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# THE $p$-PART OF THE GROUP OF COMPONENTS OF A NÉRON MODEL

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Let $K$ be a complete field with a discrete valuation $v$. Let $\mathcal{O}_K$ denote its ring of integers. Let $k$ denote the residue field of $\mathcal{O}_K$, and assume that it is algebraically closed. Let $p \geq 0$ denote the characteristic of $k$. Let $A/K$ be an abelian variety of dimension $g$ and let $\mathcal{A}/\mathcal{O}_K$ denote its Néron model. The special fiber $\mathcal{A}_k$ of $\mathcal{A}$ is a smooth group scheme over $k$. It is an extension of a finite abelian group $\Phi_K$, called the group of components, by a connected commutative group scheme $\mathcal{A}_k^0$, the connected component of 0 in $\mathcal{A}_k$.

Let $M/K$ be any finite separable field extension. The functoriality properties of the Néron model show the existence of a natural map of groups of components:

$$\gamma_{K,M} : \Phi_K \longrightarrow \Phi_M.$$

Let $\Psi_{K,M}$ denote the kernel of $\gamma_{K,M}$. Let $L/K$ denote the extension of $K$ minimal with the property that $A_L/L$ has semistable reduction (note that $L/K$ is Galois). Recall that the abelian variety $A/K$ is said to have potentially good reduction if the special fiber of the Néron model of $A_L/L$ is an abelian variety. In this case, $\Psi_{K,L} = \Phi_K$.

In an unpublished preprint, McCallum [McC] proves that the group $\Psi_{K,M}$ is killed by the order of the group $\mathrm{Gal}(M/K)$ when $k$ is the algebraic closure of a finite field and $M/K$ is Galois. A proof that the prime-to-$p$ part of $\Psi_{K,L}$ is killed by $\mathrm{Gal}(L/K)$ can be found in [Lor1, 3.5]. Since the largest quotient of $\mathrm{Gal}(M/K)$ having prime-to-$p$ order is cyclic, we find that the prime-to-$p$ part of $\Psi_{K,M}$ is killed by the exponent of $\mathrm{Gal}(M/K)$. McCallum asks in [McC] whether the full group $\Psi_{K,M}$ is in fact killed by the exponent of $\mathrm{Gal}(M/K)$. The purpose of this paper is to present a different proof of McCallum's result, without any restriction on $k$, as well as to show that the $p$-part of $\Psi_{K,M}$ is not, in general, killed by the exponent of $\mathrm{Gal}(M/K)$.

Let us stress here the significance of McCallum's theorem. While the prime-to-$p$ part of $\Psi_{K,L}$ is well understood (see, for instance, [Lor3], [Edi2]), McCallum's result is the only known general statement regarding the $p$-part of

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$\Psi_{K,L}$. Since Serre and Tate have shown in [S-T] that $|\mathrm{Gal}(L/K)|$ is divisible only by primes $q$ with $q \leq 2g+1$, we find that McCallum's theorem implies that $p$ may divide $|\Psi_{K,L}|$ only when $p \leq 2g+1$. The next theorem generalizes McCallum's theorem to the case where $k$ is arbitrary. Contrary to the general hypotheses in this article, $K$ is not necessarily complete and $k$ is not necessarily algebraically closed in the next theorem and its proof.

**Theorem 1.** *Let $D$ be a henselian discrete valuation ring, $K$ its field of fractions, $k$ its residue field and $A$ an abelian variety over $K$. Let $K \to K'$ be a finite separable field extension, $D' \subset K'$ the integral closure of $D$ and $k'$ the residue field of $D'$. Let $\mathcal{A}$ and $\mathcal{A}'$ denote the Néron models of $A$ and $A' := A_{K'}$ over $D$ and $D'$, respectively. Let $\Phi := \mathcal{A}_k/\mathcal{A}_k^0$ and $\Phi' := \mathcal{A}'_{k'}/(\mathcal{A}'_{k'})^0$. Then the kernel of the morphism $\gamma_{K,K'}\colon \Phi_{k'} \to \Phi'$ is killed by $n := [K' : K]$.*

*Proof.* Let $S := \mathrm{Spec}\,(D)$ and $S' := \mathrm{Spec}\,(D')$. Replacing $K$ by the largest unramified extension of $K$ in $K'$ we reduce to the case where $K \to K'$ is completely ramified, i.e., where $k \to k'$ is purely inseparable. So from now on we assume that $k \to k'$ is purely inseparable.

