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On a Generalization of the Frobenius Number

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Abstract

We consider a generalization of the Frobenius problem, where the object of interest is the greatest integer having exactly j representations by a collection of positive relatively prime integers. We prove an analogue of a theorem of Brauer and Shockley and show how it can be used for computation.

1 Introduction

The *linear diophantine problem of Frobenius* has long been a celebrated problem in number theory. Most simply put, the problem is to find the *Frobenius number* of k positive relatively prime integers $(a_1, \ldots, a_k)$, i.e., the greatest integer M for which there is no way to express M as the non-negative integral linear combination of the given a_i.

A generalization, which has drawn interest both from classical study of the Frobenius problem ([1, Problem A.2.6]) and from the perspective of partition functions and integer points in polytopes (as in Beck and Robins [2]), is to ask for the greatest integer M that can be expressed in exactly j different ways. We make this precise with the following definitions:

A *representation* of M by a k-tuple $(a_1, \ldots, a_k)$ of non-negative, relatively prime integers is a solution $(x_1, \ldots, x_k) \in \mathbb{N}^k$ to the equation $M = \sum_{i=1}^{k} a_i x_i$.

We define the *j-Frobenius number* of a k-tuple $(a_1, \ldots, a_k)$ of relatively prime positive integers to be the greatest integer M with exactly j representations of M by $(a_1, \ldots, a_k)$ if such a positive integer exists and zero otherwise. We refer to this quantity as $g_j(a_1, \ldots, a_k)$.

Finally, we define $f_j(a_1, \ldots, a_k)$ exactly as we defined $g_j(a_1, \ldots, a_k)$, except that we consider only *positive representations* $(x_1, \ldots, x_k) \in \mathbb{Z}^k_{>0}$.

Note that the 0-Frobenius number of $(a_1, \ldots, a_k)$ is just the classical Frobenius number. The purpose of this paper is to prove a generalization of a result of Brauer and Shockley [3] on the classical Frobenius number.

2 The Main Results

Our main result is the following:

Theorem 1. *If $d = \gcd(a_2, \ldots, a_k)$ and $j \geq 0$, then either*

$$g_j(a_1, a_2, \ldots, a_k) = d \cdot g_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}) + (d-1)a_1$$

or $g_j(a_1, a_2, \ldots, a_k) = g_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}) = 0$.

Lemma 2. *If $f_j(a_1, \ldots, a_k)$ is nonzero, there exist integers $x_2, \ldots, x_k > 0$ such that*

$$f_j(a_1, \ldots, a_k) = \sum_{i=2}^{k} a_i x_i.$$

Proof. Let $f_j := f_j(a_1, \ldots, a_k)$. By the definition of f_j, we can write $f_j = \sum_{i=1}^{k} a_i x_{i,\ell}$ with $x_{i,\ell} > 0$ for $1 \leq \ell \leq j$. Since

$$f_j + a_1 = \sum_{i=1}^{k} a_i x_{i,\ell} + a_1 = a_1(x_{1,\ell} + 1) + \sum_{i=2}^{k} a_i x_{i,\ell},$$

we obtain at least j positive representations of $f_j + a_1$. As f_j is the largest number with exactly j positive representations, there must be at least $j+1$ distinct ways to represent $f_j + a_1$. Specifically, we have $f_j + a_1 = \sum_{i=1}^{k} a_i x'_{i,\ell}$ with $x'_{i,\ell} > 0$ for all $1 \leq \ell \leq j+1$. Subtract a_1 from both sides of these $j+1$ equations to obtain $f_j = (x'_{1,\ell} - 1)a_1 + \sum_{i=2}^{k} a_i x'_{i,\ell}$. Evidently, there exists some $\ell_0 \in [1, j+1]$ for which $x'_{1,\ell_0} - 1 = 0$ because f_j cannot have $j+1$ positive representations. Therefore, $f_j(a_1, \ldots, a_k) = \sum_{i=2}^{k} a_i x'_{i,\ell_0}$. □

Theorem 3. *If $\gcd(a_2, \ldots, a_k) = d$, then*

$$f_j(a_1, a_2, \ldots, a_k) = d \cdot f_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}).$$

Proof. Let $a_i = da'_i$ for $i = 2, \ldots, k$ and $N = f_j(a_1, \ldots, a_k)$.
Assuming $N > 0$, we know by Lemma 2 that

$$N = \sum_{i=2}^{k} a_i x_i = d \sum_{i=2}^{k} a'_i x_i$$

with $x_i > 0$. Let $N' = \sum_{i=2}^{k} a'_i x_i$. We want to show that $N' = f_j(a_1, a'_2, \ldots, a'_k)$ and will do this in three steps.

