is described in terms of combinatorial data associated with the special fiber X_k of X (see [Ray], Sect. 8 or [BLR], 9.6/1; see also [B-L], 1.1, when $\mathfrak{R}$ is not necessarily strictly henselian).

We will write the special fiber X_k/k as a Weil divisor $X_k = \sum_C r(C)C$, where C runs through the irreducible components of X_k, and where $r(C)$ is the multiplicity of C in X_k. Furthermore, let $e(C)$ denote the geometric multiplicity of C (see [BLR], 9.1/3). For divisors D, D' intersecting properly on X, one can define the intersection multiplicity $(D \cdot D')$. To recall the definition, consider first two prime divisors D, D' on X. Then $(D \cdot D)$ is the sum of all local intersection numbers

$$(D \cdot D')_{x_k} := [k(x_k) : k] \cdot \mathrm{len}(\mathcal{O}_{X,x_k}/(h_D, h_{D'}))$$

at closed points $x_k \in X$, where $h_D, h_{D'}$ are functions representing D, D' at x_k. Using the terminology of [BLR], 9.1, we can consider the line bundles $\mathcal{L}$, $\mathcal{L}'$ associated to D, D' and observe that $(D \cdot D')$ is the degree $\deg_{D'}(\mathcal{L})$

of the restriction of $\mathcal{L}$ to D', provided the support of D' is contained in X_k. Of course, for more general divisors intersecting properly on X, the intersection multiplicity $(D \cdot D')$ is defined via linear expansion. In addition, self-intersection is defined for divisors with support on the special fiber X_k, via the equations $(X_k \cdot C) = 0$, for any irreducible component C of X_k.

Writing $\mathbb{Z}^I$ for the free $\mathbb{Z}$-module generated by the irreducible components C of X_k, we consider the complex of $\mathbb{Z}$-modules

$$\mathbb{Z}^I \xrightarrow{\alpha} \mathbb{Z}^I \xrightarrow{\beta} \mathbb{Z},$$

where the $\mathbb{Z}$-linear maps α, β are given by

$$\alpha(D) := \sum_C (D \cdot C)C, \qquad \beta(C) := r(C).$$

Furthermore, we consider the $\mathbb{Z}$-linear map

$$\lambda\colon \mathbb{Z}^I \longrightarrow \mathbb{Z}^I, \qquad C \longmapsto e(C)C.$$

Then λ admits a $\mathbb{Q}$-inverse λ^{-1} where, by [BLR], 9.1/8, we may view $\lambda^{-1} \circ \alpha$ as a map from $\mathbb{Z}^I$ to $\mathbb{Z}^I$. So we can just as well look at the complex

$$\mathbb{Z}^I \xrightarrow{\lambda^{-1}\circ\alpha} \mathbb{Z}^I \xrightarrow{\beta\circ\lambda} \mathbb{Z}.$$

By [BLR], 9.6/1, the quotient $\mathrm{Ker}\,(\beta\lambda)/\,\mathrm{Im}(\lambda^{-1}\alpha)$ is canonically identified with the component group Φ_J. To be more precise, let us introduce the degree map

$$\rho\colon\ \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^I, \qquad \mathcal{L} \longmapsto \sum_C \deg_C(\mathcal{L})C,$$

where $\deg_C(\mathcal{L})$ denotes the degree of a line bundle $\mathcal{L}$ on the component C. Let $P(X)$ be the subgroup in $\mathrm{Pic}(X)$ consisting of all line bundles of total degree 0 on X. We obtain from [BLR], 9.6/1, and the proof of 9.5/9:

Proposition 2.1. *In the above situation, the following diagram is commutative:*

$$\begin{array}{ccccc}
P(X) & \xrightarrow{\text{res}} & \mathrm{Pic}^0(X_K) & = & J_K(K) = J(\mathfrak{R}) \\
\Big\downarrow{\scriptstyle \lambda^{-1}\circ\rho} & & & & \Big\downarrow \\
\mathrm{Ker}\,(\beta\lambda) & \longrightarrow & \mathrm{Ker}\,(\beta\lambda)/\,\mathrm{Im}(\lambda^{-1}\alpha) & = & \Phi_J,
\end{array}$$

where the vertical map on the right is the natural composition $J(\mathfrak{R}) \to J_k(k) \to \Phi_J$.

Thus, given any point $a_K \in J_K(K)$, its image in $\mathrm{Ker}\,(\beta\lambda)/\,\mathrm{Im}(\lambda^{-1}\alpha)$ is constructed as follows. Choose a divisor D_K of degree 0 on X_K representing a_K. Consider the schematic closure D of D_K in X_K, and let $[D]$ be the line bundle on X associated to the Weil divisor D. Then the image of a_K in Φ_J is given by the class of $\lambda^{-1}\rho([D])$ in $\mathrm{Ker}\,(\beta\lambda)/\,\mathrm{Im}(\lambda^{-1}\alpha)$.

In order to describe the above maps in terms of matrices, choose a numbering $C_1, \ldots, C_v$ of the irreducible components of the special fiber X_k, and consider the intersection matrix $M := (C_i \cdot C_j)_{1\leq i,j\leq v}$, the vector of multiplicities $R = {}^t(r_1, \ldots, r_v)$ with $r_i := r(C_i)$, as well as the diagonal matrix $\Lambda = \mathrm{diag}(e_1, \ldots, e_v) \in M_v(\mathbb{Z})$ with diagonal entries the geometric multiplicities $e_i := e(C_i)$. Then $\alpha\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}^v$ and $\beta\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}$ are given by the matrices M and ${}^tR$, whereas $\lambda^{-1}\circ\alpha$ and $\beta\circ\lambda$ correspond to $\Lambda^{-1}M$ and ${}^t(\Lambda R)$. We thus obtain

$$\Phi_{\Lambda,M} := \mathrm{Ker}\,({}^t(\Lambda R))/\,\mathrm{Im}(\Lambda^{-1}M) = \mathrm{Ker}\,(\lambda^{-1}\alpha)/\,\mathrm{Im}(\beta\lambda)$$

as the group of components of the Jacobian of X_K.

In the same way we can consider the quotient

$$\Phi_M := \mathrm{Ker}\,({}^tR)/\,\mathrm{Im}(M) = (\mathbb{Z}^v/\,\mathrm{Im}(M))_{\mathrm{tors}} = \mathrm{Ker}\,\beta/\,\mathrm{Im}\,\alpha.$$

We call the latter finite group the *component group* of M; note that $\mathrm{Im}(M)$ has rank $v-1$ (see, e.g., [BLR], 9.5/10), and that Φ_M is completely determined by M. Viewing the canonical diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^v & = & \mathbb{Z}^v & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda^{-1}\alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^v & \xrightarrow{\lambda} & \mathbb{Z}^v & \longrightarrow & \mathrm{coker}\,\lambda & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta\lambda} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

as a short exact sequence of (vertical) complexes, the relation between Φ_M and $\Phi_{\Lambda,M}$ becomes obvious:

Lemma 2.2. *The middle row of the above diagram gives rise to an exact sequence*

$$0 \longrightarrow \Phi_{\Lambda,M} \longrightarrow \Phi_M \longrightarrow \mathrm{coker}\,\lambda,$$

and the diagram of 2.1 *extends to a commutative diagram*

$$\begin{array}{ccccc}
P(X) & \xrightarrow{\mathrm{res}} & \mathrm{Pic}^0(X_K) & = & J_K(K) = J(\mathfrak{R}) \\
\downarrow{\scriptstyle \lambda^{-1}\circ\rho} & & & & \downarrow \\
\mathrm{Ker}\,(\beta\lambda) & \longrightarrow & \mathrm{Ker}\,(\beta\lambda)/\,\mathrm{Im}(\lambda^{-1}\alpha) & = & \Phi_{\Lambda,M} = \Phi_J \\
\downarrow{\scriptstyle \lambda} & & \downarrow & & \downarrow \\
\mathrm{Ker}\,(\beta) & \longrightarrow & \mathrm{Ker}\,(\beta)/\,\mathrm{Im}(\alpha) & = & \Phi_M
\end{array}$$

where the three bottom vertical maps are injective, and where ρ is the degree map $\mathcal{L} \longmapsto (\deg_{C_i}(\mathcal{L}))_i$.

We may now restrict to $\Phi_{\Lambda,M} \times \Phi_{\Lambda,M}$ the pairing $\langle\ ,\ \rangle_M : \Phi_M \times \Phi_M \to \mathbb{Q}/\mathbb{Z}$ attached to M in Sect. 1 and get a symmetric pairing

$$\langle\ ,\ \rangle_{\Lambda,M} : \Phi_{\Lambda,M} \times \Phi_{\Lambda,M} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Theorem 2.3. *As before, consider a smooth proper geometrically connected curve X_K and a flat proper $\mathfrak{R}$-model X, which is regular. Choose a numbering $C_1, \dots, C_v$ of the irreducible components of X_k and let M be the associated intersection matrix. Let $\Lambda = \operatorname{diag}(e_1, \dots, e_v)$, with $e_i = e(C_i)$. Assume that either k is perfect, or that X_K admits a rational point. Then:*

(i) *The component group Φ_J of the Jacobian J_K of X_K is canonically identified with $\Phi_{\Lambda,M}$, and there is a canonical injection $\Phi_J = \Phi_{\Lambda,M} \hookrightarrow \Phi_M$, which is induced by λ, respectively by multiplication with Λ; see* 2.2.

(ii) *The pairing $\langle\ ,\ \rangle_{\Lambda,M}$ on $\Phi_{\Lambda,M}$, restriction of the pairing $\langle\ ,\ \rangle_M$ on Φ_M, gives rise to a well-defined symmetric pairing*

$$\langle\ ,\ \rangle_J = \langle\ ,\ \rangle_{\Lambda,M} : \Phi_J \times \Phi_J \longrightarrow \mathbb{Q}/\mathbb{Z},$$

which is independent of the chosen numbering of the components of X_k.

(iii) *The pairing $\langle\ ,\ \rangle_M$ is perfect. Hence, the pairings $\langle\ ,\ \rangle_{\Lambda,M}$ and $\langle\ ,\ \rangle_J$ are perfect if Λ is the unit matrix; for example, the latter is the case if k is algebraically closed.*

(iv) *The restriction of the pairing $\langle\ ,\ \rangle_J$ to the prime-to-p part of $\Phi_J \times \Phi_J$ is always perfect.*

Proof. Assertion (i) is clear. To verify (ii), let $A \in M_n(\mathbb{Z})$ be a permutation matrix, so that ${}^tA = A^{-1}$. Set $M' := AM({}^tA)$ and $R' = AR$. Furthermore, let $\Lambda' := A\Lambda({}^tA)$. Then Λ' is again a diagonal matrix since A is a permutation matrix. The isomorphism $\Phi_M \xrightarrow{\sim} \Phi_{M'}$ described in 1.2, induced by $V \longmapsto AV$, yields by restriction an isomorphism $\Phi_{\Lambda,M} \xrightarrow{\sim} \Phi_{\Lambda',M'}$ which is compatible with the pairings $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{M'}$.

The assertion on the perfectness of $\langle\ ,\ \rangle_M$ follows from 1.3. Since $\langle\ ,\ \rangle_M$ is perfect on $\Phi_M \times \Phi_M$, it is also perfect when restricted to the ℓ-part of $\Phi_M \times \Phi_M$ for any prime ℓ. Thus, to prove assertion (iv), it is sufficient to show that the canonical injection $\Phi_{\Lambda,M} \hookrightarrow \Phi_M$ is an isomorphism on prime-to-p parts. That the latter is true follows from the exact sequence $0 \longrightarrow \Phi_{\Lambda,M} \longrightarrow \Phi_M \longrightarrow \operatorname{coker}\lambda$ of 2.2, since $\operatorname{coker}\lambda = \bigoplus_i \mathbb{Z}/(e_i)$ is a p-group by [BLR], 9.1/4 (c). □

3. Grothendieck's pairing

As before, let $\mathfrak{R}$ be a strictly henselian discrete valuation ring with field of fractions K, uniformizing element π, and residue field k. Write $i\colon \operatorname{Spec} k \longrightarrow \operatorname{Spec} \mathfrak{R}$ for the canonical morphism. Let A_K be an abelian variety over K with dual A'_K. Denote by A and A' the corresponding Néron models, and by Φ_A and $\Phi_{A'}$ their component groups. Grothendieck introduced in [Gr], IX, 1.2, a canonical pairing

$$\langle\ ,\ \rangle\colon \Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

which represents the obstruction of extending the Poincaré bundle $\mathcal{P}_K$ on $A_K \times A'_K$ to a biextension of $A \times A'$ by $\mathbb{G}_{m,\mathfrak{R}}$. Our aim in this section is to produce in 3.7 an explicit formula for $\langle a, x\rangle$ in terms of the orders of vanishing of a certain rational function associated with $a \in \Phi_A$ and $x \in \Phi_{A'}$.

Let $\mathcal{G}$ denote the Néron model of the multiplicative group $\mathbb{G}_{m,K}$; see [BLR], 10.1/5 for its construction[1]. Consider the exact sequence

$$0 \longrightarrow \mathbb{G}_{m,\mathfrak{R}} \longrightarrow \mathcal{G} \longrightarrow i_*\mathbb{Z} \longrightarrow 0,$$

as well as the associated Biext sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Biext}^1(A, A'; \mathbb{G}_{m,\mathfrak{R}}) &\longrightarrow \operatorname{Biext}^1(A, A'; \mathcal{G}) \\ &\longrightarrow \operatorname{Biext}^1(A, A'; i_*\mathbb{Z}) \longrightarrow 0, \end{aligned}$$

which is obtained by interpreting Biext as Ext groups; cf. [Gr], VII, 3.6.5. Due to [Gr], VIII, 6.7, restriction to generic fibers yields an isomorphism

$$\operatorname{Biext}^1(A, A'; \mathcal{G}) \xrightarrow{\sim} \operatorname{Biext}^1(A_K, A'_K; \mathbb{G}_{m,K}),$$

and by [Gr], VIII, 5.6 and 5.10, there is a canonical isomorphism

$$\operatorname{Biext}^1(A, A'; i_*\mathbb{Z}) \xrightarrow{\sim} \operatorname{Biext}^1(\Phi_A, \Phi_{A'}; \mathbb{Z}).$$

Furthermore, using the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$, we get an isomorphism

$$\begin{aligned} \operatorname{Biext}^1(\Phi_A, \Phi_{A'}; \mathbb{Z}) \xleftarrow{\sim} &\operatorname{Biext}^0(\Phi_A, \Phi_{A'}; \mathbb{Q}/\mathbb{Z}) \\ &= \operatorname{Hom}(\Phi_A \otimes_{\mathbb{Z}} \Phi_{A'}, \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Thus, viewing the Poincaré bundle $\mathcal{P}_K$ as an element in $\operatorname{Biext}^1(A_K, A'_K; \mathbb{G}_{m,K})$ or $\operatorname{Biext}^1(A, A'; \mathcal{G})$, we can look at its image in $\operatorname{Biext}^1(A, A'; i_*\mathbb{Z})$ and interpret it as a morphism $\Phi_A \otimes_{\mathbb{Z}} \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$. The latter is Grothendieck's pairing of component groups; it represents the obstruction of extending $\mathcal{P}_K$ to an element of $\operatorname{Biext}^1(A, A'; \mathbb{G}_{m,\mathfrak{R}})$.

[1] As we do not require a Néron model to be of finite type, our notion of Néron model corresponds to the notion of Néron lft-model in [BLR].