The Néron mapping property of $\mathcal{A}'$ gives a morphism $\alpha\colon \mathcal{A}_{S'} \to \mathcal{A}'$ inducing the identity on the generic fibres. Let $\mathcal{B} := \prod_{S'/S} \mathcal{A}'$ denote the Weil restriction of $\mathcal{A}'$ from $S'$ to $S$ (see [BLR, §7.6], or [Edi1, §2], for some properties of this construction). By the definition of $\mathcal{B}$, $\alpha$ induces a morphism $\alpha\colon \mathcal{A} \to \mathcal{B}$. Let $B := \mathcal{B}_K$ and let $K \to K^{\mathrm{sep}}$ be a separable closure. Since $B_{K^{\mathrm{sep}}}$ is canonically isomorphic to the product of $n$ copies, indexed by $\mathrm{Hom}_K(K', K^{\mathrm{sep}})$, of $A_{K^{\mathrm{sep}}}$, we have a morphism $\beta_{K^{\mathrm{sep}}}\colon B_{K^{\mathrm{sep}}} \to A_{K^{\mathrm{sep}}}$ which takes the sum. This morphism $\beta_{K^{\mathrm{sep}}}$ is compatible with the canonical descent data from $K^{\mathrm{sep}}$ to $K$ for both source and target; hence, there is a unique morphism $\beta\colon B \to A$ which, by base change from $K$ to $K^{\mathrm{sep}}$, gives $\beta_{K^{\mathrm{sep}}}$. From the fact that $\mathcal{A}$ is a Néron model of $A$ it follows that $\beta$ extends (uniquely) to $\beta\colon \mathcal{B} \to \mathcal{A}$. By construction we have that $\beta \circ \alpha\colon \mathcal{A} \to \mathcal{A}$ is the "multiplication by $n$" morphism. Let $\Psi := \mathcal{B}_k/\mathcal{B}_k^0$. The composition $\beta \circ \alpha$ of the induced morphisms $\alpha\colon \Phi \to \Psi$ and $\beta\colon \Psi \to \Phi$ is multiplication by $n$. To finish the proof we will show the existence of a canonical isomorphism between $\Psi_{k'}$ and $\Phi'$ that identifies $\alpha_{k'}\colon \Phi_{k'} \to \Psi_{k'}$ with $\gamma_{K,K'}$.

We define $R := D' \otimes_D k'$. Note that $R$ is an Artinian local $k'$-algebra with residue field $k'$ and of $k'$-dimension $n$; we consider $R$ as a $D'$-algebra in the usual way. We have $\mathcal{B}_{k'} = \prod_{R/k'} \mathcal{A}'_R$, since Weil restriction commutes with base change. Let $m \subset R$ be the maximal ideal; note that $m^n = 0$. As in §5.1 of [Edi1] we define, for any $k'$-algebra $C$, and for any $i$ with $0 \leq i \leq n$:

$$(\mathrm{F}^i\mathcal{B}_{k'})C = \ker\left(\mathcal{B}_{k'}(C) = \mathcal{A}'(C \otimes_{k'} R) \longrightarrow \mathcal{A}'(C \otimes_{k'} (R/m^i))\right).$$

This defines a filtration of $\mathcal{B}_{k'}$ by subfunctors:

$$\mathcal{B}_{k'} = \mathrm{F}^0\mathcal{B}_{k'} \supset \cdots \supset \mathrm{F}^n\mathcal{B}_{k'} = 0.$$

Each functor $C \mapsto \mathcal{A}'(C \otimes_{k'} (R/m^i))$ is represented by the group scheme $\prod_{(R/m^i)/k'} \mathcal{A}'_{R/m^i}$; hence, the functors $\mathrm{F}^i\mathcal{B}_{k'}$ are represented by closed subgroup schemes of $\mathcal{B}_{k'}$. For $0 \leq i < n$ and any $k'$-algebra $C$ we define

$$(\mathrm{Gr}^i\mathcal{B}_{k'})C := ((\mathrm{F}^i\mathcal{B}_{k'})C)/((\mathrm{F}^{i+1}\mathcal{B}_{k'})C).$$

For all $i$ and $C$ the maps $\mathcal{A}'(C \otimes_{k'} R) \to \mathcal{A}'(C \otimes_{k'} (R/m^i))$ are surjective, since $\mathcal{A}'$ is smooth over $D'$. It follows that $\mathrm{Gr}^0\mathcal{B}_{k'} = \mathcal{A}'_{k'}$, and as in §5.1 of [Edi1] one shows that for $0 < i < n$ there are canonical isomorphisms

$$\mathrm{Gr}^i\mathcal{B}_{k'} \xrightarrow{\sim} \mathrm{Tan}_0(\mathcal{A}'_{k'}) \otimes_{k'} (m^i/m^{i+1}),$$

where the $k'$-vector space on the right-hand side should be regarded as a variety over $k'$. Since $\mathrm{F}^1\mathcal{B}_{k'}$ is a repeated extension of the $\mathrm{Gr}^i\mathcal{B}_{k'}$ with $i > 0$, it is connected and the projection $\mathcal{B}_{k'} \to \mathrm{Gr}^0\mathcal{B}_{k'} = \mathcal{A}'_{k'}$ induces an isomorphism from $\Psi_{k'}$ to $\Phi'$. The composition of $\alpha_{k'}\colon \mathcal{A}_{k'} \to \mathcal{B}_{k'}$ with the projection $\mathcal{B}_{k'} \to \mathrm{Gr}^0\mathcal{B}_{k'} = \mathcal{A}'_{k'}$ is the canonical morphism $\mathcal{A}_{k'} \to \mathcal{A}'_{k'}$ and hence induces $\gamma_{K,K'}\colon \Phi_{k'} \to \Phi'$. □

