PERIOD, INDEX, AND AN INVARIANT OF GROTHENDIECK FOR RELATIVE CURVES

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1. An invariant of Grothendieck for relative curves

Let S be a noetherian regular connected scheme of dimension 1. Let $f : \mathcal{X} \to S$ be a proper morphism. Consider the relative Picard functor $\mathbf{P} := \mathrm{Pic}_{\mathcal{X}/S}$ of $\mathcal{X}$ over S ([2], 8.1/2). Let K be the field of rational functions on S, and denote by X/K the generic fiber of f. We have a commutative diagram of abelian groups:

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{X}) & \longrightarrow & \mathbf{P}(S) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \longrightarrow & \mathbf{P}(K) \end{array}$$

Assume from now on that X/K is a smooth proper geometrically connected curve of genus g, and consider the degree morphism $\deg : \mathbf{P}(K) \to \mathbb{Z}$. The *period* $\delta'(X/K)$ of X/K is defined to be the positive generator of the image of the degree map $\mathbf{P}(K) \to \mathbb{Z}$. The index $\delta(X/K)$ is the positive generator of the image of the degree map $\mathrm{Pic}(X) \to \mathbb{Z}$. We let $\gamma(\mathcal{X}/S)$ denote the positive generator of the image of the composition $\mathbf{P}(S) \to \mathbf{P}(K) \to \mathbb{Z}$. Clearly, it follows from the definitions that $\delta'(X) \mid \gamma(\mathcal{X})$ and $\delta'(X) \mid \delta(X)$.

Assume in addition that $\mathcal{X}$ is regular. Then $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$ is surjective, and

$$\delta'(X) \mid \gamma(\mathcal{X}) \mid \delta(X).$$

Let $\mathrm{Br}(S)$ denote the Brauer group of S. As $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$, we find that $\gamma(\mathcal{X}) = \delta(X)$ if $\mathrm{Br}(S) = 0$[1] ([6], Prop. 4.1). In general, the invariant $\gamma(\mathcal{X})$ depends only on the generic curve X/K and the base scheme S, as our next lemma shows.

The integer $\gamma(\mathcal{X})$ was considered by Grothendieck and in [6], Remarque 4.2 (a), he states that "j'ignore si on a toujours $\delta(X) = \gamma(\mathcal{X})$" (I do not know if one always has $\delta(X) = \gamma(\mathcal{X})$). We give below examples where $\delta(X/K) > \gamma(\mathcal{X}/S) > \delta'(X/K)$.

2. Behavior of the invariant under restriction

Lemma 2.1. *Let $\mathcal{X}/S$ and $\mathcal{X}'/S$ be two regular proper models of the curve X/K. Then $\gamma(\mathcal{X}'/S) = \gamma(\mathcal{X}/S)$.*

Proof. Without loss of generality, we can suppose that $\mathcal{X}'$ dominates $\mathcal{X}$ and $\mathcal{X}' \to \mathcal{X}$ has only one exceptional divisor E. Comparing the canonical exact sequence

$$1 \to \mathrm{Pic}(S) \to \mathrm{Pic}(\mathcal{X}) \to \mathbf{P}(S) \to \mathrm{Br}(S) \to \mathrm{Br}(\mathcal{X})$$

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[1] This is true if for instance S is a proper curve either over a separably closed field, or over a finite field, or if S is strictly local.

([2], 8.1/4) to the one associated to $\mathcal{X}' \to S$, we find a canonical isomorphism

$$\mathrm{Coker}(\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}')) \simeq \mathrm{Coker}(\mathbf{P}(S) \to \mathbf{P}'(S))$$

where $\mathbf{P}' = \mathrm{Pic}_{\mathcal{X}'/S}$. Now the cokernel on the left is generated by the class of the exceptional divisor on $\mathcal{X}'$, which maps to 0 by the degree map. Therefore, $\mathbf{P}(S)$ and $\mathbf{P}'(S)$ have the same image in $\mathbb{Z}$, and $\gamma(\mathcal{X}) = \gamma(\mathcal{X}')$. □

Lemma 2.2. *Let $S = \mathrm{Spec}(\mathcal{O}_K)$ with $\mathcal{O}_K$ a Henselian discrete valuation ring with residue field k. Let $\mathcal{X} \to S$ be smooth and projective, with generic fiber X/K a geometrically connected curve. Then $\gamma(\mathcal{X}) = \delta'(X/K) = \delta'(\mathcal{X}_k/k)$ and $\delta(X/K) = \delta(\mathcal{X}_k/k)$.*