In the following we want to write the pairing in the form of a homomorphism $\Phi_{A'} \longrightarrow \mathrm{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z})$. We claim that there is a commutative diagram

$$\begin{array}{ccccc} A'(\mathfrak{R}) & \longrightarrow & \mathrm{Ext}^1(A, \mathcal{G}) & & \\ \downarrow & & \downarrow & & \\ \Phi_{A'} & \longrightarrow & \mathrm{Ext}^1(\Phi_A, \mathbb{Z}) & \xleftarrow{\sim} & \mathrm{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z}) \end{array}$$

with the pairing homomorphism occurring in the lower row. Here Ext groups are meant with respect to the fppf-topology; they may also be interpreted in the sense of group extensions. To define the map in the first row, start out from the isomorphism $A'_K(K) \xrightarrow{\sim} \mathrm{Ext}^1(A_K, \mathbb{G}_{m,K})$ given by the duality between A_K and A'_K. Using [Gr], VIII, 6.6, this isomorphism induces an isomorphism $A'(\mathfrak{R}) \xrightarrow{\sim} \mathrm{Ext}^1(A, \mathcal{G})$. The first vertical map is the projection of $A'(\mathfrak{R})$ onto its component group, whereas the second is induced from the projection $\mathcal{G} \longrightarrow i_*\mathbb{Z}$, using the fact that $\mathrm{Ext}^1(A, i_*\mathbb{Z})$ coincides with $\mathrm{Ext}^1(\Phi_A, \mathbb{Z})$ by [Gr], 5.5 and 5.9. That the diagram is commutative, follows from [Gr], VIII, 7.3.4.

We will identify $\mathrm{Ext}^1(\Phi_A, \mathbb{Z})$ with the cohomology group $H^2(\Phi_A, \mathbb{Z})_s$ in the sense of [Ser], VII, §1.4. In order to specify the cohomology class associated to an element $\Phi \in \mathrm{Ext}^1(\Phi_A, \mathbb{Z})$ represented by an extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Phi \xrightarrow{q} \Phi_A \longrightarrow 0,$$

we can choose a section $s\colon \Phi_A \longrightarrow \Phi$ of q (not additive, in general) and consider the class associated to the cocycle γ given by

$$\gamma(a, b) = s(a + b) - s(a) - s(b), \qquad a, b \in \Phi_A.$$

We are especially interested in the case where, as in 3.2 below, Φ is induced by some element $\mathcal{L}_K \in \mathrm{Ext}^1(A_K, \mathbb{G}_{m,K})$. In this situation, we want to show that the choice of a $\mathbb{G}_{m,\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$ extending $\mathcal{L}_K$ on A determines a section $s\colon \Phi_A \longrightarrow \Phi$ of q.

Due to [Gr], VIII, 6.5, there is an equivalence between the category of $\mathbb{G}_{m,K}$-torsors on A_K and the category of $\mathcal{G}$-torsors on A, whose inverse is given by restriction to the generic fiber. Given a $\mathbb{G}_{m,K}$-torsor $\mathcal{L}_K$ on A_K, we obtain an associated $\mathcal{G}$-torsor $\mathcal{L}$ on A by choosing any $\mathbb{G}_{m,\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$ on A extending $\mathcal{L}_K$ and considering its push-out via $\mathbb{G}_{m,\mathfrak{R}} \longrightarrow \mathcal{G}$. As A is regular, such a torsor $\overline{\mathcal{L}}$ can always be found; for instance, when $\mathcal{L}_K$ is associated to a prime divisor D_K on A_K, define $\overline{\mathcal{L}}$ as the torsor associated to the schematic closure of D_K in A.

Lemma 3.1. *Let $\overline{\mathcal{L}}$ be a $\mathbb{G}_{m,\mathfrak{R}}$-torsor on A. Write $\mathcal{L}_K$ for its generic fiber and $\mathcal{L}$ for its push-out via $\mathbb{G}_{m,\mathfrak{R}} \longrightarrow \mathcal{G}$, so that $\mathcal{L}$ is the $\mathcal{G}$-torsor on A associated to $\mathcal{L}_K$ under the equivalence described above. Then:*

(i) *In terms of total spaces, $\mathcal{L}$ is obtained by glueing copies of $\overline{\mathcal{L}}$, parametrized by $n \in \mathbb{Z}$, along multiplication by $\pi^n \in \mathbb{G}_{m,K}(K)$ on the generic fiber. In particular, there is a canonical open immersion $\overline{\mathcal{L}} \hookrightarrow \mathcal{L}$.*
(ii) *On sets of components of special fibers, $\overline{\mathcal{L}} \hookrightarrow \mathcal{L}$ induces an injection $\Phi_{\overline{\mathcal{L}}} \hookrightarrow \Phi_{\mathcal{L}}$ over the group of components Φ_A.*
(iii) *The projection $\Phi_{\overline{\mathcal{L}}} \longrightarrow \Phi_A$ is bijective; hence, composing its inverse with the map of* (ii)*, we get a section $s\colon \Phi_A \longrightarrow \Phi_{\mathcal{L}}$ of the projection $q\colon \Phi_{\mathcal{L}} \longrightarrow \Phi_A$.*

Proof. Note that any $\mathbb{G}_{m,\mathfrak{R}}$-torsor on A is locally trivial with respect to the Zariski topology. Thus, there is a Zariski-open covering $\mathfrak{U}$ of A on which $\overline{\mathcal{L}}$ is given by a cocycle η with values in $\mathbb{G}_{m,\mathfrak{R}}$. The push-out $\mathcal{L}$ of $\overline{\mathcal{L}}$ is given by the same cocycle η, however, viewed now as a cocycle with values in $\mathcal{G}$. Using the fact that $\mathcal{G}$ is obtained by glueing copies of $\mathbb{G}_{m,\mathfrak{R}}$, parametrized by $n \in \mathbb{Z}$, along multiplication by $\pi^n \in \mathbb{G}_{m,K}(K)$ on the generic fiber, the assertion of (i) follows. The same argumentation shows (ii) and (iii) where, in the latter case, we have to use that, for any irreducible k-scheme X_k, also $\mathbb{G}_{m,k} \times_k X_k$ is irreducible. □

Of course, we want to apply the assertions of 3.1 to the setting of extensions.

Lemma 3.2. *Consider an extension*

$$(*) \qquad 0 \longrightarrow \mathbb{G}_{m,K} \longrightarrow \mathcal{L}_K \longrightarrow A_K \longrightarrow 0$$

of (commutative) K-group schemes and let

$$(**) \qquad 0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{L} \longrightarrow A \longrightarrow 0$$

be the extension of $\mathfrak{R}$-group schemes associated to $(*)$*, using the equivalence of categries described in* [Gr], VIII, 6.6. *Furthermore, consider the image*

$$(***) \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \Phi_{\mathcal{L}} \xrightarrow{\ q\ } \Phi_A \longrightarrow 0$$

of $(**)$ *under the canonical map*

$$\operatorname{Ext}^1(A, \mathcal{G}) \longrightarrow \operatorname{Ext}^1(A, i_*\mathbb{Z}) \simeq \operatorname{Ext}^1(\Phi_A, \mathbb{Z}),$$

which is induced from push-out with respect to $\mathcal{G} \longrightarrow i_*\mathbb{Z}$; cf. [Gr], VIII, 5.5 *and* 5.9. *Then:*

(i) *The sequence* $(**)$ *is the sequence of Néron models associated to* $(*)$.
(ii) *The sequence* $(***)$ *is the sequence of component groups associated to* $(**)$.
(iii) *Using* 3.1*, the choice of a $\mathbb{G}_{m,\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$ extending $\mathcal{L}_K$ on A determines a section $s\colon \Phi_A \longrightarrow \Phi_{\mathcal{L}}$ of q.*

Proof. For assertion (i) we have only to show that $\mathcal{L}$ is the Néron model of $\mathcal{L}_K$. In terms of torsors, $\mathcal{L}$ is the $\mathcal{G}$-torsor associated to $\mathcal{L}_K$, as described in 3.1 (i) and its proof. From this we read that the construction of $\mathcal{L}$ is compatible with extensions $\mathfrak{R}'/\mathfrak{R}$ of ramification index 1 in the sense of [BLR], 3.6/1, and that, furthermore, the canonical map $\mathcal{L}(\mathfrak{R}) \longrightarrow \mathcal{L}_K(K)$ is surjective. But then, as $\mathcal{L}$ is a smooth and separated $\mathfrak{R}$-group scheme, the assertion follows from [BLR], 10.1/2.

In order to verify (ii), let us start with the sequence $(**)$ and investigate how the associated sequence of component groups changes when we pass to $(***)$, always keeping in mind that, as a $\mathcal{G}$-torsor, $\mathcal{L}$ is locally trivial with respect to the Zariski topology on A. First we take the push-out of $(**)$ via $\mathcal{G} \longrightarrow i_*\mathbb{Z}$, a process which leaves component groups untouched, as $p\colon \mathcal{G} \longrightarrow i_*\mathbb{Z}$ is an isomorphism on component groups. Then we restrict to special fibers, which certainly does not affect component groups, and, finally, we use the fact that the resulting extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow (p_*\mathcal{L})_k \longrightarrow A_k \longrightarrow 0$$

is the pull-back of $(***)$ with respect to the projection $A_k \longrightarrow \Phi_A$. Trivially, this map is an isomorphism on component groups and, thus, this process preserves component groups. As $(***)$ is already a sequence of constant k-groups, we are done. □

In the situation of 3.2 and, in particular, of 3.2 (iii), we will consider the map

$$\mathrm{ord}_{\overline{\mathcal{L}}}\colon \Phi_{\mathcal{L}} \longrightarrow \mathbb{Z}, \qquad c \longmapsto c - s \circ q(c),$$

and we will call $\mathrm{ord}_{\overline{\mathcal{L}}}\, c = c - s \circ q(c)$ the order of c (relative to the section s, or relative to the $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$ extending $\mathcal{L}_K$). By composition we then get an order function

$$\mathcal{L}_K(K) = \mathcal{L}(\mathfrak{R}) \longrightarrow \Phi_{\mathcal{L}} \xrightarrow{\mathrm{ord}_{\overline{\mathcal{L}}}} \mathbb{Z},$$

on K-valued points of $\mathcal{L}_K$, which we will also denote by $\mathrm{ord}_{\overline{\mathcal{L}}}$.

If f is a rational function on A_K, its order $\mathrm{ord}_c\, f$ on a component $c \in \Phi_A$ is defined as usual. Namely, let ζ be the generic point of c viewed as an irreducible component of the special fiber A_k. Then the local ring $\mathcal{O}_{A,\zeta}$ is a discrete valuation ring with uniformizing element π, the same we have in $\mathfrak{R}$, and with field of fractions $K(A_K)$, thus giving rise to a valuation ord_c on $K(A_K)$, which extends the one we have on K.

Now let us fix an element $x \in \Phi_{A'}$ and show how to describe its image under Grothendieck's pairing map

$$\Phi_{A'} \longrightarrow \mathrm{Ext}^1(\Phi_A, \mathbb{Z}) \simeq H^2(\Phi_A, \mathbb{Z})_s$$

in terms of a cocycle $\gamma = \gamma_x \in Z^2(\Phi_A, \mathbb{Z})_s$. First, choose a point $\mathcal{L}_K \in A'_K(K)$ representing x. So $\mathcal{L}_K$ is a primitive $\mathbb{G}_{\mathrm{m},K}$-torsor on A_K, and we can select a divisor D_K on A_K inducing $\mathcal{L}_K$. Then the closure D of D_K in A defines a $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$ extending $\mathcal{L}_K$ and, thus, by 3.2 (iii), defines a section $s\colon \Phi_A \longrightarrow \Phi_{\mathcal{L}}$ of the associated map of component groups $q\colon \Phi_{\mathcal{L}} \longrightarrow \Phi_A$. The image of x in $H^2(\Phi_A, \mathbb{Z})_s$ is then given by the cocycle

$$\gamma(a,b) = s(a+b) - s(a) - s(b), \qquad a, b \in \Phi_A,$$

which we want to compute in more detail.

Theorem 3.3. *Let D_K be a divisor on A_K giving rise to a primitive $\mathbb{G}_{\mathrm{m},K}$-torsor $\mathcal{L}_K$ on A_K and, thus, to an extension in* $\mathrm{Ext}^1(A_K, \mathbb{G}_{\mathrm{m},K})$. *Let $\overline{\mathcal{L}}$ be the $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor associated to the schematic closure D of D_K in A and $s\colon \Phi_A \longrightarrow \Phi_{\mathcal{L}}$ the section which is induced from $\overline{\mathcal{L}}$ in the sense of* 3.2 (iii). *For any element $a \in \Phi_A$, fix a representative $a_K \in A_K(K)$ of a and a rational function $f_a \in K(A_K)$ with divisor* $\mathrm{div}(f_a) = T_{a_K}^{-1}(D_K) - D_K$, *where T_{a_K} is the translation by a_K on A_K. Then, for $a, b \in \Phi_A$,*

$$-\gamma(a,b) = s(a) + s(b) - s(a+b) = \mathrm{ord}_b\, f_a - \mathrm{ord}_0\, f_a + s(0),$$

where 0 *indicates the identity in Φ_A.*

In particular, if $x \in \Phi_{A'}$ is the image of $\mathcal{L}_K$ under the projection $A'_K(K) \longrightarrow \Phi_{A'}$, then, replacing the section s by $s' = s - s(0)$, the image of x with respect to the pairing map $\Phi_{A'} \longrightarrow$ $\mathrm{Ext}^1(\Phi_A, \mathbb{Z})$ *consists of the extension given by the cocycle γ', where*

$$-\gamma'(a,b) = s'(a) + s'(b) - s'(a+b) = \mathrm{ord}_b\, f_a - \mathrm{ord}_0\, f_a.$$

Proof. As a primitive $\mathbb{G}_{\mathrm{m},K}$-torsor, $\mathcal{L}_K$ is equipped with the structure of an extension of A_K by $\mathbb{G}_{\mathrm{m},K}$. The multiplication on $\mathcal{L}_K$ is a composition of maps

$$\mathcal{L}_K \times \mathcal{L}_K \longrightarrow p_1^*\mathcal{L}_K \otimes p_2^*\mathcal{L}_K \xrightarrow{\sim} \mu^*\mathcal{L}_K \longrightarrow \mathcal{L}_K,$$

where $p_1, p_2\colon A_K \times A_K \longrightarrow A_K$ are the two projections and $\mu\colon A_K \times A_K \longrightarrow A_K$ is the multiplication map. The first map in the composition is the canonical map to the tensor product, the last one the canonical projection, and the middle one is the actual "multiplication map", derived from the condition of $\mathcal{L}_K$ being primitive. This isomorphism involves a certain choice and is determined up to a global section in $\mathcal{O}^*_{A_K}$, i. e., up to a constant in K^*.

Now choose a point $a_K \in A_K(K)$ representing $a \in \Phi_A$. Writing $\mathcal{L}_K(a_K)$ for the fiber of $\mathcal{L}_K$ over a_K and restricting first factors to $\mathcal{L}_K(a_K)$, we see that multiplication on $\mathcal{L}_K$ by points in $\mathcal{L}_K(a_K)$ is given by the first row of the following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{L}_K(a_K) \times \mathcal{L}_K & \longrightarrow & p_1^*\mathcal{L}_K \otimes p_2^*\mathcal{L}_K|_{\{a_K\}\times A_K} & \longrightarrow & T^*_{a_K}\mathcal{L}_K & \longrightarrow & \mathcal{L}_K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{a_K\} \times A_K & = & \{a_K\} \times A_K & \xrightarrow{p_2} & A_K & \xrightarrow{T_{a_K}} & A_K
\end{array}$$

whose right square is cartesian. In order to prove the assertion of the theorem, we need to know how this composition behaves with respect to order functions. The right isomorphism is obtained from pull-back with respect to translation by a_K. So, in terms of $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsors, it extends to an isomorphism[2]

$$\left[T_{a_{\mathfrak{R}}}^{-1} D\right] \xrightarrow{\sim} [D]$$

over the translation $T_{a_{\mathfrak{R}}}$ on the Néron model A, where $a_{\mathfrak{R}} \in A(\mathfrak{R})$ is the point induced from a_K. In particular, this map maintains orders if, on $T_{a_K}^* \mathcal{L}_K$, we base these on the $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor $T_{a_{\mathfrak{R}}}^* \overline{\mathcal{L}}$.