**Lemma 2.** *Let $E/K$ be an elliptic curve. Then $\Psi_{K,L}$ is killed by the exponent of* $\mathrm{Gal}(L/K)$.

*Proof.* The possible groups $\Psi_{K,L}$ are $(1), \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/4\mathbb{Z}$. It is shown in [Lor1, 4.2], using Tate's algorithm, that, if $p = 2$ or 3 and if $p \mid |\Phi_K|$, then $p \mid |\mathrm{Gal}(L/K)|$. Hence, Lemma 2 is true if $p = 3$. When $p = 2$, it remains only to show that, if $\Psi_{K,L} = \mathbb{Z}/4\mathbb{Z}$, then $\mathrm{Gal}(L/K)$ is not killed by 2. When $K$ is the maximal unramified extension of $\mathbb{Q}_2$, the fact that the exponent of $\mathrm{Gal}(L/K)$ is divisible by 4 can be verified using Tate's algorithm and the tables of Kraus [Kra]. However, when $K$ is any field, it seems very difficult to prove Lemma 2 using only the "equation" of the elliptic curve. We will therefore provide a proof here that uses McCallum's result. Since $|\mathrm{Gal}(L/K)|$ kills $\Psi_{K,L}$, we are left, again, to consider only the case where $p = 2$ and $\Psi_{K,L} = \mathbb{Z}/4\mathbb{Z}$. In this case, 4 must divide $|\mathrm{Gal}(L/K)|$. It is well known that $\mathrm{Gal}(L/K)$ is a subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$. One easily checks that every subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ of order divisible by 4 has exponent divisible by 4. Hence, Lemma 2 follows. □

Let us now describe an example that shows that the group $\Psi_{K,M}$ is not, in general, killed by the exponent of $\mathrm{Gal}(M/K)$. Let $t \in \mathcal{O}_K$ be a uniformizing parameter. Let $p > 2$ and $q := p^f$. Let $X/K$ denote the smooth projective model of the plane curve

$$y^2 = (x^q + t)^2 + At^m x^{2r},$$

with $A \in \mathcal{O}_K^*$, $m, r \in \mathbb{N}$, and $0 \leq r \leq q$, and $m \geq 2$. Note that $(0, \pm t)$ is a $K$-rational point of $X$. Let $J/K$ denote the Jacobian of $X$. Our aim in the remainder of this paper is:

(1) To describe the minimal model of $X/K$ over $\mathcal{O}_K$. This model is independent of the residual characteristic of $\mathcal{O}_K$. We will show, using this model, that the group of components $\Phi_K$ of $J$ is cyclic of order $q^2$.
(2) To show that, if $v(p)$ is large enough and if $r$ is appropriately chosen, then the Jacobian $J$ has potentially good reduction. In particular, $\Phi_K = \Psi_{K,L}$.
(3) To show that the first ramification subgroup of the Galois group $\mathrm{Gal}(L/K)$ associated to $J$ is an elementary abelian $p$-group of order $q^2$.

The statements (2) and (3) will follow from an explicit description of the stable reduction of the curve $X_L/L$ over $\mathcal{O}_L$. In our search for this example, we were guided by the case where $q = 3$ and the curve $X/K$ has genus 2. In this case, the reduction of $X/K$ over $\mathcal{O}_K$ can be computed using the algorithm in [Liu].

Let $\mathcal{X}/\mathcal{O}_K$ denote the regular minimal model of $X/K$ over $\mathcal{O}_K$. Its special fiber is a Cartier divisor and, as such, can be written as $\mathcal{X}_k = \sum_{i=1}^{s} r_i C_i$, where $C_i$ is an irreducible component of multiplicity $r_i$. Let $(G, M, R)$ denote the associated arithmetical graph (see [Lor2, 1.2]). We describe below an arithmetical graph $G(\nu, n, a, b, c, d)$, and we will show that the graph associated to the special fiber $\mathcal{X}_k$ is of the form $G(\nu, n, a, b, c, d)$ for certain values of the parameters $\nu, n, a, b, c$, and $d$.

Let $\nu \geq 0$ and $n, a, b, c, d \geq 1$ be integers. Let $G(\nu, n, a, b, c, d)$ denote the following arithmetical graph:

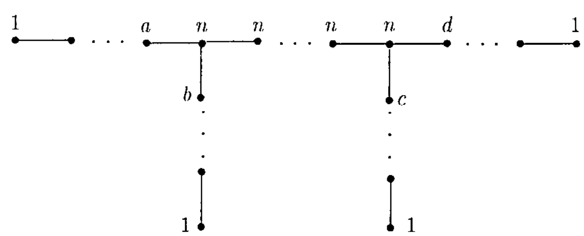


FIGURE 1

Recall that $\overset{r}{\bullet}$ denotes a vertex of multiplicity $r$. The symbols