Proof. As $\mathcal{X} \to S$ is smooth, $\mathbf{P}$ is a scheme and its connected components are proper and smooth over S. The properness of $\mathbf{P}$ implies that the canonical map $\mathbf{P}(S) \to \mathbf{P}(K)$ is an isomorphism, so that $\gamma(\mathcal{X}) = \delta'(X)$. The smoothness of $\mathbf{P}$, with the facts that S is Henselian and $\mathrm{Pic}_{\mathcal{X}_k/k} = \mathbf{P} \times_S \mathrm{Spec}(k)$, implies that $\mathbf{P}(S) \to \mathrm{Pic}_{\mathcal{X}_k/k}(k)$ is surjective. Therefore $\gamma(\mathcal{X}) = \delta'(\mathcal{X}_k/k)$. Finally the equality $\delta(X/K) = \delta(\mathcal{X}_k/k)$ is a special case of [5], 9.4 (a). □

Let K be a number field or the function field of a curve S/k over a finite field k. The existence of curves X/K (of genus 1) such that $\delta'(X/K) < \delta(X/K)$ is established for instance in [3], Theorem 3, and Concluding Remark IV.

Example 2.3 Let $\mathcal{O}_K$ be a complete discrete valuation ring with residue field k and fraction field K. Let C/k be a proper smooth geometrically connected curve with distinct index and period $\delta'(C/k) < \delta(C/k)$. Because $\mathcal{O}_K$ is complete, we can lift C to a proper smooth curve $\mathcal{X} \to S = \mathrm{Spec}\,\mathcal{O}_K$ with generic fiber X/K. Lemma 2.2 shows that $\gamma(\mathcal{X}) = \delta'(X) < \delta(X)$.

Consider two smooth proper geometrically connected curves X/K and Y/K with K-morphisms $f : X \to \mathbb{P}^1_K$ and $g : Y \to \mathbb{P}^1_K$ of degrees d_X and d_Y, respectively. Assume that $\gcd(d_X, d_Y) = 1$, and that the induced field extension $K(Y)/K(\mathbb{P}^1)$ is Galois. Then the $K(\mathbb{P}^1)$-algebra $K(X) \otimes_{K(\mathbb{P}^1)} K(Y)$ is a field of degree $d_X d_Y$ over $K(\mathbb{P}^1)$. The field K is algebraically closed in this field, and we denote by Z/K the corresponding smooth proper geometrically connected curve.

Recall (2.1) that the invariant $\gamma(\mathcal{X}/S)$ of a regular model $\mathcal{X}/S$ of X/K only depends on S and on X/K. We now fix S, and denote this invariant below simply by $\gamma(X)$.

Lemma 2.4. *Keep the above notation and assumptions. Suppose that the invariants of the curves X/K and Y/K satisfy $\delta(X) \mid d_X$ and $\delta(Y) \mid d_Y$. Then the curve Z/K has invariants*

$$(\delta'(Z), \gamma(Z), \delta(Z)) = (\delta'(X)\delta'(Y), \gamma(X)\gamma(Y), \delta(X)\delta(Y)).$$

Proof. The natural morphism $i : Z \to X$ of degree d_Y induces $i^* : \mathrm{Pic}_{X/K}(K) \to \mathrm{Pic}_{Z/K}(K)$ with $\deg_Z \circ i^* = d_Y \deg_X$, and $i_* : \mathrm{Pic}_{Z/K}(K) \to \mathrm{Pic}_{X/K}(K)$ with $\deg_X \circ i_* = \deg_Z$. Similarly, $j : Z \to Y$ of degree d_X induces $j^* : \mathrm{Pic}_{Y/K}(K) \to \mathrm{Pic}_{Z/K}(K)$ with $\deg_Z \circ j^* = d_X \deg_Y$, and $j_* : \mathrm{Pic}_{Z/K}(K) \to \mathrm{Pic}_{Y/K}(K)$ with $\deg_Y \circ j_* = \deg_Z$.