On the left-hand side of the above diagram, we identify $\{a_K\} \times A_K$ with A_K and $p_1^* \mathcal{L}_K \otimes p_2^* \mathcal{L}_K|_{\{a_K\}\times A_K}$ with $\mathcal{L}_K$, which is possible since $p_1^* \mathcal{L}_K|_{\{a_K\}\times A_K}$ is trivial. Then the resulting map $\mathcal{L}_K(a_K) \times \mathcal{L}_K \longrightarrow \mathcal{L}_K$ over $\{a_K\} \times A_K = A_K$ is bi-additive on orders, basing the definition of these on the $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$. Thus, it remains to discuss the middle map of the above diagram, which we can now view as an isomorphism

$$\varphi\colon \mathcal{L}_K = [D_K] \xrightarrow{\sim} \left[T_{a_K}^{-1} D_K\right] = T_{a_K}^* \mathcal{L}_K$$

of $\mathbb{G}_{\mathrm{m},K}$-torsors on A_K; note that $\mathcal{L}_K$ still carries orders induced from $\overline{\mathcal{L}} = [D]$, whereas on $T_{a_K}^* \mathcal{L}_K$ we consider orders derived from $T_{a_{\mathfrak{R}}}^* \overline{\mathcal{L}} = [T_{a_{\mathfrak{R}}}^{-1} D]$. To abbreviate, let us write $\mathcal{L}' = T_{a_{\mathfrak{R}}}^* \mathcal{L}$ and $\overline{\mathcal{L}}' = T_{a_{\mathfrak{R}}}^* \overline{\mathcal{L}}$. Then φ induces an isomorphism of $\mathbb{Z}$-torsors $\tilde{\varphi}\colon \Phi_{\mathcal{L}} \xrightarrow{\sim} \Phi_{\mathcal{L}'}$, and we claim that the formula

$$\operatorname{ord}_{\overline{\mathcal{L}}}(s(a) + s(b)) = \operatorname{ord}_{\overline{\mathcal{L}}'} \tilde{\varphi}(s(b)) = \operatorname{ord}_b f_a$$

holds for a particular rational function $f_a \in K(A_K)$ having divisor $T_{a_K}^{-1} D_K - D_K$.

To be more precise, switch to invertible sheaves and recall the fact we have used already, that, on schemes X we are considering, there is a bijective correspondence between invertible sheaves, line bundles, and $\mathbb{G}_{\mathrm{m}}$-torsors. Namely, to an invertible sheaf $\mathcal{I}$ associate the line bundle $\operatorname{Spec} S(\mathcal{I})$ corresponding to the symmetric $\mathcal{O}_X$-algebra $S(\mathcal{I})$ of $\mathcal{I}$, and to pass from line bundles to $\mathbb{G}_{\mathrm{m}}$-torsors, just remove the zero section. In particular, the functor from invertible sheaves to line bundles or $\mathbb{G}_{\mathrm{m}}$-torsors is contravariant.

Now let $[D_K]^{\mathrm{in}}$ and $[T_{a_K}^{-1} D_K]^{\mathrm{in}}$ be the invertible sheaves of rational functions in $K(A_K)$ which are canonically attached to D_K and $T_{a_K}^{-1} D_K$.

---

[2] A word on notation: Given a Weil or Cartier divisor D on a regular noetherian scheme X, the corresponding divisor class modulo linear equivalence is denoted by $[D]$, as usual. However, as done below, when it is convenient and poses no problems, we will make no difference between $[D]$ and other constructs associated to $[D]$, like the associated line bundle, $\mathbb{G}_{\mathrm{m}}$-torsor, or invertible sheaf.

Then φ corresponds to an isomorphism

$$\varphi^{\text{in}}\colon \left[T_{a_K}^{-1} D_K\right]^{\text{in}} \xrightarrow{\sim} [D_K]^{\text{in}},$$

and the latter consists of multiplication by a certain rational function $f_a \in K(A_K)$ having divisor $T_{a_K}^{-1} D_K - D_K$. To justify the above claimed formula, just observe that a local generator g of $[D_K]^{\text{in}}$ in a neighborhood of some point $c_K \in A_K(K)$, whose closure in A is disjoint from D, will extend to a local generator of $[D]^{\text{in}}$ in a neighborhood of the corresponding point of $A(\mathfrak{R})$ if and only if we have $\text{ord}_c\, g = 0$, where $c \in \Phi_A$ is induced from c_K. The corresponding fact is true for $[T_{a_K}^{-1} D_K]^{\text{in}}$ and $[T_{a_{\mathfrak{R}}}^{-1} D]^{\text{in}}$, and we thereby see that changes of orders under the map φ are realized by addition of the orders which f_a assumes on the components of Φ_A. This is precisely the assertion of the formula, due to the fact that $\text{ord}_{\overline{\mathcal{L}}}\, s(b) = 0$. Note also that $\text{ord}_0\, f_a = s(0)$ for $b = 0$, as multiplication with $s(0) \in \mathbb{Z}$ has the effect of adding $s(0)$ to orders on $\Phi_{\mathcal{L}}$.

Recalling again the fact that the map φ records changes in $\overline{\mathcal{L}}$-order on $\mathcal{L}_K$ under multiplication by points in $\mathcal{L}_K(a_K)$ having trivial $\overline{\mathcal{L}}$-order, we get

$$\begin{aligned} -\gamma(a, b) &= s(a) + s(b) - s(a+b) = \text{ord}_{\overline{\mathcal{L}}}(s(a) + s(b)) \\ &= \text{ord}_b\, f_a = \text{ord}_b\, f_a - \text{ord}_0\, f_a + s(0). \end{aligned}$$

Certainly, the value of $\text{ord}_b\, f_a - \text{ord}_0\, f_a$ remains unchanged if f_a is replaced by any multiple $t f_a$ with $t \in K^*$. Therefore it follows that γ is as stated in the assertion of the theorem. By its definition, the cocycle γ' differs from γ by a coboundary and, thus, both give rise to the same cohomology class in $H^2(\Phi_A, \mathbb{Z})_s$. □

Remark 3.4. If, in the situation of the above proof, we are given a primitive $\mathbb{G}_{m,K}$-torsor $\mathcal{L}_K$ on A_K and a $\mathbb{G}_{m,\mathfrak{R}}$-torsor $\overline{\mathcal{L}}$ extending it, there is some freedom in choosing the isomorphism $p_1^*\mathcal{L}_K \otimes p_2^*\mathcal{L}_K \xrightarrow{\sim} \mu^*\mathcal{L}_K$ giving rise to the structure of $\mathcal{L}_K$ as an extension of A_K by $\mathbb{G}_{m,K}$. In fact, this isomorphism can be scaled in such a way that the unit section of $\mathcal{L}_K$ is positioned at a place where it extends to an $\mathfrak{R}$-valued point of $\overline{\mathcal{L}}$. This implies $s(0) = 0$ and has the effect that then γ', as occurring in 3.3, coincides with γ. Thus, γ' may be viewed as a cocycle, which is canonically attached to $\overline{\mathcal{L}}$ or to the section s.

For any element $x \in \Phi_{A'}$, we have described in 3.3 its image in $\text{Ext}^1(\Phi_A, \mathbb{Z})$ under Grothendieck's pairing map. We now want to consider the full pairing morphism

$$\Phi_{A'} \longrightarrow \text{Ext}^1(\Phi_A, \mathbb{Z}) \xleftarrow{\sim} \text{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z})$$

and specify the image of x as an element $\varphi_x \in \text{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z})$. To do this, recall that the isomorphism on the right is obtained from the long Ext sequence associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

using the fact that $\mathrm{Hom}(\Phi_A, \mathbb{Q})$ and $\mathrm{Ext}^1(\Phi_A, \mathbb{Q})$ are trivial.[3]

Lemma 3.5. *For $x \in \Phi_{A'}$ consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Phi_x & \xrightarrow{q} & \Phi_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \| & & \\ 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \tilde{\Phi}_x & \xrightarrow{\tilde{q}} & \Phi_A & \longrightarrow & 0, \end{array}$$

where the upper row is the extension associated to x in the sense of 3.3 *and the lower one is its push-out via $\mathbb{Z} \longrightarrow \mathbb{Q}$. Then:*

(i) *The lower row of the diagram splits via a unique additive section $\tilde{s}\colon \Phi_A \longrightarrow \tilde{\Phi}_x$; let $\tilde{p} = \mathrm{id} - \tilde{s} \circ \tilde{q} : \tilde{\Phi}_x \longrightarrow \mathbb{Q}$ be the associated projection.*

(ii) *Choosing a (set-theoretic) section $s\colon \Phi_A \longrightarrow \Phi_x$ of $q\colon \Phi_x \longrightarrow \Phi_A$ and an integer $n > 0$ satisfying $n \cdot \Phi_A = 0$, the splitting is given by*

$$\tilde{s}(a) = (\iota \circ s)(a) - \frac{1}{n}(n \cdot (\iota \circ s)(a)), \qquad a \in \Phi_A,$$

where $n \cdot s(a)$ is an element in $\mathbb{Z}$ and as such is uniquely divisible by n in the image of $\mathbb{Q}$ in $\tilde{\Phi}_x$.

(iii) *The upper row of the above diagram is the pull-back of the short exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

via the homomorphism

$$\varphi_x\colon \Phi_A \xrightarrow{s} \Phi_x \xrightarrow{\iota} \tilde{\Phi}_x \xrightarrow{\tilde{p}} \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

where, for $a \in \Phi_A$,

$$\varphi_x(a) = \frac{1}{n}(n \cdot (\iota \circ s)(a)) = \frac{1}{n}(n \cdot s(a)) \mod \mathbb{Z}.$$

In particular, the pairing morphism $\Phi_{A'} \longrightarrow \mathrm{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z})$ maps x to φ_x.

---

[3] Note that the isomorphism $\mathrm{Ext}^1(\Phi_A, \mathbb{Z}) \xleftarrow{\sim} \mathrm{Hom}(\Phi_A, \mathbb{Q}/\mathbb{Z})$ and, likewise, the definition of Grothendieck's pairing, involves a certain choice of sign. As we are viewing $\mathrm{Ext}^1(\Phi_A, \mathbb{Z})$ as the group of extensions of Φ_A by $\mathbb{Z}$, the isomorphism is supposed to attach to a homomorphism $\varphi\colon \Phi_A \longrightarrow \mathbb{Q}/\mathbb{Z}$ the pull-back of $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ with respect to φ.

Proof. Assertion (i) follows from the fact that $\mathrm{Ext}^1(\Phi_A, \mathbb{Q})$ and $\mathrm{Hom}(\Phi_A, \mathbb{Q})$ are trivial. Furthermore, $\iota \circ s$ is a set-theoretic section of $\tilde{q}$. To any set-theoretic section t of $\tilde{q}$ is associated the additive section $t - \frac{1}{n}(n \cdot t)$, where $n \cdot t$ takes values in the image of $\mathbb{Q}$ only. As $q(n \cdot s(a)) = n \cdot q(s(a)) = 0$, we have $n \cdot s(a) \in \mathbb{Z}$, and (ii) follows.

Thus, it remains to justify assertion (iii). First observe that φ_x is a homomorphism as, indeed, $s(a+b) - s(a) - s(b)$ belongs to $\mathbb{Z}$ for all $a, b \in \Phi_A$. Then consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Phi_x & \longrightarrow & \Phi_A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \tilde{p} \circ \iota} & & \downarrow{\scriptstyle \varphi_x} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

which is commutative. It is easy to check that Φ_x satisfies the properties of a fibered product of $\mathbb{Q}$ and Φ_A over $\mathbb{Q}/\mathbb{Z}$, say in the category of sets and, thus, also in the category of groups. But then the upper row of the preceding diagram is the pull-back with respect to φ_x of the lower one, as claimed in (iii). Finally, the formula for φ_x follows from (ii) and the equation $\tilde{p} = \mathrm{id} - \tilde{s} \circ \tilde{q}$. □

Instead of describing φ_x in 3.5 (iii) via a section $s \colon \Phi_A \longrightarrow \Phi_x$, we can just as well use cocycles of the type derived in 3.3.

Lemma 3.6. *As in* 3.5*, consider an extension*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Phi \xrightarrow{q} \Phi_A \longrightarrow 0$$

of a finite abelian group Φ_A *and a section* $s \colon \Phi_A \longrightarrow \Phi$ *of* $q \colon \Phi \longrightarrow \Phi_A$. *Set*

$$\gamma(a,b) = s(a+b) - s(a) - s(b), \qquad a, b \in \Phi_A,$$

and let $n > 1$ *be an integer such that* $n \cdot \Phi_A = 0$. *Then*

$$n \cdot s(a) = -\sum_{i=0}^{n-1} \gamma(a, i \cdot a), \qquad a \in \Phi_A.$$

Proof. For integers $i > 0$ we have

$$s(i \cdot a) = s(a) + s((i-1) \cdot a) + \gamma(a, (i-1) \cdot a)$$

and, hence,

$$s(0) = s(n \cdot a) = n \cdot s(a) + \sum_{i=1}^{n-1} \gamma(a, i \cdot a).$$

As $s(0) = -\gamma(a, 0)$, the assertion follows. □

Now we can combine 3.3, 3.5, and 3.6, in order to obtain an explicit description of Grothendieck's pairing.

Theorem 3.7. *As before, consider Grothendieck's pairing*

$$\langle\cdot,\cdot\rangle\colon \Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

associated to an abelian variety A_K *and its dual* A'_K. *Let* $n > 0$ *be an integer satisfying* $n \cdot \Phi_A = 0$.

For $a \in \Phi_A$ *and* $x \in \Phi_{A'}$, *fix respresentatives* $a_K \in A_K(K)$ *and* $[D_K] \in A'_K(K)$, *where* D_K *is a divisor on* A_K. *Let* $f_a \in K(A_K)$ *be a rational function with divisor* $\operatorname{div}(f_a) = T_{a_K}^{-1}(D_K) - D_K$, *where* T_{a_K} *is the translation by* a_K *on* A_K. *Then*

$$\langle a, x\rangle = \frac{1}{n}\sum_{i=1}^{n-1}(\operatorname{ord}_{i\cdot a} f_a - \operatorname{ord}_0 f_a) \mod \mathbb{Z}.$$

Proof. We use the section $s'\colon \Phi_A \longrightarrow \Phi_x$ of 3.3, as well as the associated cocycle γ' given by $\gamma'(a,b) = s'(a+b) - s'(a) - s'(b)$ for $a, b \in \Phi_A$. Then, due to 3.5 and 3.6, we get

$$\begin{aligned}
\langle a, x\rangle = \varphi_x(a) &= \frac{1}{n}(n \cdot s'(a)) && \mod \mathbb{Z}\\
&= -\frac{1}{n}\sum_{i=0}^{n-1}\gamma'(a, i\cdot a) && \mod \mathbb{Z}\\
&= \frac{1}{n}\sum_{i=0}^{n-1}(\operatorname{ord}_{i\cdot a} f_a - \operatorname{ord}_0 f_a) && \mod \mathbb{Z}\\
&= \frac{1}{n}\sum_{i=1}^{n-1}(\operatorname{ord}_{i\cdot a} f_a - \operatorname{ord}_0 f_a) && \mod \mathbb{Z}.
\end{aligned}$$

□

Remark 3.8. Alternatively, the pairing of 3.7 is described by

$$\langle a, x\rangle = \frac{1}{n}\sum_{i=0}^{n-1}\operatorname{ord}_{i\cdot a} f_a \mod \mathbb{Z}.$$

4. Grothendieck's pairing and Néron's local symbols

We start this section by recalling some basic facts about Néron's height functions and the attached local symbols, in order to be able to express the values of Grothendieck's pairing in terms of certain values of Néron's symbols. For reference we will use Néron's original article [Nér], as well as [Lan1], Chaps. 10 and 11. Our notation will be similar to that of [Lan1].