FIGURE 2

indicate that the chain is continued as in [Lor2, 2.4], using Euclid's algorithm on $r$ and $r_1$, and that $\gcd(r, r_1) = 1$. The integer $\nu$ is one less than the number of vertices of multiplicity $n$. Implicit in the above graph is the fact that $n$ divides $a+b$ and $c+d$. The integer $(a+b+n)/n$ is the "self-intersection" of the node on the left (a node is a vertex with at least three adjacent edges). Similarly, $(c+d+n)/n$ is the self-intersection of the node on the right. When no confusion may occur, we will denote the graph $G(\nu, n, a, b, c, d)$ simply by $G$. We let $M$ denote the intersection matrix associated to $G$, and we denote by $\Phi$ the group of components of $G$. Recall that if $\mathrm{diag}(e_1, \ldots, e_s, 1, \ldots, 1, 0)$ is an integer matrix row and column equivalent to $M$, then $\Phi \cong \prod_{i=1}^{s} \mathbb{Z}/e_i\mathbb{Z}$. The order of $\Phi$ can be computed using the formula in [Lor1, 1.5]. We find that $|\Phi| = n^2$.

Recall that the graph $G(\nu, 2, 1, 1, 1, 1)$ corresponds to the graph $I_\nu^*$ in Kodaira's notation for the reduction types of elliptic curves. In particular, the associated group $\Phi$ is $\mathbb{Z}/4\mathbb{Z}$ if $\nu$ is odd, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if $\nu$ is even. These facts are generalized as follows:

**Lemma 3.** *The group of components $\Phi$ of the graph $G(\nu, n, a, b, c, d)$ is isomorphic to the product of $\mathbb{Z}/\gcd(n, \nu)\mathbb{Z}$ and $\mathbb{Z}/\frac{n^2}{\gcd(n,\nu)}\mathbb{Z}$.*

*Proof.* Let us only briefly sketch the row and column operations needed to compute $\Phi$. Order the vertices of $G$ as follows. First order the vertices of the chain "$a$" from left to right, from the end point to the vertex of multiplicity $a$; then order the vertices of the chain "$b$" from bottom to top, from the end point to the vertex of multiplicity $b$; then order the vertices of multiplicity $n$ from left to right; then order the vertices of the chain "$c$" from top to bottom; and finally order the vertices of the chain "$d$" from left to right. Use Lemma 2.5 in [Lor2] to "shrink" the four terminal chains. Let $sn = a+b+n$, and let $tn = c+d+n$. We are left to compute the row and column reduction of the following matrix:

$$\begin{pmatrix}
-n & 0 & 1 & & & & & & & \\
0 & -n & 1 & & & & & & & \\
a & b & -s & 1 & & & & & & \\
 & & 1 & -2 & 1 & & & & & \\
 & & & 1 & -2 & 1 & & & & \\
 & & & & 1 & \ddots & \ddots & & & \\
 & & & & & \ddots & -2 & 1 & & \\
 & & & & & & 1 & -t & c & d \\
 & & & & & & & 1 & -n & 0 \\
 & & & & & & & 1 & 0 & -n
\end{pmatrix}.$$

The entries that are not explicitly indicated in the above matrix are all null, except for the ones on the diagonal, and on the lower and upper diagonals. To deal with the central part of this $(\nu+5)\times(\nu+5)$-matrix, proceed as follows. Assume that $\nu \geq 3$. Add the columns $4, 5, \ldots, \nu+1$ to the $(\nu+2)$-column, add the columns $4, 5, \ldots, \nu$ to the $(\nu+1)$-column, etc., and finally add column 4 to column 5. Then add column $(\nu+2)$ to column $(\nu+1)$, etc. and, lastly, add column 5 to column 4. The new matrix has the form:

$$\begin{pmatrix}
-n & 0 & 1 & & & & & & & \\
0 & -n & 1 & & & & & & & \\
a & b & -s & \nu-1 & \nu-2 & \ldots & 1 & & & \\
 & & 1 & -\nu & -(\nu-2) & \ldots & -1 & & & \\
 & & & 0 & -1 & 0 & 0 & & & \\
 & & & & 0 & \ddots & \vdots & & & \\
 & & & \vdots & \vdots & \ddots & -1 & 1 & & \\
 & & & 1 & 1 & \ldots & 1 & -t & c & d \\
 & & & & & & & 1 & -n & 0 \\
 & & & & & & & 1 & 0 & -n
\end{pmatrix}.$$

Hence, we are reduced to finding the row and column reduction of the $8\times 8$ matrix

$$\begin{pmatrix}
-n & 0 & 1 & & & & & \\
0 & -n & 1 & & & & & \\
a & b & -s & (\nu-1) & 1 & & & \\
 & & 1 & -\nu & -1 & & & \\
 & & & 0 & -1 & 1 & & \\
 & & & 1 & 1 & -t & c & d \\
 & & & & & 1 & -n & 0 \\
 & & & & & 1 & 0 & -n
\end{pmatrix}.$$