Let us prove now that $\gamma(Z) = \gamma(X)\gamma(Y)$. Choose a regular model $\mathcal{X}/S$ of X/K and a regular model $\mathcal{Z}/S$ of Z/K endowed with a morphism $i : \mathcal{Z} \to \mathcal{X}$ over S extending the natural map $i : Z \to X$. (We can choose $\mathcal{Z}$ to be a desingularization of the normalization of $\mathcal{X}$ in $\mathrm{Spec}(K(Z))$.) Similarly, we choose a regular model $\mathcal{Y}/S$ of Y/K and a regular model $\mathcal{Z}'/S$ of Z/K endowed with a morphism $j : \mathcal{Z}' \to \mathcal{X}$ over S extending the natural map $j : Z \to Y$. We have a natural maps $i^* : \mathrm{Pic}_{\mathcal{X}/S}(S) \to \mathrm{Pic}_{\mathcal{Z}/S}(S)$ and $i_* : \mathrm{Pic}_{\mathcal{Z}/S}(S) \to \mathrm{Pic}_{\mathcal{X}/S}(S)$. Consider an element of $\mathrm{Pic}_{\mathcal{Z}/S}(S)$: its image in $\mathrm{Pic}_{\mathcal{X}/S}(S)$ has degree divisible by $\gamma(X)$ by hypothesis. Hence, $\gamma(X)$ divides $\gamma(Z)$. Similarly, $\gamma(Y)$ divides $\gamma(Z)$, and since $\gcd(\gamma(X), \gamma(Y)) = 1$, $\gamma(X)\gamma(Y)$ divides $\gamma(Z)$. Consider now an element of degree $\gamma(X)$ in $\mathrm{Pic}_{\mathcal{X}/S}(S)$. By pull-back, we get an element of degree $d_Y\gamma(X)$ in $\mathrm{Pic}_{\mathcal{Z}/S}(S)$. Similarly, we get an element of degree $d_X\gamma(Y)$ in $\mathrm{Pic}_{\mathcal{Z}'/S}(S)$. Hence, $\gamma(Z)$ divides $\gcd(d_X\gamma(Y), d_Y\gamma(X)) = \gamma(X)\gamma(Y)$.

The equalities $\delta(Z) = \delta(X)\delta(Y)$ and $\delta'(Z) = \delta'(X)\delta'(Y)$ follows immediately from the equalities $\gcd(d_X\delta(Y), d_Y\delta(X)) = \delta(X)\delta(Y)$ and $\gcd(d_X\delta'(Y), d_Y\delta'(X)) = \delta'(X)\delta'(Y)$. We leave the details to the reader. □

Example 2.5 We use 2.4 to produce an example of a Henselian discrete valuation ring $\mathcal{O}_K$ and of a smooth projective geometrically connected curve Z/K with regular model $\mathcal{Z}/\mathcal{O}_K$ such that $\delta'(Z) < \gamma(\mathcal{Z}/\mathcal{O}_K) < \delta(Z)$.

Let k be a number field such that there exists an elliptic curve E/k with all points of order 3 defined over k. The existence of an E-torsor X_0/k such that $\delta'(X_0/k) = 3$ and $\delta(X_0/k) = 9$ is established in [3], Theorem 3. Let K denote the henselization of $k[t]_{(t)}$, let $S := \mathrm{Spec}\,\mathcal{O}_K$, and let $\mathcal{X}/S$ denote the constant family $X_0 \times_k \mathrm{Spec}\,\mathcal{O}_K$, with generic fiber X/K. Since $\mathcal{X}/S$ is smooth, we find (2.2) that $(\delta', \gamma, \delta) = (3, 3, 9)$. Consider a divisor of degree 9 on X, and its associated line bundle $\mathcal{L}$. Two linearly independent sections in $\mathcal{L}(X)$ define a K-morphism $X \to \mathbb{P}^1$. The degree of this morphism is divisible by 9 because $\delta(X/K) = 9$, and since $\deg(\mathcal{L}) = 9$, we find that $X \to \mathbb{P}^1$ has degree $d_X = 9$.

Let $d \in k^*$, and consider now a quadratic extension $k(\sqrt{d})/k$. Let $\mathcal{Y}/S$ denote the projective plane curve with equation $x^2 - dy^2 = tz^2$ in $\mathbb{P}^2_S/S$. Let Y/K denote its generic fiber. We have a Galois K-morphism $Y \to \mathbb{P}^1$ of degree $d_Y = 2$. It is well-known that $\delta'(Y) = 1$ and $\delta(Y) = 2$. We claim that $\gamma(\mathcal{Y}) = 2$. Once this claim is proved, we may apply 2.4 to obtain the curve Z/K with the desired properties.

To show that $\gamma(\mathcal{Y}) = 2$, we proceed as follows. Let $L := K(\sqrt{d})/K$ and let $\mathcal{Y}' := \mathcal{Y} \times_S S'$ denote the étale base change of $\mathcal{Y}$ to S', with $S' := \mathrm{Spec}\,\mathcal{O}_L$. The morphism $\mathcal{Y}' \to S'$ has now a section. The special fiber of $\mathcal{Y}'/S'$ consists of two irreducible components permuted by the action of the Galois group $\mathbb{Z}/2\mathbb{Z}$ of S'/S. The functor $\mathrm{Pic}_{\mathcal{Y}'/S'}$ is representable by a scheme, while $\mathrm{Pic}_{\mathcal{Y}/S}$ is not (see [2], 8.2, after Thm. 1). The scheme $\mathrm{Pic}_{\mathcal{Y}'/S'}$ is the disjoint union of schemes $\mathrm{Pic}^d_{\mathcal{Y}'/S'}$, $d \in \mathbb{Z}$. The scheme $\mathrm{Pic}^1_{\mathcal{Y}'/S'}$ is the gluing over the generic point of S' of copies $S'_{a,b}$ of S', $a + b = 1$, $a, b \in \mathbb{Z}$. The action of the Galois group of S'/S sends $S'_{a,b}$ to $S'_{b,a}$. The image of the map $\mathrm{Pic}^1_{\mathcal{Y}/S}(S) \to \mathrm{Pic}^1_{\mathcal{Y}'/S'}(S')$ must be invariant under the action of Galois, and thus $\mathrm{Pic}^1_{\mathcal{Y}/S}(S) = \emptyset$. Since $\delta(Y) = 2$, we have $\gamma(\mathcal{Y}) = \delta(Y)$.