As before, let K be the field of fractions of a strictly henselian discrete valuation ring $\mathfrak{R}$, and let v be the valuation on K, normalized in such a way that the value group of K is $\mathbb{Z}$. Fixing an algebraic closure K^a of K, there is a unique extension of ν to K^a, again denoted by ν. We shall assume, unless otherwise indicated, that for any finite extension L/K, we have $[L : K] = [L_\nu : K_\nu]$, where L_ν and K_ν denote the completions of L and K with respect to ν. In other words, the absolute value on K corresponding to ν is required to be *well behaved* in the terminology of [Nér], Chap. I.1, or [Lan1], Chap. 1, page 14, a notion which corresponds to the notion of *weakly stable* field in [BGR], 3.5.2. A general ν will not enjoy this property, even if the residue field of the strictly henselian discrete valuation ring $\mathfrak{R}$ is algebraically closed. However, ν is well behaved if $\mathfrak{R}$ is excellent, for example, if K is complete, or if $\operatorname{char} K = 0$. The assumption that K is well behaved is needed when dealing with Néron's symbols. In later results, where we derive consequences for Grothendieck's pairing, it can be removed by passing to the completion of K.

Given a smooth proper and geometrically irreducible K-scheme X_K, let us write $\operatorname{Div}_a(X_K)$ for the group of (Cartier) divisors on X_K which are algebraically equivalent to 0; by definition, such divisors are rational over K, using the terminology of [Nér] or [Lan1]. A Weil function on X_K with divisor $D_K \in \operatorname{Div}_a(X_K)$ is a map

$$\lambda_{D_K} : (X_K - \operatorname{supp} D_K)(K^a) \longrightarrow \mathbb{R}$$

satisfying the following condition: If D_K is represented by a rational function f on some open subset $U_K \subset X_K$, there is a locally bounded continuous function

$$\alpha : U_K(K^a) \longrightarrow \mathbb{R}$$

such that, for any K^a-valued point x of $U_K - \operatorname{supp} D_K$, we have

$$\lambda_{D_K}(x) = \nu(f(x)) + \alpha(x).$$

In this context, locally bounded means bounded on any bounded subset in the sense of [Lan1], Chap. 10, §1, p. 250, and continuous is meant with respect to the ν-topology, *op. cit.* p. 251. For example, any non-trivial rational function $f \in K(X_K)$ determines a Weil function λ_f on X_K, given by

$$\lambda_f(x) = \nu(f(x)).$$

Writing Γ for the group of constant functions $X_K(K^a) \longrightarrow \mathbb{R}$, Néron's height functions on X_K are characterized as follows; see [Nér], Chap. II.8, Thm. 2, or [Lan1], Chap. 11, Thm. 3.1.

Theorem 4.1 (Néron). *For any smooth projective and geometrically irreducible K-scheme X_K and any divisor $D_K \in \operatorname{Div}_a(X_K)$, there exists a Weil function λ_{D_K} on X_K with divisor D_K which satisfies the following conditions:*

(i) *If* $D_K, D'_K \in \mathrm{Div}_a(X_K)$, *then* $\lambda_{D_K+D'_K} \equiv \lambda_{D_K} + \lambda_{D'_K} \mod \Gamma$.
(ii) *If* D_K *is principal, say* $D_K = (f)$, *then* $\lambda_{D_K} \equiv \lambda_f \mod \Gamma$.
(iii) *If* $\varphi\colon X_K \longrightarrow Y_K$ *is a* K*-morphism of* K*-schemes of the mentioned type, and if* $D'_K \in \mathrm{Div}_a(Y_K)$ *is such that* $D_K = \varphi^{-1}(D'_K)$ *is defined, then*

$$\lambda_{D_K} \equiv \lambda_{D'_K} \circ \varphi \mod \Gamma.$$

The Weil function λ_{D_K} *is unique mod* Γ. *It will be called a Néron function on* X_K *with divisor* D_K.

Néron functions are used to define Néron's local height pairing at v as follows. For X_K as above, let $Z_0(X_K)$ be the group of zero cycles of degree 0 on X_K (cycles called 'rational over K' in [Nér] or [Lan1]). Then, identifying any prime zero cycle $\mathfrak{a}_K$ (= closed point) of X_K with the induced zero cycle $\mathfrak{a}_K \otimes_K K^a$ on $X_K \otimes K^a$, we can write any element $\mathfrak{a}_K \in Z_0(X_K)$ in the form $\mathfrak{a}_K = \sum_{i=1}^r n_i z_i$ with K^a-valued points z_i of X_K, where the $n_i \in \mathbb{Z}$ satisfy $\sum_{i=1}^r n_i = 0$ and the expression $\sum_{i=1}^r n_i z_i$ is invariant under the action of the Galois group of K^a/K. For any such $\mathfrak{a}_K$ and any $D_K \in \mathrm{Div}_a(X_K)$ with support disjoint from the support of $\mathfrak{a}_K$, we set

$$(\mathfrak{a}_K, D_K) := \lambda_{D_K}(\mathfrak{a}_K) = \sum_{i=1}^r n_i \lambda_{D_K}(z_i),$$

where λ_{D_K} is a Néron function with divisor D_K. We call (,) Néron's local symbol at v.[4]

Corollary 4.2. *Néron's local symbol* (,) *has the following properties:*

(i) $(\mathfrak{a}_K, D_K)$ *is bilinear in* $\mathfrak{a}_K$ *and* D_K.
(ii) *For* $\mathfrak{a}_K = \sum_{i=1}^r n_i z_i \in Z_0(X_K)$ *with* $z_i \in X_K(K^a)$ *and* $D_K = \mathrm{div}\, f$ *with* $f \in K(X_K)^*$, *one has*

$$(\mathfrak{a}_K, D_K) = \sum_{i=1}^r n_i v(f(z_i)).$$

(iii) *If* $\varphi\colon X'_K \longrightarrow X_K$ *is a* K*-morphism, then*

$$(\varphi(\mathfrak{a}'_K), D_K) = (\mathfrak{a}'_K, \varphi^{-1}(D_K))$$

for any zero cycle $\mathfrak{a}'_K$ *on* X'_K *and any divisor* D_K *on* X_K *such that both sides are defined. The latter requires that* $\varphi(\mathfrak{a}'_K)$ *is disjoint from* D_K, *and that we have* $\varphi(X'_K) \not\subset D_K$, *in which case,* $\varphi^{-1}(D_K)$ *is a well-defined divisor.*

[4] Actually, Néron considers symbols of type $(D_K, \mathfrak{a}_K)$, whereas we have chosen to reverse the order of arguments. We thereby avoid a switching of arguments in all formulas describing Grothendieck's pairing in terms of Néron's symbols.

On curves, Néron's symbol can be described via intersection theory on regular models. This fact will be used as a key ingredient for the computation of Grothendieck's pairing on Jacobians. The interpretation of Néron's symbol in terms of intersection theory has been explained by Gross [Gro] over local fields, whereas the more general version we will need is attributed to Hriljac [Hr]; see [Lan2], Chap. III, Thm. 5.2. For the convenience of the reader, we have included below a direct proof of the statement needed for our main result (Thm. 4.6).

Theorem 4.3. *Let X be a flat proper $\mathfrak{R}$-scheme which is regular and whose generic fiber X_K is smooth and geometrically irreducible. Write $C_1, \dots, C_v$ for the irreducible components of the special fiber X_k of X, and $M = ((C_i \cdot C_j))_{i,j=1,\dots,v}$ for the associated intersection matrix. For any divisor D on X, let*

$$\rho([D]) := ((D \cdot C_i))_{i=1,\dots,v} = (\deg_{C_i}[D])_{i=1,\dots,v}$$

be the vector of degrees on the components C_i of X_k. Then, identifying $Z_0(X_K)$ with $\mathrm{Div}_a(X_K)$, Néron's symbol (D_K, D'_K) on X_K for divisors $D_K, D'_K \in \mathrm{Div}_a(X_K)$ with disjoint supports is given by

$$\big(D_K\ ,\ D'_K\big) = -(A \cdot D') + (D \cdot D') \in \mathbb{Q}.$$

In this formula, D and D' are the schematic closures of D_K and D'_K in X, and $A \in \sum_{i=1}^{v} \mathbb{Q} \cdot C_i = \mathbb{Q}^v$ is a rational divisor on X satisfying $\rho([D]) = \rho([A]) = MA$. That such a divisor A always exists follows, for instance, from [BLR], 9.5/10.

Proof. We define a symbol $[D_K, D'_K]$ for $D_K, D'_K \in \mathrm{Div}_a(X_K)$ with disjoint supports by the formula

$$\big[D_K\ ,\ D'_K\big] := -(A \cdot D') + (D \cdot D'),$$

where A, D, D' are as described in the statement of the theorem. The symbol is well-defined, as the kernel of M consists of multiples of the divisor "special fiber" of X. Let us show that [,] coincides with Néron's symbol (,) using the criterion of [Lan1], Chap. 11, Thm. 3.7. To do this, we must check the following conditions:

(1) *The symbol* [,] *is bilinear.*
(2) *If D_K is principal, say $D_K = (f)$, then $[D_K, D'_K] = \nu(f(D'_K))$.*
(3) *The symbol is symmetric; i. e., $[D_K, D'_K] = [D'_K, D_K]$.*
(4) *Let $\tau(D_K, D'_K) := [D_K, D'_K] - (D_K, D'_K)$. Then for D_K fixed and $\deg^+(D'_K)$ bounded, the values $\tau(D_K, D'_K)$ are bounded; $\deg^+(D'_K)$ is the degree of the positive part of D'_K.*

Obviously, the symbol is bilinear. To establish condition (2), let D, D' be the schematic closures of D_K, D'_K, and view $f \in K(X_K)$ as a rational function on X. Then its divisor is

$$\operatorname{div}_X f = D - A \qquad \text{with} \qquad A = -\sum_{i=1}^{v} \operatorname{ord}_{C_i}(f) C_i,$$

where we have written ord_{C_i} for the extension of ν corresponding to the valuation ring $\mathcal{O}_{X,C_i}$ of $K(X)$. Since $\rho([\operatorname{div}_X f]) = 0$, it follows that

$$\rho([D]) = \rho([A]) = MA$$

and, by definition,

$$[D_K, D'_K] = \sum_{i=1}^{v} \operatorname{ord}_{C_i}(f)(C_i \cdot D') + (D \cdot D') = ((\operatorname{div}_X f) \cdot D').$$

To compute such an intersection multiplicity, assume that D' is a prime divisor. Then the support of D' consists of a point $x_K \in X_K$ and of a unique point $x_k \in X_k$, since $\mathfrak{R}$ is henselian. Choosing an affine open neighborhood $U \subset X$ of x, let $\mathfrak{p} \subset \mathcal{O}_X(U)$ be the prime ideal corresponding to D' and $\mathfrak{p}_K$ its extension to $\mathcal{O}_X(U_K)$. There is a canonical commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_X(U) & \xrightarrow{\sigma} & \mathcal{O}_X(U)/\mathfrak{p} & =: R' \hookrightarrow R'^{\text{nor}} \\
\downarrow & & \downarrow & \\
\mathcal{O}_X(U_K) & \xrightarrow{\sigma_K} & \mathcal{O}_X(U_K)/\mathfrak{p}_K & =: K'
\end{array}$$

with vertical injections, where R' is a local ring with maximal ideal corresponding to x_k, and where R'^{nor} is the normalization of R' in its field of fractions K'. We denote by $d = [K' : K]$ the degree of the extension K'/K, by $e(K'/K)$ its ramification index and by $f(K'/K)$ its residue degree, so that $d = e(K'/K) f(K'/K)$, due to the fact that the valuation on K is well behaved. For any element $h \in \mathcal{O}_X(U)$ we have $\nu(h(D'_K)) = d \cdot \nu'(\sigma_K(h))$, where ν' is the unique extension of ν to K'. On the other hand, as we can interpret R'^{nor} as the valuation ring corresponding to ν' and as

$$\operatorname{len} R'/(\sigma(h)) = \operatorname{len} R'^{\text{nor}}/(\sigma(h))$$

(see the beginning of [BLR], 9.1), we get

$$\begin{aligned}
((\operatorname{div}_X h) \cdot D')_{x_k} &= f(K'/K) \cdot \operatorname{len} R'/(\sigma(h)) \\
&= e(K'/K) f(K'/K) \cdot \nu'(\sigma(h)) = d \cdot \nu'(\sigma(h)),
\end{aligned}$$

which shows $((\operatorname{div}_X h) \cdot D')_{x_k} = \nu(h(D'_K))$. Applying this reasoning to f and f^{-1} on suitable affine open neighborhoods of closed points in X_k belonging to the support of $\operatorname{div}_X f$, condition (2) follows.

Next we verify condition (3). Again, let D, D' be the schematic closures of D_K, D'_K in X, and let $A, B \in \sum_{i=1}^{v} \mathbb{Q} \cdot C_i = \mathbb{Q}^v$ satisfy $\rho([D]) = MA$ and $\rho([D']) = MB$. Then

$$\begin{aligned}(A \cdot D') &= {}^tA \cdot \rho([D']) = {}^tA \cdot M \cdot B \\ &= {}^tB \cdot M \cdot A = {}^tB \cdot \rho([D]) = (B \cdot D)\end{aligned}$$

and, as the intersection symbol $(D \cdot D')$ is commutative, we get

$$[D_K, D'_K] = -(A \cdot D') + (D \cdot D') = -(B \cdot D) + (D' \cdot D) = [D'_K, D_K],$$

as required.

It remains to justify the boundedness in condition (4). To do this, we look at divisors $D_K, D'_K \in \mathrm{Div}_a(X_K)$ and consider their schematic closures D, D' on X, assuming that D_K and D are fixed. Furthermore, let $A \in \sum_{i=1}^{v} \mathbb{Q} \cdot C_i = \mathbb{Q}^v$ satisfy $\rho([D]) = MA$ so that, for variable D'_K, we have

$$[D_K, D'_K] = -(A \cdot D') + (D \cdot D') = -{}^tA \cdot \rho([D']) + (D \cdot D').$$

In order to show that $(A \cdot D')$ is bounded if $\deg^+ D'_K$ is bounded, let D'^+_K be the positive part of D'_K and D'^+ its schematic closure on X. Then the degree of D'^+ is constant on Spec R by [BLR], 9.1/2, and, on the special fiber, it is the sum of all products $r_i \deg_{C_i}[D'^+]$ by [BLR], 9.1/5, where r_i is the multiplicity of C_i in X_k. In particular, all components of $\rho([D'^+])$ and, consequently, $(A \cdot D'^+)$, are bounded if $\deg^+ D'_K$ is bounded. In the same way one can proceed with the negative part of D'_K.