Using, among other operations, operations with the first line and third column, and operations on the sixth column and the last line, one is reduced to considering the matrix

$$\begin{pmatrix}
0 & -n & & & & \\
-n & b & \nu-1 & 1 & & \\
n & 0 & -\nu & -1 & & \\
 & & 0 & -1 & 0 & n \\
 & & 1 & 1 & c & -n \\
 & & & & -n & 0
\end{pmatrix}, \quad \text{and then} \quad \begin{pmatrix}
0 & -n & 0 & 0 \\
0 & b & c & 0 \\
n & 0 & c\nu & 0 \\
0 & 0 & -n & 0
\end{pmatrix}.$$

The determinant of this $4\times 4$ matrix is 0. The greatest common divisor of the determinants of its $3\times 3$ minors is equal to $n^2$. The greatest common divisor of the determinants of its $2\times 2$ minors is equal to $\gcd(n,\nu)$. The greatest common divisor of the coefficients of this matrix is equal to 1. Hence, this matrix is row and column equivalent to $\mathrm{diag}(1, \gcd(n,\nu), n^2/\gcd(n,\nu), 0)$. Lemma 3 follows. □

**Lemma 4.** *Let $K$ be a field with a discrete valuation $v$. Assume that the associated residue field is not of characteristic 2. Let $X/K$ denote the smooth projective model of the plane curve given by the equation*

$$y^2 = (x^n + t + txf(x))^2 - At^m x^s,$$

*where $f(x) \in \mathcal{O}_K[x]$ is of degree less than $n$, and $v(A) = 0$. Assume also that $m, n \geq 2$, and that $0 \leq s \leq 2n$. Then the graph associated to the special fiber of the minimal model of $X/K$ over $\mathcal{O}_K$ is of the form $G(\nu, n, n-1, 1, n-1, 1)$, with $\nu = (m-2)n + s$.*

*Proof.* Let $g(x) := (x^n + t + txf(x))^2 + At^m x^s$. The plane curve given by $y^2 - g(x) = 0$ has a smooth model obtained by glueing the affine curve $y^2 - g(x) = 0$ to the affine curve $v^2 - \tilde{g}(u) = 0$, where $u := 1/x$, $v := y/x^n$, and $g(x)/x^{2n} =: \tilde{g}(1/x)$. Let $\mathcal{X}_0/\mathcal{O}_K$ denote the model of $X/K$ obtained by glueing in the obvious manner the two affine schemes $U_1 := \operatorname{Spec} \mathcal{O}_K[x,y]/(y^2 - g(x))$ and $U_2 := \operatorname{Spec} \mathcal{O}_K[u,v]/(v^2 - \tilde{g}(u))$.

We claim that the scheme $\mathcal{X}_0$ has a unique singular point $P_0$ belonging to the special fiber $\mathcal{X}_{0,k}$, and that $\mathcal{X}_{0,k}$ is the union of two smooth rational curves $C$ and $C'$ intersecting at $P_0$. We shall represent $\mathcal{X}_{0,k}$ by the following diagram:

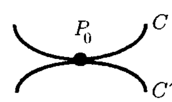


FIGURE 3

To prove our claim, recall that if a point $P \in \mathcal{X}_{0,k}$ is a singular point of $\mathcal{X}$, then it must be a singular point on the scheme $\mathcal{X}_{0,k}$. The special fiber of $U_2$ is $\operatorname{Spec} k[u,v]/(v^2 - 1)$, the disjoint union of two affine lines. Hence, $U_2$ is a regular scheme. The special fiber of $U_1$ is $\operatorname{Spec} k[x,y]/(y^2 - x^{2n})$, the union of two smooth affine lines intersecting at the point $P_0$ corresponding to the maximal ideal $M := (x,y,t)$. We will justify the fact that $P_0$ is singular on $\mathcal{X}_0$ by showing below that the exceptional fiber of the blow-up $\mathcal{X}_1$ of $P_0$ on $\mathcal{X}_0$ is not a smooth rational curve. We could also verify directly that $\dim_k M/M^2 > 2$.

Let us now describe the special fiber of the blow-up $\mathcal{X}_1$. It consists of the strict transforms of $C$ and $C'$, and of two smooth projective lines $E_1$ and

$E_1'$. These four components have multiplicity one in $\mathcal{X}_{1,k}$, and all intersect in a point $P_1$, which is the unique singular point of $\mathcal{X}_1$. We represent $\mathcal{X}_{1,k}$ as follows:

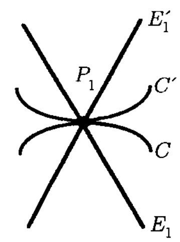


FIGURE 4

The scheme $\mathcal{X}_1$ can be covered by four affine charts. Let us briefly describe the chart $W_1$ that contains $P_1$. Let $y = xy_1$ and $t = xt_1$. Substitute these new expressions in $y^2 - g(x) = 0$ to obtain

$$x^2[y_1^2 - (x^{n-1} + t_1 *)^2 - At_1^m x^{m+s-2}] = 0.$$

Let $g_1(x, t_1) := (x^{n-1} + t_1 *)^2 + At_1^m x^{m+s-2}$. Let

$$B_1 := \mathcal{O}_K[x, y_1, t_1]/(y_1^2 - g_1(x, t_1), t - xt_1).$$

Let $W_1 = \operatorname{Spec} B_1$. The point $P_1$ corresponds to the maximal ideal $(x, y, t_1)$. The exceptional fiber is $\operatorname{Spec} B/(x)$, and the union of the strict transforms of $C$ and $C'$ is $\operatorname{Spec} B/(t_1)$.