Step 1: First, we know that N' does not have $j+1$ or more positive representations by $a_1, a'_2, \ldots, a'_k$. If N' could be so represented, then for $1 \le l \le j+1$ we would have

$$N' = a_1 y_{1,\ell} + \sum_{i=2}^{k} a'_i y_{i,\ell}.$$

Multiplying this equation by d immediately produces too many representations of N and thus a contradiction.

Step 2: Next, we know that

$$f_j(a_1, \ldots, a_k) = N = a_1 x_{1,\ell} + \sum_{i=2}^{k} a_i x_{i,\ell}$$

for $1 \le l \le j$ and $x_i > 0$, so

$$\frac{N}{d} = \frac{a_1 x_{1,\ell}}{d} + \sum_{i=2}^{k} \frac{a_i x_{i,\ell}}{d}.$$

Since $d|N$ and $d|a_i$ for $i \ge 2$, we must have $d|a_1 x_{1,\ell}$ for $1 \le \ell \le j$. In addition, $\gcd(a_1, d) = 1$ so we must have $d|x_{1,\ell}$ for $1 \le \ell \le j$. So

$$N' = a_1 \frac{x_{1,\ell}}{d} + \sum_{i=2}^{k} a'_i x_{i,\ell},$$

hence N' has at least j distinct positive representations. But we have already shown that N' cannot have $j+1$ or more positive representations, thus N' has exactly j positive representations.

Step 3: Finally we will show that N' is the largest number with exactly j positive representations by $a_1, a'_2, \ldots, a'_k$. Consider any $n > N'$. Since $dn > dN' = N$, we know that dn can be represented as a linear combination of $a_1, \ldots, a_k$ in exactly X ways with $X \ne j$. Thus, for $1 \le l \le X$ and $X \ne j$ we have

$$dn = a_1 x_{1,\ell} + \sum_{i=2}^{k} a_i x_{i,\ell}$$

and as in Step 2,

$$n = a_1(\frac{x_{1,\ell}}{d}) + \sum_{i=2}^{k} a_i' x_{i,\ell}.$$

If $X > j$ then we certainly do not have exactly j representations, so assume $X < j$. Assume now that we can write $n = a_1 y_1 + \sum_{i=2}^{k} a_i' y_i$ where $y_i \neq x_{i,\ell}$ for any such ℓ. By multiplying by d we get a new representation for dn, which is a contradiction because dn is represented in exactly $X \neq j$ ways.

Therefore N' is the greatest number with exactly j positive representations and so

$$N' = f_j(a_1, a_2', \ldots, a_k').$$

Thus

$$f_j(a_1, a_2, \ldots, a_k) = d \cdot f_j(a_1, \frac{a_2}{d}, \ldots, \frac{a_k}{d}).$$

□

Having established our results about $f_j(a_1, \ldots, a_k)$, we show that we can translate these results to results about the j-Frobenius numbers.

Lemma 4. *Either* $f_j(a_1, \ldots, a_k) = g_j(a_1, \ldots, a_k) = 0$ *or,*

$$f_j(a_1, \ldots, a_k) = g_j(a_1, \ldots, a_k) + \sum_{i=1}^{k} a_i.$$

Proof. For ease, write f_j for $f_j(a_1, \ldots, a_k)$, g_j for $g_j(a_1, \ldots, a_k)$, and $K = \sum_{i=1}^{k} a_i$.