The intersection multiplicity $(D \cdot D')$ is not bounded for variable D', but it will compensate against a certain part of (D_K, D'_K). To justify this, let us consider an affine open convering $(U_i)_{i=1\dots n}$ of X together with rational functions f_i on U_i, such that the collection $(U_i, f_i)_{i=1\dots n}$ represents the divisor D on X. Let $E_i \subset X(K^a)$ be the subset of those K^a-valued points which extend to integral points of U_i, with values in the valuation ring of K^a. In particular, each E_i is bounded in $U_{i,K}$ and we have $X_K(K^a) = \bigcup_{i=1}^{n} E_i$, since X is proper. Furthermore, let $\alpha_i : U_{i,K}(K^a) \longrightarrow \mathbb{R}$ be locally bounded continuous functions such that the collection $(U_{i,K}, f_i, \alpha_i)_{i=1\dots n}$ represents the Néron divisor corresponding to the Néron function f_{D_K} we have on X_K. Then each map α_i is bounded on E_i and, as

$$f_{D_K}(z) = v(f_i(z)) + \alpha_i(z) \quad \text{for } z \in E_i - (\mathrm{supp}\, D_K)(K^a),$$

the assertion of (4) will follow if we can show that $(D \cdot D') - f_i(D'_K)$ is trivial for effective divisors D'_K having support in E_i and with schematic closure D'. However, the latter is clear by our discussion of local intersection multiplicities in (2). Namely, for a prime divisor D' on X, which is induced by some point $x_K \in E_i$ specializing into a point $x_k \in X_k$, we get

$$(D \cdot D')_{x_k} = ((\mathrm{div}_X f_i) \cdot D')_{x_k} = [K(x_K) : K]v(f_i(x_K)) = v(f_i(D'))$$

from the computation of (2). □

For any abelian variety $X_K = A_K$, Néron's symbol $(\mathfrak{a}_K, D_K)$ can be related to the Néron model A of A_K; see [Nér], Chap. III.4, Thm. 1. or [Lan1], Chap. 11, Thm. 5.1. Namely, if $\mathfrak{a}_K$ is a linear combination of K-valued points of A_K, there is a decomposition

$$(\mathfrak{a}_K, D_K) = i(\mathfrak{a}_K, D_K) + j(\mathfrak{a}_K, D_K),$$

where $i(\mathfrak{a}_K, D_K)$ takes values in $\mathbb{Z}$ and, moreover, is trivial if the schematic closures of $\mathfrak{a}_K$ and D_K on A have disjoint supports. Furthermore, for any rational function $f \in K(A_K)$, one has

$$j\Big(\sum_{i=1}^{r} n_i z_i \ , \ \operatorname{div} f\Big) = \sum_{i=1}^{r} n_i \operatorname{ord}_{C(z_i)} f,$$

with $C(z_i)$ denoting the component of the special fiber of A on which z_i specializes; cf. [Nér], Chap. III.2 and, in particular, Chap. III.3, Prop. 1. As can be read from this formula in the special case of principal divisors, it follows more generally from [Nér], Chap. III.3, Prop. 2 (ii) that, for fixed D_K, the symbol $j(\mathfrak{a}_K, D_K)$ depends only on the specialization of $\mathfrak{a}_K$ on Φ_A; that is, $j\big(\sum_{i=1}^r n_i z_i \ , \ D_K\big)$ remains unchanged if each z_i is replaced by $z_i' \in A_K(K)$ such that both z_i and z_i' have the same image in Φ_A.

Using the symbol j, the formula of 3.7, which reads

$$(*) \qquad \langle a, x\rangle = \frac{1}{n}\sum_{i=1}^{n-1} j\big((i \cdot a_K) - (0) \ , \ \operatorname{div} f_a\big) \mod \mathbb{Z},$$

for any representative a_K of a, can be rewritten in a more convenient way as follows:

Theorem 4.4. *As in* 3.7*, consider Grothendieck's pairing*

$$\langle\ ,\ \rangle : \Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

associated to an abelian variety A_K and its dual A'_K. Then, for $a \in \Phi_A$ and $x \in \Phi_{A'}$, we have

$$\langle a, x\rangle = -j\big((a_K) - (0) \ , \ D_K\big) \mod \mathbb{Z},$$

where $a_K \in A_K(K)$ is a representative of a, and where D_K is a divisor on A_K such that $[D_K] \in A'_K(K)$ represents x.

Proof. We start out from the formula $(*)$, where n is a positive integer satisfying $n \cdot \Phi_A = 0$, and where

$$\operatorname{div} f_a = T_{a_K}^{-1}(D_K) - D_K.$$

Using the bilinearity and translation invariance of the symbol j, as stated in [Nér], Chap. III.3, Prop. 1, in combination with the fact that $j(\mathfrak{a}_K, D_K)$ depends only on the specialization of $\mathfrak{a}_K$ on Φ_A, we can write:

$$
\begin{aligned}
&\sum_{i=1}^{n-1} j\big((i \cdot a_K) - (0)\ ,\ \operatorname{div} f_a\big) \\
&= \sum_{i=1}^{n-1} j\big((i \cdot a_K) - (0)\ ,\ T_{a_K}^{-1}(D_K) - D_K\big) \\
&= \sum_{i=1}^{n-1} j\big(((i+1) \cdot a_K) - (a_K)\ ,\ D_K\big) \ - \ \sum_{i=1}^{n-1} j\big((i \cdot a_K) - (0)\ ,\ D_K\big) \\
&= j\big((0) - (a_K) - (n-1) \cdot (a_K) + (n-1) \cdot (0)\ ,\ D_K\big) \\
&= -\, n \cdot j\big((a_K) - (0)\ ,\ D_K\big)
\end{aligned}
$$

Indeed, to go from line 3 to line 4, we use that $n \cdot a_K$ and 0 both specialize into $0 \in \Phi_A$. Thus, the formula of 4.4 follows from $(*)$. □

Remark 4.5. Let A be a smooth $\mathfrak{R}$-group scheme of finite type and A_K its generic fiber. In [MB], II.1.1, Moret-Bailly constructs an obstruction for extending cubical line bundles from A_K to A. In our situation, where in place of A_K we consider the product $A_K \times A'_K$ of an abelian variety A_K with its dual A'_K, as well as the associated product of Néron models $A \times A'$ in place of A, the obstruction to extend the Poincaré bundle as a cubical line bundle from $A_K \times A'_K$ to $A \times A'$ corresponds to a bilinear pairing $\Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$. In II.1.1.6, Moret-Bailly suggests that it is quite likely that his pairing coincides with Grothendieck's pairing up to sign.

Furthermore, in [MB], III.1.4, Moret-Bailly expresses his pairing via Néron's symbols, and the formula he obtains amounts to the one given in our Theorem 4.4, although without the introduction of a minus sign. Thus, we can conclude from 4.4 that, in fact, Grothendieck's pairing coincides with Moret-Bailly's pairing up to sign and that, given the conventions we have used, the sign is a minus sign.

From now on we drop the assumption that the valuation on K is well behaved. Furthermore, we assume that $A_K = J_K$ is the Jacobian of a smooth proper geometrically connected curve X_K of genus g, admitting a rational point P. Let $h\colon X_K \longrightarrow J_K$, $Q \longmapsto [Q]-[P]$, be the associated map from X_K into its Jacobian. We write $\mathcal{M}$ for the universal line bundle on $X_K \times J_K$ (satisfying $\mathcal{M}|_{\{P\}\times J_K} = 0$ and $\deg \mathcal{M}|_{X_K\times\{y\}} = 0$ for all points y of J_K) and $\mathcal{P}$ for the Poincaré bundle on $J_K \times J'_K$, where J'_K is the dual of J_K. There is a unique morphism $h'\colon J'_K \longrightarrow J_K$ satisfying $(\mathrm{id} \times h')^*\mathcal{M} = (h \times \mathrm{id})^*\mathcal{P}$ on $X_K \times J'_K$. It is given by the pull-back of line bundles with respect to $h\colon X_K \longrightarrow J_K$ and is an isomorphism (see for instance [Mil], Thm. 6.9).

To describe the inverse of h', we consider the maps $h^{(i)}: X_K^{(i)} \longrightarrow J_K$, $i \in \mathbb{N}$, induced from h, where $X_K^{(i)}$ is the i-fold symmetric product of X_K. The image of $h^{(g-1)}$ gives rise to a divisor Θ on J_K, the so-called theta divisor, and one knows that the morphism

$$\varphi_{[\Theta]}: J_K \longrightarrow J'_K, \qquad a_K \longmapsto \left[T_{a_K}^{-1}\Theta\right] - [\Theta],$$

is an isomorphism. In fact, $-\varphi_{[\Theta]}$ and h' are inverse to each other by [Mil], Thm. 6.9. Also note that $\varphi_{[\Theta]}$ and, hence, h' are independent of the choice of the rational point P on X_K, as any change of P leads to a translate of Θ.

In the remainder of this paper, we will always identify J'_K with J_K using the isomorphisms $h': J'_K \longrightarrow J_K$ and its inverse $-\varphi_{[\Theta]}$. Induced by this identification is an identification of the corresponding Néron models and their component groups so that Grothendieck's pairing associated to J_K and J'_K becomes a pairing

$$\langle\ ,\ \rangle: \Phi_J \times \Phi_J \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Theorem 4.6. *Let K be the field of fractions of an arbitrary strictly henselian discrete valuation ring $\mathfrak{R}$. Let J_K be the Jacobian of a smooth proper and geometrically connected curve X_K having a rational point P. Identify J_K with its dual J'_K via the map $h': J'_K \longrightarrow J_K$ introduced above, which is given by pull-back of line bundles with respect to $h: X_K \longrightarrow J_K$, $Q \longmapsto [Q] - [P]$.*

Let X be a proper flat $\mathfrak{R}$-model of X_K which is regular. Let Λ be the diagonal matrix with entries the geometric multiplicities of the irreducible components of X_k, and let M be the intersection matrix of X_k. As in 2.3, *identify the component group Φ_J with the group $\Phi_{\Lambda,M}$.*

Then Grothendieck's pairing

$$\langle\ ,\ \rangle: \Phi_J \times \Phi_J \longrightarrow \mathbb{Q}/\mathbb{Z}$$

coincides with the pairing

$$\langle\ ,\ \rangle_{\Lambda,M}: \Phi_J \times \Phi_J \longrightarrow \mathbb{Q}/\mathbb{Z}$$

considered in 2.3.

Corollary 4.7. *Let K, $\mathfrak{R}$, X_K, and J_K be as in* 4.6. *Then Grothendieck's pairing $\langle\ ,\ \rangle$ is perfect when restricted to the prime-to-p part of $\Phi_J \times \Phi_{J'}$. Furthermore, the pairing is perfect on all of $\Phi_J \times \Phi_{J'}$ when k is algebraically closed or, more generally, when X_K has a proper flat $\mathfrak{R}$-model X which is regular and with special fiber X_k all of whose irreducible components are geometrically reduced.*

Proof. Use 4.6 in conjunction with 2.3, (iii) and (iv). □

Corollary 4.8. *Let E_K be an elliptic curve, where K is the field of fractions of an arbitrary strictly henselian discrete valuation ring $\mathfrak{R}$. If the reduction type of E_K over $\mathfrak{R}$ is a classical Kodaira type, then Grothendieck's pairing is perfect.*

Proof. The assertion follows from 4.7, since any elliptic curve over K having a classical Kodaira type as reduction, is such that all the irreducible components of the special fiber are geometrically reduced, except possibly when $p = 2$, the Kodaira type is III, and the intersection matrix is $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$. In the latter case, if some geometric multiplicity is equal to 2, then one easily checks that the associated group of components is trivial and, thus, that the pairing is perfect in this case, too. Let us add that it is possible, to provide a more direct proof of 4.8 via 3.7, without using Néron's symbols. □

When the residue field k is not perfect, there are several additional possible reduction types of elliptic curves in addition to the classical Kodaira types, and when $p = 2$, the pairing is not always perfect (see for instance [Lo5]). The additional reduction types are listed in [Szy].

Note that the statement regarding the prime-to-p part of $\Phi_J \times \Phi_{J'}$ is true in general for any abelian variety and any residue field. A proof of this statement was sketched in [Gr], and completed in [Ber].

Proof of 4.6. Let us first show that it is sufficient to prove the theorem in the case where K is complete and, thus, where the valuation is well behaved. Let $\widehat{\mathfrak{R}}$ and $\widehat{K}$ denote the completions of $\mathfrak{R}$ and K, respectively. The formation of Néron models commutes with the base change $\widehat{\mathfrak{R}}/\mathfrak{R}$ by [BLR], 7.2/2. It follows that the component groups of A_K and $A_{\widehat{K}}$ are canonically isomorphic. Using this isomorphism, we find that Grothendieck's pairing for A_K is canonically equal to Grothendieck's pairing for $A_{\widehat{K}}$; see [Gr], VIII, 7.3.5.3. Consider now a proper flat $\mathfrak{R}$-model X of X_K which is regular. Then $X \times_{\mathfrak{R}} \widehat{\mathfrak{R}}$ and X have the same special fiber since $\mathfrak{R}$ and $\widehat{\mathfrak{R}}$ have same uniformizing parameter and residue field. Furthermore $X \times_{\mathfrak{R}} \widehat{\mathfrak{R}}$ is regular. Namely, due to the properness of $X \times_{\mathfrak{R}} \widehat{\mathfrak{R}}$ over $\widehat{\mathfrak{R}}$, all closed points of $X \times_{\mathfrak{R}} \widehat{\mathfrak{R}}$ are situated on the special fiber, and completions of local rings at such points may be viewed as completions of the corresponding local rings of X. In addition, the intersection theory on X is the same as the intersection theory on $X \times_{\mathfrak{R}} \widehat{\mathfrak{R}}$. From all this it follows that Theorem 4.6 is true in general once it is proven in the case where the valuation is well behaved.

For the rest of the proof we assume that the valuation of K is well behaved. In particular, the results of this section on Néron's symbols become applicable. We fix a theta divisor Θ on J_K. As we have explained above, the map

$$\varphi_{[\Theta]}\colon J_K \longrightarrow J'_K, \qquad a_K \longmapsto \left[T_{a_K}^{-1}\Theta\right] - [\Theta],$$

is an isomorphism, and $-\varphi_{[\Theta]}$ is the inverse of the isomorphism $h'\colon J'_K \to J_K$ we are using in order to identify J'_K with J_K.

Identifying the associated Néron models and their component groups via h', Thm. 4.4 allows us to write Grothendieck's pairing on $\Phi_J \times \Phi_J$ as

$$\begin{aligned}\langle a, x\rangle &= -j\big((a_K) - (b_K)\,,\ -\big(T_{x_K}^{-1}\Theta - \Theta\big)\big) \quad \bmod \mathbb{Z}\\ &= j\big((a_K) - (b_K)\,,\ T_{x_K}^{-1}\Theta - \Theta\big) \quad \bmod \mathbb{Z},\end{aligned}$$

where $a_K, b_K, x_K \in J_K(K)$ are representatives of $a, 0, x \in \Phi_J$. Let Θ^- be the pull-back of Θ under the inverse map on J_K and let $\Theta^-_{y_K}$ be the translate of Θ^- by some point $y_K \in J_K(K)$. As $\varphi_{[\Theta]}$ coincides with $\varphi_{[\Theta^-_{y_K}]}$, we may just as well replace Θ by $\Theta^-_{y_K}$ in the above formula. Choose now $y_K \in J_K(K)$ specializing into $0 \in \Phi_J$. Then, in addition, we may also replace x_K by $x_K - y_K$, and it follows that

$$\begin{aligned}\langle a, x\rangle &= j\big((a_K) - (b_K)\,,\ T_{x_K-y_K}^{-1}\Theta^-_{y_K} - \Theta^-_{y_K}\big) \quad \bmod \mathbb{Z}\\ &= -j\big((a_K) - (b_K)\,,\ T_{-x_K+y_K}^{-1}\Theta^-_{y_K} - \Theta^-_{y_K}\big) \quad \bmod \mathbb{Z}\\ &= -j\big((a_K) - (b_K)\,,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big) \quad \bmod \mathbb{Z}\end{aligned} \tag{1}$$

for $a_K, b_K, x_K, y_K \in J_K(K)$ specializing into $a, 0, x, 0 \in \Phi_J$.