Let $\mathcal{X}/S$ be a regular model, proper over S. Let U be an open subset of S. The composition $\mathbf{P}(S) \to \mathbf{P}(U) \to \mathbf{P}(K)$ shows that $\delta'(X) \mid \gamma(\mathcal{X}_U) \mid \gamma(\mathcal{X})$.

Lemma 2.6. *There exists a non-empty open subset U of S such that $\gamma(\mathcal{X}_U/U) = \delta'(X/K)$.*

Proof. Since X/K is assumed to be geometrically smooth and proper, we find that there exists a non-empty open subset $V \subseteq S$ such that $\mathcal{X}_V \to V$ is smooth and proper. Then the Picard functor of $\mathcal{X}_V \to V$ is representable and proper over V ([1], XII, 1.2). Let $\xi \in \mathbf{P}(K)$ be of degree $\delta'(X)$. Then there exists an open neighborhood U in V such that ξ belongs to the image of $\mathbf{P}(U) \to \mathbf{P}(K)$. Therefore $\gamma(\mathcal{X}_U) \mid \delta'(X)$ and, hence, $\gamma(\mathcal{X}_U) = \delta'(X)$. □

Corollary 2.7. *Let K be the field of fractions of a Dedekind domain $\mathcal{O}_K$. Suppose that there exists a proper smooth geometrically connected curve X/K with $\delta'(X/K) < \delta(X/K)$. Then there exists a non-empty open subset U of* $\mathrm{Spec}(\mathcal{O}_K)$ *and a regular model $\mathcal{X} \to U$ of X/K such that $\gamma(\mathcal{X}/U) = \delta'(X/K) < \delta(X/K)$.*

Remark 2.8 Let S be global (i.e., smooth and proper over a finite field, or the spectrum of a ring of integers). The above corollary shows that $\delta'(X/K)$ belongs to the set $\{\gamma(\mathcal{X}_U/U)\}_U$, where U runs through the dense open subsets of S. Does the latter set contain all integers divisible by $\delta'(X/K)$ and dividing $\delta(X/K)$?

As noted at the beginning of this section, $\gamma(\mathcal{X}/U) = \delta(X/K)$ when $\mathrm{Br}(U) = (0)$. When S is a smooth proper curve over a finite field, and $P \in S$ is any closed point, both $\mathrm{Br}(S)$ and $\mathrm{Br}(S \setminus \{P\})$ are trivial ([4]). Let K be a number field with ring of integers $\mathcal{O}_K$. If K is purely imaginary or has only one real place, then $\mathrm{Br}(\mathcal{O}_K) = (0)$ ([6], Prop. 2.4).

Remark 2.9 Let X be any smooth projective irreducible variety over a field K. In [7], van Hamel introduces the period $\delta'(X/K)$ of X/K and the pseudo-index $\mathrm{PsI}(X)$, with the divisibility relations $\delta'(X/K) \mid \mathrm{PsI}(X) \mid \delta(X/K)$ ([7], Lemma, p. 104). When K is a p-adic field and X/K is a curve of genus $g \geq 1$, $\mathrm{PsI}(X) = \delta(X/K)$ ([7], Theorem 2, p. 103). We note that in this case $\mathrm{PsI}(X) = \gamma(\mathcal{X})$, where $\mathcal{X}/\mathcal{O}_K$ is any regular model of X/K.

Let X be any smooth geometrically integral variety over a field K. In [8], Wittenberg introduces the period $\delta'(X/K)$ of X/K and the generic period $P_{\mathrm{gen}}(X)$, with the divisibility relations $\delta'(X/K) \mid P_{\mathrm{gen}}(X) \mid \delta(X/K)$ ([8], 2.1). The author claims at the top of page 809 that one can show that $P_{\mathrm{gen}}(X) = \delta(X/K)$ when $\dim(X) = 1$, over any field K. Thus, in general, $\gamma(\mathcal{X}) \neq P_{\mathrm{gen}}(X)$.

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