The reader will check that the scheme $\mathcal{X}_2$, obtained as the blow-up of $P_1$, has a unique singular point $P_2$, and that $\mathcal{X}_{2,k}$ can be represented as follows:

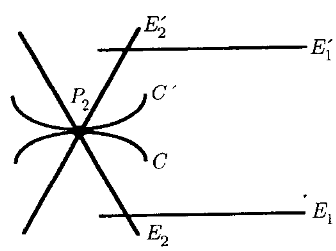


FIGURE 5

The exceptional components $E_2$ and $E_2'$ have multiplicity 2 in $\mathcal{X}_{2,k}$. A similar process can be repeated $n-3$ more times to obtain a scheme $\mathcal{X}_{n-1}$ with a unique singular point $P_{n-1}$ and a special fiber $\mathcal{X}_{n-1,k}$ represented as follows:

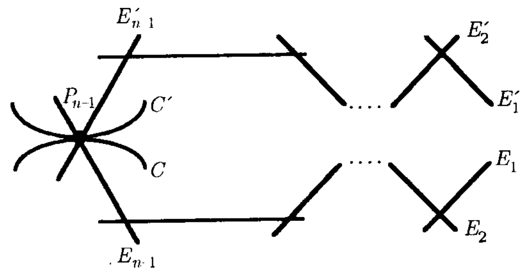


FIGURE 6

The multiplicity of the exceptional fiber in the special fiber $\mathcal{X}_{i,k}$ is $i$.

Blowing-up $\mathcal{X}_{n-1}$ at $P_{n-1}$ separates the components $C$ and $C'$, and gives a scheme $\mathcal{Z}_0$ with a unique singular point $Q_0$ and a special fiber of the form:

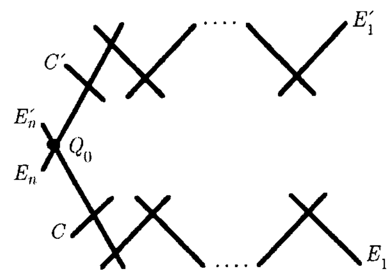


FIGURE 7

We may describe an affine open set $V_n$ of $\mathcal{Z}_0$ that contains $Q_0$ as follows. Let $y = x^n y_n$ and $t = x^n t_n$. Substitute these expressions into $y^2 - g(x) = 0$ to get

$$x^{2n}[y_n^2 - (1+t_n\sigma)^2 - At_n^m x^{(m-2)n+s}] = 0.$$

Let $g_n(x, t_n) := (1+t_n\sigma)^2 + At_n^m x^{(m-2)n+s}$. Let

$$B_n := \mathcal{O}_K[x, y_n, t_n]/(y_n^2 - g_n(x,t_n), t - x^n t_n).$$

Let $V_n := \operatorname{Spec} B_n$. The point $Q_0 \in V_n$ corresponds to the maximal ideal $(x, y_n, 1+t_n\sigma)$. The reader will check that by localizing, one can make the change of variable $z := 1 + t_n\sigma$, and obtain a main equation of the form $y_n^2 = z^2 + Dx^{(m-2)n+s}$, with $D \notin (x, y_n, z)$. This is the equation of an ordinary double point (see [Des, 2.2]), which is resolved by a chain of $n(m-2)+s-1$ smooth rational curves. Hence, $\nu + 1 = n(m-2)+s+1$ is the number of components of multiplicity $n$ in the special fiber of the regular model of $X/K$. □

**Remark 1.** If either $s > 2n$ or $\deg(f) \geq n$, then the genus of $X/K$ is larger than $n-1$. The graph associated to a regular model $\mathcal{X}/\mathcal{O}_K$ in this case will not be simply connected. This fact can already be seen on the corresponding model $\mathcal{X}_0$: when $s > 2n$ or $\deg(f) \geq n$, the special fiber of $\mathcal{X}_0$ is the union of two rational components meeting in two distinct points, one in the chart $U_1$ and the other one in the chart $U_2$ (the notation is as in the proof of Lemma 4). In particular, the graph associated to $\mathcal{X}/\mathcal{O}_K$ cannot be of the form $G(\nu, n, a, b, c, d)$.

**Remark 2.** Fix a graph $G(\nu, n, a, b, c, d)$. Let $\ell$ be a prime that does not divide any of the multiplicities of the vertices of the graph $G$. Only for such $\ell$ does the Existence Theorem of Winters ([Win, Corollary 4.3]) predict the existence of a discrete valuation field $F$ of equicharacteristic $\ell$ and of a curve $Y/F$ such that the graph associated to the minimal model of $Y/F$ over $\mathcal{O}_F$ is $G(\nu, n, a, b, c, d)$. Lemma 4 shows that the graph $G(\nu, n, n-1, 1, n-1, 1)$ occurs as the graph associated to the minimal model of a curve in any odd residual characteristic.