We return now to Néron's symbols of type $(\mathfrak{a}_K, D_K)$. In fact, we know that $(\mathfrak{a}_K, D_K)$ coincides with $j(\mathfrak{a}_K, D_K)$ for cycles $\mathfrak{a}_K$ with rational components if the schematic closures of the supports of $\mathfrak{a}_K$ and D_K in the Néron model J of J_K are disjoint. By reasons of dimension, such schematic closures are nowhere dense on the special fiber of J. Thus, fixing $a, x \in \Phi_J$ and representatives $x_K, y_K \in J_K(K)$ of $x, 0 \in \Phi_J$, this implies

$$j\big((a_K) - (b_K)\,,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big) = \big((a_K) - (b_K)\,,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big) \tag{2}$$

for all representatives $a_K, b_K \in J_K(K)$ of $a, 0 \in \Phi$, provided we avoid that a_K (respectively b_K) specializes into a certain lower dimensional closed subset of the component a (respectively 0). In a similar way, we can fix representatives a_K, b_K of $a, 0$ and choose the representatives x_K, y_K of $x, 0$ appropriately. Actually, it is enough to keep the supports of $(a_K) - (b_K)$ and $\Theta^-_{x_K} - \Theta^-_{y_K}$ disjoint, because then $j\big((a_K) - (b_K)\,,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big)$ differs from $\big((a_K) - (b_K)\,,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big)$ by an integer, which is of no importance when we take residue classes in $\mathbb{Q}/\mathbb{Z}$.

Next we want to use the functoriality of Néron's symbols in order to express Grothendieck's pairing on J_K via data on X_K. Consider the maps $h\colon X_K \longrightarrow J_K$, $Q \longmapsto [Q] - [P]$, and $h^{(g)}\colon X_K^{(g)} \longrightarrow J_K$ induced from h on the g-th symmetric power $X_K^{(g)}$ of X_K. There is a non-trivial open subset $U_K \subset J_K$ satisfying the following conditions; see [Mil], Lemma 6.7:

(i) *For any $z_K \in U_K(K)$, the inverse image $(h^{(g)})^{-1}(z_K)$ consists of a single point $D(z_K) \in X_K^{(g)}(K)$.*
(ii) *$h^{-1}(\Theta^-_{z_K})$ is defined as a Cartier divisor on X_K and, interpreting $D(z_K)$ as an effective Cartier divisor on X_K, we have $h^{-1}(\Theta^-_{z_K}) = D(z_K)$.*

We claim that, given $a, x \in \Phi_J$, there are representatives $a_K, b_K, x_K, y_K \in U_K(K)$ of $a, 0, x, 0 \in \Phi_J$ such that

$$(**)\qquad \begin{aligned} \langle a, x\rangle &= -j\big((a_K) - (b_K)\ ,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big) && \mod \mathbb{Z} \\ &= -\big((a_K) - (b_K)\ ,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big) && \mod \mathbb{Z} \\ &= -\big(D(a_K) - D(b_K)\ ,\ D(x_K) - D(y_K)\big) && \mod \mathbb{Z}. \end{aligned}$$

To justify this, start with representatives $a_K, b_K \in U_K(K)$ of $a, 0 \in \Phi_J$ and write

$$D(a_K) = \sum_{i=1}^{g}(a_i), \qquad D(b_K) = \sum_{i=1}^{g}(b_i)$$

with points a_i, b_i of X_K which might have values in some finite separable extension of K. Choose representatives $x_K, y_K \in U_K(K)$ such that a_K, b_K, as well as all images $\overline{a}_i = h(a_i)$ and $\overline{b}_i = h(b_i)$, do not belong to the support of the divisor $\Theta^-_{x_K} - \Theta^-_{y_K}$. Then all Néron symbols in $(**)$ are well-defined, and the first two equalities of $(**)$ are clear from (1) and (2). Furthermore, the functoriality of Néron's symbol, as stated in 4.2 (iii), yields

$$\begin{aligned} &\big(D(a_K) - D(b_K)\ ,\ D(x_K) - D(y_K)\big) \\ &= \Big(\sum_{i=1}^{g}(a_i) - (b_i)\ ,\ h^{-1}(\Theta^-_{x_K}) - h^{-1}(\Theta^-_{y_K})\Big) \\ &= \Big(\sum_{i=1}^{g}(\overline{a}_i) - (\overline{b}_i)\ ,\ \Theta^-_{x_K} - \Theta^-_{y_K}\Big). \end{aligned}$$

Now recall the following translation property of Néron's symbols ([Lan1], Chap. 11, Thm. 4.1). Let D be any divisor on J_K and $\mathfrak{a}$, $\mathfrak{b}$ any zero cycles of degree 0 on J_K. Set $D_{\mathfrak{a}} = \sum_{i=1}^{r} n_i \cdot T_{z_i} D$ if $\mathfrak{a} = \sum_{i=1}^{r} n_i \cdot (z_i)$ and write D^- for the pull-back of D under the inverse map of J_K. Assume that $\mathfrak{a}$ and $D_{\mathfrak{b}}$ have disjoint supports. Then

$$\big(\mathfrak{a}\ ,\ D_{\mathfrak{b}}\big) = \big(\mathfrak{b}\ ,\ D^-_{\mathfrak{a}}\big).$$

This relation is used to justify the first and third equalities below:

$$\begin{aligned} &\Big(\sum_{i=1}^{g}(\overline{a}_i) - (\overline{b}_i)\ ,\ \Theta^-_{x_K} - \Theta^-_{y_K}\Big) \\ &= \Big((x_K) - (y_K)\ ,\ \sum_{i=1}^{g}(\Theta_{\overline{a}_i} - \Theta) - \sum_{i=1}^{g}(\Theta_{\overline{b}_i} - \Theta)\Big) \\ &= \big((x_K) - (y_K)\ ,\ \Theta_{a_K} - \Theta_{b_K}\big) + \delta \\ &= \big((a_K) - (b_K)\ ,\ \Theta^-_{x_K} - \Theta^-_{y_K}\big) + \delta, \end{aligned}$$

for some $\delta \in \mathbb{Z}$. In addition, to pass from the second to the third line, we use the equations

$$a_K = \sum_{i=1}^{g} \overline{a}_i, \qquad b_K = \sum_{i=1}^{g} \overline{b}_i$$

and the fact that the divisors $\sum_{i=1}^{g} \Theta_{\overline{a}_i}$ and $\sum_{i=1}^{g} \Theta_{\overline{b}_i}$ are defined over K. As the divisors on the right hand sides in lines 2 and 3 differ by a principal divisor which, when evaluated on any cycle with rational components on J_K, yields values is $\mathbb{Z}$ (using 4.2 (ii)), we deduce that the congruences in $(**)$ are valid.

We now interpret the quantities occurring in the last line of $(**)$, which concern Néron's symbol on X_K, in terms of the description of Φ_J given in 2.3. To do this, we use the $\mathfrak{R}$-model X of X_K whose existence we have required. Let $C_1, \dots, C_v$ be the irreducible components of the special fiber X_k of X and $e_i = e(C_i)$, respectively $r_i = r(C_i)$, the geometric multiplicity of C_i, respectively the multiplicity of C_i in X_k. Set

$$\Lambda = \mathrm{diag}(e_1, \dots, e_v), \qquad M = (C_i \cdot C_j)_{1 \le i,j \le v}, \qquad {}^tR = (r_1, \dots, r_v),$$

and consider the maps

$$M\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}^v, \qquad {}^tR\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}, \qquad \rho\colon \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^v,$$

as well as their $\mathbb{Q}$-extensions obtained from tensoring with $\mathbb{Q}$ over $\mathbb{Z}$ where, as before, ρ is the degree map $\mathcal{L} \longmapsto (\deg_{C_i}(\mathcal{L}))_i$. Then the component group Φ_J can be canonically identified with a subgroup of the quotient $\Phi_M = \mathrm{Ker}\,({}^tR)/\,\mathrm{Im}(M)$; cf. 2.2. In fact, given any point $a_K \in J_K(K)$, its image in $\mathrm{Ker}\,({}^tR)/\,\mathrm{Im}(M)$ is constructed as follows. Choose a divisor D_K of degree 0 on X_K representing a_K and pass to the schematic closure D of D_K in X_K. Then the image of a_K in Φ_M is given by the class of $\rho([D])$ in $\mathrm{Ker}\,({}^tR)/\,\mathrm{Im}(M)$.

At this point we recall the description of Néron's symbol on X_K, as given in 4.3. For a zero cycle (or divisor) Z_K of degree 0 and a divisor D_K on X_K, we consider the schematic closure Z of Z_K in X and choose a rational divisor $A \in \sum_{i=1}^{v} \mathbb{Q} \cdot C_i = \mathbb{Q}^v$ on X such that $\rho([Z]) = \rho([A]) = MA$. Then, according to 4.3, if D is the schematic closure of D_K in X, Néron's symbol is given by

$$\big(Z_K\,,\; D_K\big) = -(A \cdot D) + (Z \cdot D).$$

Furthermore, note that

$$(Z_K\,,\; D_K) \equiv -(A \cdot D) \mod \mathbb{Z}$$

for residue classes in $\mathbb{Q}/\mathbb{Z}$, since $(Z \cdot D) \in \mathbb{Z}$, and that

$$(A \cdot D) = \sum_{i=1}^{v} c_i \cdot \deg_{C_i}[D] = {}^tA \cdot \rho([D])$$

for $A = \sum_{i=1}^{v} c_i C_i$.

Let us apply this to $Z_K = D(a_K) - D(b_K)$ and $D_K = D(x_K) - D(y_K)$ as occurring in the computation (∗∗) above, and thereby determine the value of Grothendieck's pairing $\langle a, x\rangle$. Passing from Z_K to Z and then to $T = (\deg_{C_i}[Z])_i = \rho([Z]) = MA$, we arrive at a representative $T \in \operatorname{Ker}({}^tR)$ of $a \in \Phi_J \subset \Phi_M$ and at a rational M-inverse A of T. Likewise, $T' = \rho([D])$ is a representative of $x \in \Phi_J$. Now a comparison with the pairing $\langle\ ,\ \rangle_M$ defined in Sect. 1 shows

$$\langle a, x\rangle = (A \cdot D) \bmod \mathbb{Z} = {}^tAT' \bmod \mathbb{Z} = \langle a, x\rangle_M = \langle a, x\rangle_{\Lambda,M},$$

since, by its definition, $\langle\ ,\ \rangle_{\Lambda,M}$ is the restriction of $\langle\ ,\ \rangle_M$ to $\Phi_J = \Phi_{\Lambda,M}$. □

Remark 4.9. Let X be a regular model of a smooth proper geometrically connected curve X_K. Raynaud's result gives a description of the group of components Φ_J of J when k is algebraically closed, even when X_K does not have a K-rational point. It is thus natural to wonder whether an analog of Theorem 4.6, which would describe Grothendieck's pairing on $\Phi_J \times \Phi_{J'}$ only in terms of the combinatorics of the special fiber X_k, still holds in this case. The following provides some evidence that such an analog might hold.

Assume that the residue field k is algebraically closed. Let X_K be a curve of genus 1 without a rational point. Let X over $\mathfrak{R}$ be its regular minimal model. As a divisor, the special fiber X_k is of the form mF, where $m > 1$ is an integer and F is the special fiber of the regular minimal model of some elliptic curve. Let J_K denote the Jacobian of X_K. We would like to describe Grothendieck's pairing on $\Phi_J \times \Phi_{J'}$ in terms of the special fiber X_k. An indirect way to achieve this is to proceed as follows. Let $J^{\min}$ denote the regular minimal model of J_K over $\mathfrak{R}$. Since J_K is an elliptic curve, Theorem 4.6 allows us to compute Grothendieck's pairing on $\Phi_{J'} \times \Phi_{J''}$ using $J_k^{\min}$. Since J_K is autodual, Grothendieck's pairing on $\Phi_J \times \Phi_{J'}$ is thus understood in terms of the special fiber $J_k^{\min}$. It is likely that the intersection matrices associated with F and $J_k^{\min}$ are the same, a fact which is more or less known in the function field case; see [C-D], 5.3.1. When this statement holds, we find that Grothendieck's pairing on $\Phi_J \times \Phi_{J'}$ can indeed be described in terms of the special fiber X_k.

5. Explicit examples

Let X_K be a smooth proper geometrically connected curve over a discrete valuation field K with algebraically closed residue field. Let X be a proper flat $\mathfrak{R}$-model of X_K, which is regular. When no confusion may ensue, X_K will be simply called a *curve* over K and X will be called a *regular model* of X_K. Theorem 4.6 reduces the computation of Grothendieck's pairing $\langle\ ,\ \rangle$ for the Jacobian J_K of X_K to the computation of the pairing $\langle\ ,\ \rangle_M$ of Sect. 1 associated with the intersection matrix M attached to the special fiber X_k. Thus, the computation of Grothendieck's pairing is reduced to linear algebra and can be theoretically computed explicitly in each particular case. However, when X_k is not simply connected, it becomes quite difficult to provide explicit general formulas already for the order of the group Φ_M, not to mention for values of the pairing $\langle\ ,\ \rangle_M$. On the other hand, when X_k is tree-like, then we can provide explicit formulas for the values of $\langle\ ,\ \rangle_M$ in terms of the combinatorics of X_k; this is done in 5.1. Formulae for $|\Phi_M|$ in this case can already be found in [Lo2], 1.5. In particular, assuming that the residue field k is algebraically closed, Proposition 5.1 below enables us to compute Grothendieck's pairing in the case of Jacobians J_K with potentially good reduction or, more generally, in the case where the special fiber of the Néron model J of J_K has toric rank equal to zero (see [Lo2], 1.4). Now able to compute explicit examples of pairings, we can address in 5.2 the question of the existence of Jacobians with specified dimension, group of components and pairing. Among the many examples that are worked out explicitly in this section, the reader will find in 5.8 a chart exhibiting all possible pairings attached to the Néron models of elliptic curves. Note that in the case of elliptic curves, Grothendieck's pairing can also be computed directly using 3.7.

Let X_K be a curve and X a regular model of X_K. Recall that associated with the special fiber $X_k = \sum_{i=1}^{v} r_i C_i$ of X is a triple (G, M, R), where $M \in M_v(\mathbb{Z})$ is the intersection matrix of X_k, where $R = {}^t(r_1, \dots, r_v)$ is the vector of multiplicities of the irreducible components of X_k, and where G is the following graph on vertices $C_1, \dots, C_v$: the vertex C_i is linked in G to the vertex C_j by $C_i \cdot C_j$ edges ($i \neq j$). For convenience, let us call a triple (G, M, R) an *arithmetical graph* if the following conditions hold:

- G is a connected graph with vertices $C_1, \dots, C_v$;
- $M = ((c_{ij})) \in M_v(\mathbb{Z})$ is symmetric. Its coefficient c_{ij}, $i \neq j$, is equal to the number of edges between the vertex C_i and the vertex C_j. The coefficients c_{ii} are all negative integers;
- The vector $R = {}^t(r_1, \dots, r_v)$ has positive integers as coefficients. In addition, we assume that $\gcd(r_1, \dots, r_v) = 1$ and $MR = 0$.

Let (G, M, R) be any arithmetical graph. Let $M : \mathbb{Z}^v \to \mathbb{Z}^v$ and ${}^tR : \mathbb{Z}^v \to \mathbb{Z}$ be the linear maps associated to the matrices M and R. The group of components of (G, M, R) is defined as

$$\Phi_G := \mathrm{Ker}({}^tR)/\mathrm{Im}(M) = (\mathbb{Z}^v/\mathrm{Im}(M))_{\mathrm{tors}}.$$

We denote by $\langle\ ,\ \rangle_G : \Phi_G \times \Phi_G \to \mathbb{Q}/\mathbb{Z}$ the perfect pairing $\langle\ ,\ \rangle_M$ attached in 1.1 to the symmetric matrix M. In particular, if (G, M, R) is the arithmetical graph associated to a regular model X of a curve X_K where all irreducible components of X_k have trivial geometric multiplicities, then we know from 2.3 and 4.6 that Φ_G coincides with the component group Φ_J of the Jacobian J_K of X_K and that Grothendieck's pairing $\langle\ ,\ \rangle$ coincides with $\langle\ ,\ \rangle_M$ on Φ_J.