Any representation $(y_1, \ldots, y_k)$ of M gives a representation $(y_1 + 1, \ldots, y_k + 1)$ of $M + K$. Moreover, adding or subtracting K preserves the distinctness of representations because it adjusts every coefficient y_i by 1. Therefore if M has j representations, $M + K$ has at least j positive representations. Likewise, every positive representation of $M + K$ gives a representation of M. Thus $f_j = 0$ if and only if $g_j = 0$. Assume now that f_j and g_j are both nonzero.

Suppose that $f_j < g_j + K$. By definition, we can find exactly j representations $(y_1, \ldots, y_k)$ for g_j and g_j has exactly j representations if and only if $g_j + K$ has exactly j *positive* representations $(x_1, \ldots, x_k)$. However, by assumption $g_j + K > f_j$ and $g_j + K$ has exactly j positive representations. This contradicts the definition of f_j, hence $f_j \geq g_j + K$.

Suppose that $f_j > g_j + K$. By definition, we can find exactly j positive representations $(x_1, \ldots, x_k)$ for f_j. The same argument as above shows that $f_j - K$ has exactly j representations in contradiction to the definition of g_j. Thus $f_j \leq g_j + K$. □

Proof of Theorem 1: Combine Theorem 3 with Lemma 4.

Corollary 5. *Let a_1, a_2 be coprime positive integers and let m be a positive integer. Suppose that $g_j = g_j(a_1, a_2, ma_1a_2) \neq 0$. Then*

- $g_j = (j+1)a_1a_2 - a_1 - a_2$ *for* $j < m+1$
- $g_{m+1} = 0$ *and*
- $g_{m+2} = (m+2)a_1a_2 - a_1 - a_2$.

Proof. Theorem 1 tells us that if $g_j(1,1,m) \neq 0$ then

$$\begin{aligned} g_j(a_1, a_2, ma_1a_2) &= a_2(g_j(a_1, 1, ma_1)) + (a_2 - 1)a_1 \\ &= a_2\left(a_1 g_j(1,1,m) + (a_1 - 1)1\right) + (a_2 - 1)a_1 \\ &= a_1a_2(g_j(1,1,m) + 2) - a_1 - a_2. \end{aligned}$$

Following Beck and Robins in their proof of [2, Proposition 1], we can use the values of the restricted partition function $p_{1,1,m}(k)$ to determine $g_j(1,1,m)$. Furthermore we can determine the relevant values with the Taylor series $\frac{1}{(1-t)^2(1-t^m)} = \sum_{k=0}^{\infty} p_{1,1,m}(k)t^k$. Now recall that for $k < m$, $p_{1,1,m}(k) = p_{1,1}(k) = k+1$ but $p_{1,1,m}(m) = m+2$ and for all $k > m$, $p_{1,1,m}(k) > m+2$. Note that no number is represented $m+1$ times. Thus $g_{m+1}(1,1,m) = 0$, $g_j(1,1,m) = j-1$ for $j < m$ and $g_{m+2}(1,1,m) = m$. □

Remark 6. It is a consequence of the asymptotics in Nathanson [4] that for a given tuple, there may be many j for which $g_j = 0$, so the ordering $g_0 < g_1 < \cdots$ may not hold. In the process of discovering the theorems of this paper, we noted the somewhat stranger occurrence of tuples where $0 < g_{j+1} < g_j$.

Take, for instance, the 3-tuple (3, 5, 8). The order $g_0 < g_1 < \cdots$ holds until $g_{14} = 52$ and $g_{15} = 51$. As should also be the case, the 3-tuple increased by a factor of $d = 2$ creates the new "dependent" 3-tuple (3, 10, 16), which fails to hold order in the same position with $g_{14} = 107$ and $g_{15} = 105$. A few independent examples are as follows:

$g_{17}(2,5,7) = 43$ and $g_{18}(2,5,7) = 42$,
$g_{38}(2,5,17) = 103$ and $g_{39}(2,5,17) = 102$,
$g_{35}(4,7,19) = 181$ and $g_{36}(4,7,19) = 180$, and
$g_{38}(9,11,20) = 376$ and $g_{39}(9,11,20) = 369$.

We do not as of yet know a lower bound on j for the above to occur. Indeed, in every case we have computed, if $g_0, g_1 > 0$ then $g_1 > g_0$, but to date neither a proof or a counterexample has presented itself.

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