Let $X/K$ be any smooth projective curve. Recall that there exists a minimal Galois extension $L/K$ such that $X$ admits a unique stable model $\mathcal{Y}$ over the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in $L$ (see for instance [Des, 1.5, and 5.10-5.16]). The special fiber $\mathcal{Y}_k$ of $\mathcal{Y}$ is called the *stable reduction* of $X$. Furthermore, for any extension $M$ of $L$, the special fiber of the stable model of $X$ over $\mathcal{O}_M$ is isomorphic to $\mathcal{Y}_k$, for the construction of stable models commutes with base change.

**Lemma 5.** *Let $q = p^f$. Let $X/K$ be the smooth projective curve corresponding to the affine equation $y^2 = (x^q + t)^2 + At^m x^{2r}$, with $A \in \mathcal{O}_K^*$, $m \geq 2$, and $0 \leq r \leq q$. Assume that $(p, r) = 1$ and that $v(p) \geq m/2$. Then the stable reduction $\mathcal{Y}_k$ of $X$ consists of two irreducible components $E$ and $F$ intersecting in a single point. Both components are isomorphic to the smooth projective curve (over $k$) corresponding to the affine equation $z^2 = u^q - u$.*

*Proof.* Denote by $\overline{K}$ an algebraic closure of $K$. The absolute value $|\cdot|$ of $K$ extends uniquely to an absolute value $|\cdot|$ of $\overline{K}$. Let $B \in K$ be a square root of $-A$. Consider the following polynomials in $\overline{K}[x]$ :

$$H_1(x) := x^q + Bt^{m/2}x^r + t,$$

$$H_2(x) := x^q - Bt^{m/2}x^r + t.$$

By definition, $y^2 = H_1(x)H_2(x)$. Let $\theta \in \overline{K}$ be a zero of $H_1(x)$. Make the change of variables $x = \lambda u + \theta$ to find that

$$H_1(x) = (\lambda^q u^q + \cdots) + Bt^{m/2}(\cdots + r\lambda u\theta^{r-1} + \theta^r) + t.$$

Let $\lambda^{q-1} := -rBt^{m/2}\theta^{r-1}$. Then

$$H_1(x) = \lambda^q(u^q - u + \varepsilon_1(u)),$$

where $\varepsilon_1(u) \in K(\theta,\lambda)[u]$ is a polynomial whose coefficients have absolute value less than 1 (this last fact uses the properties $v(p) \geq m/2$ and $m \geq 2$). After performing easy but tedious computations, the reader will find that

$$H_2(x) = -2Bt^{m/2}\theta^r + H_1(x) = (-2\lambda^{q-1}\theta/r)(1+\varepsilon_2(u)),$$

where $\varepsilon_2(u) \in K(\theta,\lambda)[u]$ is a polynomial whose coefficients have absolute value less than 1 and $\varepsilon_2(0) = 0$. Let $\alpha \in \overline{K}$ be a square root of $-2\lambda^{2q-1}\theta/r$, and set $z := y/\alpha$. We find then that

$$z^2 = (u^q - u + \varepsilon_1(u))(1+\varepsilon_2(u)).$$

The reader will note that reducing this equation modulo a uniformizing parameter of $K(\theta,\lambda,\alpha)$ produces an equation of Artin-Schreier type.

We proceed now similarly with $H_2(x)$. Let $\delta \in \overline{K}$ be a root of $H_2(x)$. Let $\mu^{q-1} := rBt^{m/2}\delta^{r-1}$ and $\beta^2 := 2\mu^{2q-1}\delta/r$. Let $v := (x-\delta)/\mu$ and $w := y/\beta$. We have $|\delta| = |t|^{1/q} = |\theta|$ and $|\lambda| = |\mu|$.

Let $M = K(\theta,\delta,\mu,\lambda,\alpha,\beta,t^{m/2})$. (It can be seen that $M = K(\theta,\delta,t^{1/4(q-1)})$). We are going to exhibit below a stable normal model $\mathcal{Y}/\mathcal{O}_M$ of $X_M/M$. By construction, $1+\varepsilon_2((\delta-\theta)/\lambda) = 0$. We find then that $|\delta-\theta| > |\lambda|$. In the function field $M(X)$ of $X_M/M$, consider

$$w_1 = \lambda/(\theta-\delta) + 1/u,$$

$$w_2 = \mu/(\delta-\theta) + 1/v.$$

Let $\pi := -\lambda\mu/(\theta-\delta)^2$, so that $w_1w_2 = \pi$, and $|\pi| < 1$. The field inclusion $M(w_1) \subseteq M(X)$ gives a natural map $X_M \to \mathbb{P}^1_M$ defined over $M$. Consider the normal model

$$\mathcal{Z} := \mathrm{Proj}\mathcal{O}_M[W_0,W_1,W_2]/(W_1W_2 - \pi W_0^2)$$

of $\mathbb{P}^1_M$, where $w_i := W_i/W_0$. Let $\mathcal{Y}$ denote the integral closure of $\mathcal{Z}$ in $X_M/M$. The scheme $\mathcal{Y}/\mathcal{O}_M$ is a normal model of $X_M/M$, which contains two affine open subsets, smooth over $\mathcal{O}_M$ :

$$U := \mathrm{Spec}\,\mathcal{O}_M[u,z], \text{ and } V := \mathrm{Spec}\,\mathcal{O}_M[v,w].$$