Let us now introduce the notation needed to state our main computational result in 5.1 below. Let (G, M, R) be an arithmetical graph with v

vertices. Motivated by the case of degenerations of curves, we shall denote by $(C, r(C))$ a vertex of G, where $r(C)$ is the coefficient of R corresponding to C. The integer $r(C)$, also denoted simply by r, is called the multiplicity of C. Fix a numbering of the vertices of G, and let $\tilde{R} := \mathrm{diag}(r_1, \dots, r_v)$. Consider the arithmetical graph $(\tilde{G}, \tilde{M}, I)$, where $I := {}^t(1, \dots, 1)$ and $\tilde{M} := \tilde{R}M\tilde{R}$. In particular, the graph $\tilde{G}$ is the graph having adjacency matrix consisting of the off-diagonal entries of the matrix $\tilde{M}$. Let $\epsilon_1, \dots, \epsilon_v$ denote the standard basis of $\mathbb{Z}^v$. Let $E_{ij} := \epsilon_i - \epsilon_j$. When two vertices of G are denoted (C, r) and (C', r') without specifying a numbering i for C and j for C', we may use $E_{CC'}$ to denote the vector E_{ij}. There is always a vector $S = {}^t(s_1, \dots, s_v) \in \mathbb{Z}^v$ such that

$$(\tilde{R}M\tilde{R})S = \mu E_{CC'},$$

with $\mu \in \mathbb{Z}$, $\mu \neq 0$. Given the above equation, then, by definition, the order of $E_{CC'}$ in $\Phi_{\tilde{G}}$ divides μ. Note also that r and r' divide μ. Define

$$E(C, C') := {}^t\left(0, \dots, 0, \frac{r'}{\gcd(r, r')}, 0, \dots, 0, \frac{-r}{\gcd(r, r')}, 0, \dots, 0\right) \in \mathbb{Z}^v,$$

where the first non-zero coefficient of $E(C, C')$ is at the position corresponding to the vertex C and, similarly, the second non-zero coefficient is at the position corresponding to the vertex C'. It follows that

$$M(\tilde{R}S) = \frac{\mu}{\mathrm{lcm}(r, r')} E(C, C').$$

Let σ be the greatest common divisor of the coefficients of the vector $\tilde{R}S$. Then the order of $E(C, C')$ in Φ_G divides (and may strictly divide) $\mu/\sigma \mathrm{lcm}(r, r')$.

Let (C, r) and (C', r') be two distinct vertices of G. We say that the pair (C, C') is *uniquely connected* if there exists a path $\mathcal{P}$ in G between C and C' such that, for each edge e on $\mathcal{P}$, the graph $G - \{e\}$ is disconnected (the terminology *weakly connected* was used in [Lo4] for the same concept). Note that when a pair (C, C') is uniquely connected, then the path $\mathcal{P}$ is the unique shortest path between C and C'. A graph is a tree if and only if every pair of vertices of G is uniquely connected.

Let (C, r) and (C', r') be a uniquely connected pair with associated path $\mathcal{P}$. While walking on $\mathcal{P} - \{C, C'\}$ from C to C', label each encountered vertex consecutively by $(C_1, r_1), (C_2, r_2), \dots, (C_n, r_n)$. Let G_i denote the connected component of C_i in $G - \{\text{edges of } \mathcal{P}\}$. The graph G_i is reduced to a single vertex if and only if C_i is not a node of G. For convenience, we write $(C, r) = (C_0, r_0)$ and $(C', r') = (C_{n+1}, r_{n+1})$ and define G_0 and G_{n+1} accordingly.

Proposition 5.1. *Let (G, M, R) be any arithmetical graph. Let (C, r) and (C', r') be two vertices such that (C, C') is a uniquely connected pair of G. Let γ denote the image of $E(C, C')$ in Φ_G. For (D, s) and (D', s') any two* *distinct vertices on G, let δ denote the image of $E(D, D')$ in Φ_G. Writing $\mathcal{P}$ for the shortest path between C and C' as above, let C_α denote the vertex of $\mathcal{P}$ closest to D in G, and let C_β denote the vertex of $\mathcal{P}$ closest to D'. In other words, $D \in G_\alpha$ and $D' \in G_\beta$. Assume that $\alpha \leq \beta$. (Note that we may have $\alpha = \beta$, and we may have $D = C_\alpha$ or $D' = C_\beta$.) Then*

$$\langle \gamma, \delta \rangle_G = \mathrm{lcm}(r, r')\mathrm{lcm}(s, s') \left(\frac{1}{r_\alpha r_{\alpha+1}} + \frac{1}{r_{\alpha+1} r_{\alpha+2}} + \dots + \frac{1}{r_{\beta-1} r_\beta} \right) \mod \mathbb{Z}.$$

In particular, if $C_\alpha = C_\beta$, then $\langle \gamma, \delta \rangle = 0$. Moreover,

$$\langle \gamma, \gamma \rangle_G = \mathrm{lcm}(r, r')^2 \left(\frac{1}{r r_1} + \frac{1}{r_1 r_2} + \dots + \frac{1}{r_n r'} \right) \mod \mathbb{Z}.$$

Proof. Consider the graph $\tilde{G}$ associated to G and introduced above. Set

$$\mu := \mathrm{lcm}(r r_1, r_1 r_2, \dots, r_{n-1} r_n, r_n r').$$

The following vector $S = {}^t(s_C, s_{C_1}, \dots)$ is such that $\tilde{M} S = \mu E_{CC'}$, where

$$\begin{aligned} s_C &:= 0, \\ s_{C_1} &:= \mu / r r_1, \\ s_{C_2} &:= \mu / r r_1 + \mu / r_1 r_2, \\ &\vdots \\ s_{C_n} &:= \mu / r r_1 + \mu / r_1 r_2 + \dots + \mu / r_{n-1} r_n, \\ s_{C'} &:= s_{C_n} + \mu / r_n r', \\ s_{C_*} &:= s_{C_i}, \text{ if } C_* \text{ is any vertex of } G_i, \text{ for all } i = 0, \dots, n+1. \end{aligned}$$

We leave it to the reader to check that $\tilde{M} S = \mu E_{CC'}$. It follows that

$$M(\tilde{R} S) = \frac{\mu}{\mathrm{lcm}(r, r')} E(C, C').$$

By definition,

$$\langle \gamma, \delta \rangle_G = (\mathrm{lcm}(r, r')/\mu)\, {}^t(\tilde{R} S) E(D, D') \mod \mathbb{Z}.$$

Proposition 5.1 follows easily from this equality. □

Proposition 5.1 was successfully used in [Lo4] to compute in some cases the exact order of the image of the element $E(C, C')$ in Φ_G. When G is a tree, the order of Φ_G is computed in [BLR], 9.6/6.

Next let us look at the problem of realizing a given symmetric pairing as a pairing associated to an arithmetical graph or to the Néron model of a Jacobian. Let Φ be any finite abelian group. It is not hard to show that there exists an arithmetical graph (G, M, R) such that $\Phi_G \cong \Phi$; in fact, one can even find such a graph with $R = {}^t(1, \ldots, 1)$ (use for instance 5.7 with 5.3 or 5.4, or see [Lo3], 4.1). Given any arithmetical graph (G, M, R), Winters' Existence Theorem [Win] then implies the existence of an equicharacteristic discrete valuation field K and a curve X_K having a regular model X whose associated arithmetical graph is (G, M, R). Thus, the Jacobian J_K of X_K is an abelian variety whose Néron model has its group of components Φ_J isomorphic to Φ.

Recall the following invariant of an arithmetical graph. If C is a vertex of G, let $d(C)$ denote the degree of C in G, that is, the number of edges of G attached to C. Let $r(C)$ denote the multiplicity of C. Then define $g(G)$ by the formula

$$2g(G) - 2 = \sum_C r(C)(d(C) - 2).$$

The integer $g(G)$ is always non-negative ([Lo2], 2.2) and when X_K has a regular model with associated graph equal to (G, M, R), then $g(G)$ is at most equal to the sum of the unipotent and toric ranks of the special fiber of the Néron model of the Jacobian of X_K ([Lo2], 2.3).

Fix an integer g. Much is known about those finite abelian groups Φ which can be interpreted as the component group Φ_G associated to an arithmetical graph (G, M, R) with $g(G) = g$ (for instance, any group Φ generated by at most g elements is such a group; see also [Lo3], 4.1). Regarding the problem of realizing a given symmetric pairing as the pairing associated with an arithmetical graph, we show:

Proposition 5.2. *Suppose that a given abelian group Φ is endowed with a perfect symmetric pairing $\langle\ ,\ \rangle : \Phi \times \Phi \to \mathbb{Q}/\mathbb{Z}$. Then there exists an arithmetical graph (G, M, R) such that $\Phi_G \cong \Phi$ and $\langle\ ,\ \rangle_G$ is equivalent to $\langle\ ,\ \rangle$.*

Proof. The classification of perfect symmetric pairings on finite abelian p-groups Φ, as described for instance in [Bar], 2.1, or [Wal], Thm. 4, shows that when p is odd, Φ decomposes as an orthogonal sum of finite cyclic groups, each endowed with a perfect pairing. When $p = 2$, the classification is more complicated. Let us introduce the following perfect pairings $\langle\ ,\ \rangle_i$ and $\langle\ ,\ \rangle'_i$ on $(\mathbb{Z}/2^i\mathbb{Z})^2$, endowed with the natural $(\mathbb{Z}/2^i\mathbb{Z})$-basis $\{\epsilon, \epsilon'\}$. Using this basis, $\langle\ ,\ \rangle_i$ is given by the matrix

$$\begin{pmatrix} \langle\epsilon, \epsilon\rangle_i & \langle\epsilon, \epsilon'\rangle_i \\ \langle\epsilon, \epsilon'\rangle_i & \langle\epsilon', \epsilon'\rangle_i \end{pmatrix} = \begin{pmatrix} 0 & 1/2^i \\ 1/2^i & 0 \end{pmatrix},$$

and $\langle\ ,\ \rangle'_i$ is given by the matrix

$$\begin{pmatrix} 1/2^{i-1} & 1/2^i \\ 1/2^i & 1/2^{i-1} \end{pmatrix}.$$

Then, when $p = 2$, the group Φ decomposes as an orthogonal sum of finite cyclic groups and of copies of $\langle\ ,\ \rangle_i$ and $\langle\ ,\ \rangle'_j$, for various values of i and j.

Our first task below is to show that all possible pairings appearing in the orthogonal decomposition of Φ mentioned above can arise as pairings attached to arithmetical graphs. To begin our series of explicit examples, let us consider the case of cyclic groups. Let n by an integer and let $\Phi = \mathbb{Z}/n\mathbb{Z}$. The classes of equivalent perfect pairings on Φ are easy to describe. Let $a \in \mathbb{Z}$ be prime to n. Then

$$\begin{aligned}\langle\ ,\ \rangle_a : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{Q}/\mathbb{Z},\\ (\overline{x}, \overline{y}) &\longmapsto axy/n \mod \mathbb{Z},\end{aligned}$$

is a perfect pairing, and any perfect pairing on Φ equivalent to $\langle\ ,\ \rangle_a$ is of the form $\langle\ ,\ \rangle_b$ for $b = ac^2$ with c prime to n. Any perfect pairing on Φ is equivalent to $\langle\ ,\ \rangle_a$ for some a.

Example 5.3. Consider the graph (I_n, M, R) consisting of a cycle of n vertices with ${}^tR = (1, \ldots, 1)$. The graph is thus the type I_n in Kodaira's notation for the reduction of elliptic curves. We have $g(I_n) = 1$. Let C and C' denote two adjacent vertices. Let γ denote the image of $E(C, C')$ in Φ_{I_n}. Then, by a straightforward verification using the definitions, one shows that $\Phi_{I_n} \cong \mathbb{Z}/n\mathbb{Z}$ and that γ is a generator with $\langle\gamma, \gamma\rangle_{I_n} = 1/n \mod \mathbb{Z}$. Alternatively, one may also obtain the result from 3.7.

Another example is the graph (J_n, M, R) consisting of two vertices C and C' linked by n edges, where

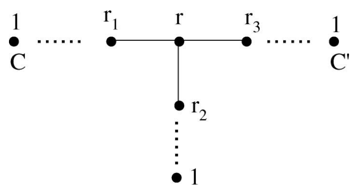
$$M = \begin{pmatrix} -n & n \\ n & -n \end{pmatrix}, \qquad R = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We have $g(J_n) = n - 1$. Then $\Phi_{J_n} = \mathbb{Z}/n\mathbb{Z}$ and the image γ of $E(C, C')$ is a generator. Moreover,

$$\langle\gamma, \gamma\rangle_{J_n} = \left(0, \tfrac{1}{n}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\tfrac{1}{n}.$$

Example 5.4. Let b and r be two coprime positive integers. Let us provide an example of a graph $(G_{r,b}, M, R)$ with cyclic group of components $\Phi_{G_{r,b}} \cong \mathbb{Z}/r\mathbb{Z}$ and endowed with a generator γ_b such that $\langle\gamma_b, \gamma_b\rangle = b/r$.

Assume first that r is odd. Consider the graph $G := G(r, r_1, r_2, r_3)$ given by



with $r \mid r_1+r_2+r_3$ and $\gcd(r, r_i) = 1$ for $i = 1, 2, 3$. The self-intersection of the node is thus $(r_1 + r_2 + r_3)/r$. The three terminal chains of G are constructed using Euclid's algorithm with the pair (r, r_i) as in [Lo1], 2.4. It is easy to check that $2g(G) = r - 1$. Proposition 9.6/6 in [BLR] shows that $|\Phi_G| = r$. Let γ denote the image of $E(C, C')$ in Φ_G. Let b_i be such that $b_i r_i \equiv 1 \mod r$. Then 5.1, together with Lemmata 2.6 and 2.8 of [Lo4], imply that

$$\langle \gamma, \gamma \rangle = \frac{b_1 + b_3}{r}.$$

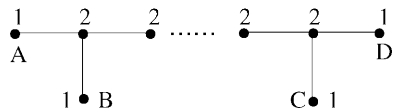
Proposition 3.7 a) in [Lo4] shows that γ has order r. Hence, Φ_G is isomorphic to $\mathbb{Z}/r\mathbb{Z}$. Note that since the pairing is perfect, $b_1 + b_3$ is coprime to r. Let a be such that $a(b_1 + b_3) \equiv b \mod r$. Let a' be such that $aa' \equiv 1 \mod r$. Consider the graph $G_{r,b} := G(r, a'r_1, a'r_2, a'r_3)$ and the corresponding element $\gamma_b \in \Phi_{G_{r,b}}$. It follows that

$$\langle \gamma_b, \gamma_b \rangle = \frac{ab_1 + ab_3}{r} = \frac{b}{r}.$$

Note that a curve X_K having a regular model with associated graph $G(r, r_1, r_2, r_3)$ has unipotent rank over K at least equal to $\frac{1}{2}(r - 1)$.

When r is even, consider the similar graph $G'(2r, r_1, r_2, r_3)$ with $2r \mid r_1 + r_2 + r_3$ and $\gcd(r_1, r) = \gcd(r_3, r) = 1$ and $\gcd(r, r_2) = 2$. The details of this case are left to the reader.