Namely, $U$ (resp. $V$) is the preimage by $\mathcal{Y} \to \mathcal{Z}$ of $D_+(W_1)$ (resp. $D_+(W_2)$). Denote by $\tilde{h}$ the image of any $h \in \mathcal{O}_M[u,z]$ in $\mathcal{O}_M[u,z] \otimes_{\mathcal{O}_M} k$. The special fiber $U_k$ (resp. $V_k$) is a smooth curve defined by the equation

$$\tilde{z}^2 = \tilde{u}^q - \tilde{u} \quad (\text{resp. } \tilde{w}^2 = \tilde{v}^q - \tilde{v}).$$

Let $E$ (resp. $F$) be the irreducible component of $\mathcal{Y}_k$ containing $U_k$ (resp. $V_k$). The geometric genus of $E$ and $F$ equals $(q-1)/2$. It is easy to see that $U_k \cup V_k$ is dense in $\mathcal{Y}_k$; so $E \cup F = \mathcal{Y}_k$. Furthermore, the sum of the geometric genera of $E$ and $F$ is $q-1 = p_a(\mathcal{Y}_k)$. This implies that $E$ and $F$ are smooth, and that they intersect in an unique point, $\infty$, which is an ordinary double point. □

**Remark 3.** Let $X/K$ be any hyperelliptic curve, and assume that the residual characteristic of $K$ is odd. The stable reduction of $X$ is completely determined if the branch locus of the canonical map $X \to \mathbb{P}^1_K$ is known or, more precisely, if the relative position of the points in the locus is known (see [Bos]). The proof of Lemma 5 given above differs from [Bos]; it avoids the use of rigid analytic geometry.

Let $L$ be the minimal Galois extension of $K$ such that $X$ admits a stable model over $\mathcal{O}_L$, and let $G$ denote the Galois group of $L$ over $K$. The next lemma gives an upper bound for the exponent of $G$.

**Lemma 6.** *Let $X/K$ be as in Lemma 5. Then the exponent of $G$ divides $4p(q-1)$.*

*Proof.* We keep the notation introduced in the proof of Lemma 5. It is well known that $G$ injects canonically into $\mathrm{Aut}_k(\mathcal{Y}_k)$ ([Des, 5.16]). So it is enough to prove that $4p(q-1)$ is divisible by the exponent of the latter group.

Let $\infty$ denote the intersection point of $E$ and $F$. This point is the pole of the rational function $\tilde{u}$ on $E$. Denote by $\mathrm{Aut}_\infty(E)$ the group of $k$-automorphisms of $E$ fixing $\infty$. Then one has an exact sequence

$$0 \to \mathrm{Aut}_\infty(E) \times \mathrm{Aut}_\infty(F) \to \mathrm{Aut}_k(\mathcal{Y}_k) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

The cyclic group of order 2 is generated by the permutation of the irreducible components of $\mathcal{Y}_k$. Any automorphism $\tau \in \mathrm{Aut}_\infty(E)$ is given by $\tau : \tilde{u} \mapsto a\tilde{u} + b$, and $\tilde{z} \mapsto c\tilde{z}$, with $a \in \mathbb{F}_q^*$, $c^2 = a$, and $b \in \mathbb{F}_q$. Therefore, the exponent of $\mathrm{Aut}_\infty(E)$ divides $2p(q-1)$ and, hence, the exponent of $\mathrm{Aut}_k(\mathcal{Y}_k)$ divides $4p(q-1)$. □

Let us now summarize the example discussed in this article. Let $X/K$ be the curve introduced in Lemma 5. Let $J/K$ denote its Jacobian. Raynaud has shown how to compute the group $\Phi_K(J)$ in terms of a regular model of $\mathcal{X}/\mathcal{O}_K$ of $X/K$ ([BLR, 9.6]). Using Raynaud's result, Lemma 3, and Lemma 4, we find that $\Phi_K = \mathbb{Z}/q^2\mathbb{Z}$. Let $\mathcal{A}/\mathcal{O}_K$ denote the Néron model of $J/K$. Raynaud has given a description of $\mathcal{A}_k^0$ in terms of $\mathcal{X}/\mathcal{O}_K$ ([BLR], Theorem 4 on page 267 and Propositions 9 and 10 on pages 248–249). Using Raynaud's result and Lemma 5, we find that $J/K$ has potentially good reduction, which implies that $\Psi_{K,L} = \Phi_K$. Lemma 6 shows that the exponent of $\mathrm{Gal}(L/K)$ does not kill the $p$-part of $\Phi_K$.

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