Example 5.5. Let $n \in \mathbb{Z}_{\geq 0}$ and consider the graph (I_n^*, M, R) given by

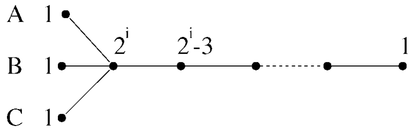


where $n + 1$ denotes the number of vertices of multiplicity 2. This graph is the graph corresponding to the type I_n^* in Kodaira's notation for the reduction of elliptic curves. It is well known that $\Phi_{I_n^*} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when $n \geq 0$ is even. Let γ and γ' denote the images in $\Phi_{I_n^*}$ of $E(A, D)$ and $E(A, B)$, respectively. Then $\{\gamma, \gamma'\}$ is a $(\mathbb{Z}/2\mathbb{Z})$-basis for $\Phi_{I_n^*}$, and with respect to this basis, the pairing $\langle\ ,\ \rangle_{I_n^*}$ is given by

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \text{ if } n = 4m \text{ and } \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \text{ if } n = 4m + 2.$$

Note that $\langle\ ,\ \rangle_{I_n^*}$ is $\langle\ ;\ \rangle_1$ when $n = 4m$ and is diagonalizable when $n = 4m + 2$.

Example 5.6. Consider the graph (G_i, M, R) with G_i as follows:

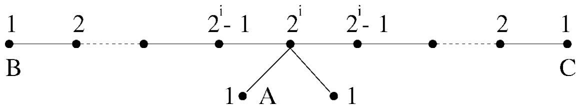


Similarly as in 5.4, the terminal chain to the right is constructed using Euclid's algorithm as in [Lo1], 2.4. Proposition 9.6/6 in [BLR] shows that $|\Phi_{G_i}| = 2^{2i}$. Let γ_{AB} and γ_{AC} denote the images of $E(A, B)$ and $E(A, C)$ in Φ_{G_i}, respectively. The reader will check, using 5.1, together with Lemmata 2.6 and 2.8 of [Lo4], that:

$$\begin{pmatrix} \langle\gamma_{AB}, \gamma_{AB}\rangle_{G_i} & \langle\gamma_{AB}, \gamma_{AC}\rangle_{G_i} \\ \langle\gamma_{AB}, \gamma_{AC}\rangle_{G_i} & \langle\gamma_{AC}, \gamma_{AC}\rangle_{G_i} \end{pmatrix} = \begin{pmatrix} 1/2^{i-1} & 1/2^i \\ 1/2^i & 1/2^{i-1} \end{pmatrix}.$$

In particular, $\Phi_{G_i} \cong (\mathbb{Z}/2^i\mathbb{Z})^2$, with γ_{AB} and γ_{AC} as generators.

Consider the graph (G_i', M, R) with G_i' as follows:



Again the terminal chains to the left and to the right are constructed using Euclid's algorithm. Proposition 9.6/6 in [BLR] shows that $|\Phi_{G_i'}| = 2^{2i}$. Let γ_{AB} and γ_{AC} denote the images of $E(A, B)$ and $E(A, C)$ in $\Phi_{G_i'}$, respectively. The reader will check that:

$$\begin{pmatrix} \langle\gamma_{AB}, \gamma_{AB}\rangle_{G_i'} & \langle\gamma_{AB}, \gamma_{AC}\rangle_{G_i'} \\ \langle\gamma_{AB}, \gamma_{AC}\rangle_{G_i'} & \langle\gamma_{AC}, \gamma_{AC}\rangle_{G_i'} \end{pmatrix} = \begin{pmatrix} 0 & 1/2^i \\ 1/2^i & 0 \end{pmatrix}.$$

In particular, $\Phi_{G_i} \cong (\mathbb{Z}/2^i\mathbb{Z})^2$, with γ_{AB} and γ_{AC} as generators.

Let us now return to the proof of 5.2. The following construction allows us to build arithmetical graphs whose associated pairings have a given orthogonal decomposition.

5.7. Given two arithmetical graphs G and G', with C a vertex of G and C' a vertex of G' both of equal multiplicity r, one obtains a new arithmetical graph H by glueing C and C' together and giving this vertex multiplicity r. When $r = 1$, one can show that $\Phi_H \cong \Phi_G \times \Phi_{G'}$ (see for instance 4.3 in [Lo4]). We shall say that H is a join of G and G'. One can also check that when $r = 1$,

$$\langle\ ,\ \rangle_H : \Phi_H \times \Phi_H \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is simply obtained as $\langle\ ,\ \rangle_H = \langle\ ,\ \rangle_G + \langle\ ,\ \rangle_{G'}$, once Φ_H is identified with $\Phi_G \times \Phi_{G'}$. Implicit in the above formula is the fact that if $x \in \Phi_G$ and $y \in \Phi_{G'}$, then $\langle x, y\rangle_H = 0$.

Since this 'join' construction is the key to the proof of 5.2, let us give here some indication of proof of the above statements. Number the vertices of G as $C_1, \dots, C_v$, and number the vertices of G' as $C'_1, \dots, C'_{v'}$. Let us assume that C_v and C'_1 have multiplicity 1, and that H is the graph obtained by glueing C_v to C'_1. Let us label the vertices of H as $C_1, \dots, C_{v-1}, D, C'_2, \dots C'_{v'}$. Let M, M', and M_H, denote the intersection matrices of G, G', and H, respectively. The images in Φ_H of the elements of the form $E(C_i, D)$ and $E(C'_j, D)$ generate Φ_H ($i \le v-1, 2 \le j \le v'$). The images in Φ_H of the elements of the form $E(C_i, D)$, $i \le v-1$, generate in Φ_H a subgroup isomorphic to Φ_G. Similarly, the images in Φ_H of the elements of the form $E(C'_j, D)$, $2 \le j \le v'$, generate in Φ_H a subgroup isomorphic to $\Phi_{G'}$.

Let $S_i := {}^t(s(C_1), \dots, s(C_v))$ be such that $MS_i = n_i E(C_i, C_v)$ for some integer n_i. Since C_v has multiplicity 1, we can always choose S_i such that $s(C_v) = 0$. Let $\bar{S}_i := {}^t(s(C_1), \dots, s(D) = 0, s(C'_1) = 0, \dots, s(C'_{v'}) = 0)$. Then $M_H \bar{S}_i = n_i E(C_i, D)$. Hence, $\langle E(C_i, D), E(C'_j, D)\rangle_H = {}^t(\bar{S}_i/n_i)E(C'_j, D) = 0$. Similarly, $\langle E(C_i, D), E(C_\ell, D)\rangle_H = \langle E(C_i, C_v), E(C_\ell, C_v)\rangle_G$.

We may now conclude the proof of 5.2. Consider first the case where Φ is a p-group and p is odd. The pairing $\langle\ ,\ \rangle$ is then always equivalent to a diagonal pairing. In 5.4, we showed that every pairing on a cyclic group can be obtained as the pairing of an arithmetical graph having a terminal vertex of multiplicity one. A diagonal pairing on Φ can be obtained by joining appropriate graphs using the construction described in 5.7. The resulting arithmetical graph also has a vertex of multiplicity one. The case where Φ is a p-group and $p = 2$ is similar; the orthogonal factors in Φ are shown to be realized by arithmetical graphs having terminal vertices of multiplicity one in 5.4 and 5.6. Consider now the general case. The canonical decomposition of Φ into a product of p-groups is easily checked to be an orthogonal decomposition. Thus, $\langle\ ;\ \rangle$ is equivalent to the pairing $\langle\ ;\ \rangle_G$ associated with an arithmetical graph G obtained by glueing arithmetical graphs whose groups of components are p-groups for appropriate primes p. □

Example 5.8. Let us use 5.1 to explicitly describe the pairing $\langle\ ,\ \rangle$ in the case of elliptic curves defined over a complete field K with algebraically closed residue field. We refer to an arithmetical graph G with $g(G) = 1$ by its Kodaira symbol $t(G) \in \{I_n, n \ge 0, I^*_n, n \ge 0, II, II^*, III, III^*, IV, IV^*\}$.

In the cases I_0, II, and II^*, the associated component group is trivial. When $t(G) \in \{III, III^*, IV, IV^*\}$, let C and C' be two distinct components of multiplicity 1 in G. Let γ denote the image of $E(C, C')$ in Φ_G. Example 5.4 shows that γ is a generator of Φ_G. When $t(G) = I_n$, let γ

be as in 5.3. Let now $t(G) = I_n^*$, with I_n^* given as in the proof of 5.5. The reader will check by direct computations, or by using 6.6 in [Lo4], that if n is odd, the image γ of $E(A, D)$ is a generator of $\Phi_G \cong \mathbb{Z}/4\mathbb{Z}$; if n is even, then the images γ and γ' of $E(A, D)$ and $E(A, B)$ are generators of $\Phi_G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It is not difficult to verify that

t	Φ	$\langle\gamma, \gamma\rangle$
III	$\mathbb{Z}/2\mathbb{Z}$	1/2
III^*	$\mathbb{Z}/2\mathbb{Z}$	1/2
IV	$\mathbb{Z}/3\mathbb{Z}$	2/3
IV^*	$\mathbb{Z}/3\mathbb{Z}$	1/3
I_{4m+1}^*	$\mathbb{Z}/4\mathbb{Z}$	1/4
I_{4m+3}^*	$\mathbb{Z}/4\mathbb{Z}$	3/4
I_n	$\mathbb{Z}/n\mathbb{Z}$	$1/n$

The pairing in the remaining two cases, I_{4m}^* and I_{4m+2}^*, is computed in 5.5.

t	Φ	basis $\{\gamma, \gamma'\}$
I_{4m}^*	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
I_{4m+2}^*	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

The reader will note that, even though every cyclic group is isomorphic to the group of components of some elliptic curve (having reduction of type I_n for some n), not every pairing on a cyclic group is equivalent to a pairing on the reduction of an elliptic curve.

6. A pairing that is not perfect

We exhibit in this section a field K with imperfect residue field, and a curve X_K such that Grothendieck's pairing $\langle\ ,\ \rangle$ associated with the Jacobian of X_K is not perfect.

Let (G, M, R) be any arithmetical graph. Let $\Lambda := \operatorname{diag}(e_1, \ldots, e_v) \in M_v(\mathbb{Z})$ be a matrix with positive entries. Assume that $\Lambda^{-1}M \in M_v(\mathbb{Z})$. Then, as in the context of 2.2, using the linear maps $\Lambda^{-1}M\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}^v$ and ${}^t(\Lambda R)\colon \mathbb{Z}^v \longrightarrow \mathbb{Z}$ associated to $\Lambda^{-1}M$ and ΛR, we define

$$\Phi_{G,\Lambda} := \operatorname{Ker}({}^t(\Lambda R))/\operatorname{Im}(\Lambda^{-1}M).$$

Recall that the map $\operatorname{Ker}({}^t(\Lambda R)) \to \operatorname{Ker}({}^t R)$, with $T \mapsto \Lambda T$, induces an injection $\Phi_{G,\Lambda} \hookrightarrow \Phi_G$.

Proposition 6.1. *Let* (G, M, R) *be an arithmetical graph, with* $M = (c_{ij})_{1\le i,j\le v}$. *Let* p *be prime. Let* $\Lambda = \operatorname{diag}(p^{a_1}, \ldots, p^{a_v})$ *with* $a_1 > a_2 \geq \cdots \geq a_v \geq 0$. *Assume that* $\Lambda^{-1}M \in M_v(\mathbb{Z})$ *and that* $\gcd(p^{a_i}r_i; i =$

$1, \ldots, v) = 1$. *Assume also that* $p^{a_1+1} \mid c_{11}$. *Then the pairing* $\langle\ ,\ \rangle_G$ *restricted to* $\Phi_{G,\Lambda} \times \Phi_{G,\Lambda}$ *is not perfect. More precisely, let*

$$Z := {}^t\left(\frac{c_{11}}{p^{a_1+1}}, \ldots, \frac{c_{v,1}}{p^{a_v+1}}\right).$$

Then $Z \in \operatorname{Ker}({}^t(\Lambda R))$ *and its image* z *in* $\Phi_{G,\Lambda}$ *has order* p. *Moreover,* $\langle z, x\rangle = 0$ *for all* $x \in \Phi_{G,\Lambda}$.

Proof. By construction, the vector pZ is the first column of the matrix $\Lambda^{-1}M$, so that $\Lambda^{-1}M\varepsilon_1 = pZ$, with ε_1 denoting the first vector of the canonical basis of $\mathbb{Z}^v$. In particular, z has order at most p in $\Phi_{G,\Lambda}$. If z is trivial in $\Phi_{G,\Lambda}$, then there exists $S \in \mathbb{Z}^v$ such that $\Lambda^{-1}MS = Z$. It follows that $\Lambda^{-1}M(pS - \varepsilon_1) = 0$, so $pS - \varepsilon_1 = \alpha R$ for some $\alpha \in \mathbb{Z}$. We find that p cannot divide α, so $p \mid r_j$ for $j = 2, \ldots, v$. This contradicts the fact that $\gcd(p^{a_i} r_i; i = 1, \ldots, v) = 1$. Hence, $z \neq 0$ in $\Phi_{G,\Lambda}$.

Consider now an arbitrary element $x \in \Phi_{G,\Lambda}$, say represented by some element $X \in \operatorname{Ker}({}^t(\Lambda R))$. Let $S \in \mathbb{Z}^v$ be such that $\Lambda^{-1}MS = mX$ for some integer $m \neq 0$. Then

$$\langle z, x\rangle = {}^t(\varepsilon_1/p)M(S/m) = {}^t(\varepsilon_1/p)\Lambda X = p^{a_1-1}({}^t\varepsilon_1 \cdot X) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

□

Example 6.2. We define an arithmetical graph (G, M, R) satisfying the hypotheses of 6.1 as follows. Let p be a prime and set $v := p + 1$. Then consider the matrix

$$M := \begin{pmatrix} -p^2 & p & \cdots & \cdots & p \\ p & 1-2p & 1 & \cdots & 1 \\ \vdots & 1 & 1-2p & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ p & 1 & \cdots & 1 & 1-2p \end{pmatrix} \in M_v(\mathbb{Z}),$$

as well as $R := {}^t(1, \ldots, 1) \in \mathbb{Z}^v$ and $\Lambda := \operatorname{diag}(p, 1, \ldots, 1) \in M_v(\mathbb{Z})$.

Let us now exhibit a curve X_K with associated arithmetical graph (G, M, R) and vector of geometric multiplicities Λ. (Note that Winters' Existence Theorem does not apply to our situation: this theorem only implies the existence of a curve X_F with associated graph (G, M, R) for some equicharacteristic discrete valuation field F with algebraically closed residue field.) Let K denote the field of fractions of $\mathfrak{R} := \mathbb{Z}[t]_{(p)}$, where t is a variable. Let X_K denote the plane projective curve given by the equation

$$F(x, y, z) := pz^{2p} - (x^p + ty^p)\prod_{i=0}^{p-1}(x - iy) = 0.$$

The reader will easily check that X_K is smooth of genus $(2p-1)(p-1)$ and that it has K-rational points at infinity. Consider the $\mathfrak{R}$-model X of X_K obtained by taking the schematic closure of X_K in $\mathbb{P}^2_{\mathfrak{R}}$. We claim that X is regular. Indeed, the only singular point of the special fiber X_k is the point $(0, 0)$ represented by the ideal $\mathfrak{p} = (p, x, y)$ in the chart $\mathrm{Spec}(A)$, with

$$A := \mathfrak{R}[x, y]/(F(x, y, 1)).$$

In $\mathrm{Spec}(A)$, the maximal ideal $\mathfrak{p}$ represents a regular point since $\mathfrak{p}A_{\mathfrak{p}}$ is generated by x and y. The reader will easily check that the intersection matrix associated to X_k is equivalent to the matrix M: In X_k, all components have multiplicity one, and the component given in $\mathrm{Spec}(A)$ by the ideal $(p, x^p + ty^p)$ is not geometrically reduced and intersects all other components of X_k with multiplicity p. It follows from Theorem 4.6 that Grothendieck's pairing associated to the Jacobian of X_K is not perfect.

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