THE CRITICAL POLYNOMIAL OF A GRAPH

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Abstract. Let G be a connected graph on n vertices with adjacency matrix A_G. Associated to G is a polynomial $d_G(x_1,\ldots,x_n)$ of degree n in n variables, obtained as the determinant of the matrix $M_G(x_1,\ldots,x_n)$, where $M_G = \mathrm{Diag}(x_1,\ldots,x_n) - A_G$. We investigate in this article the set $V_{d_G}(r)$ of non-negative values taken by this polynomial when $x_1,\ldots,x_n \geq r \geq 1$. We show that $V_{d_G}(1) = \mathbb{Z}_{\geq 0}$. We show that for a large class of graphs one also has $V_{d_G}(2) = \mathbb{Z}_{\geq 0}$. When $V_{d_G}(2) \neq \mathbb{Z}_{\geq 0}$, we show that for many graphs $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$. We give numerical evidence that in many cases, the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ might in fact be finite. As a byproduct of our results, we show that every graph can be endowed with an arithmetical structure whose associated group is trivial.

1. Introduction

Given any integer r, we let as usual $\mathbb{Z}_{\geq r}$ denote the set of integers greater than or equal to r. For any polynomial $f \in \mathbb{Z}[x_1,\ldots,x_n]$ and integer $r \geq 1$, consider the following set of non-negative values

$$V_f(r) := \{f(a_1,\ldots,a_n) \mid a_1,\ldots,a_n \in \mathbb{Z}_{\geq r}\} \cap \mathbb{Z}_{\geq 0}.$$

Given a general polynomial f, we do not know of any general results that quantify the difference between $V_f(1)$ and $V_f(2)$, and more generally, provide insights on the decreasing chain of sets $V_f(1) \supseteq V_f(2) \supseteq V_f(3) \supseteq \ldots$. It is clear that $V_f(1) \supset V_f(2)$ can be strictly decreasing. As the example below of the path A_2 on two vertices shows, the complement of $V_f(2)$ in $V_f(1)$ can be infinite. In this article, inspired by questions in algebraic geometry, we describe the sets $V_f(1) \supseteq V_f(2)$ for a large class of polynomials f associated to graphs, and give evidence that in many cases, the complement of $V_f(2)$ in $V_f(1)$ is finite for such polynomials.

Let G denote a connected (undirected) graph on n vertices $v_1,\ldots,v_n$, and no self-loops. Let A_G denote the associated symmetric adjacency matrix. Let M_G denote the matrix with coefficients in the polynomial ring $\mathbb{Z}[x_1,\ldots,x_n]$ defined as:

$$M_G := \mathrm{Diag}(x_1,\ldots,x_n) - A_G.$$

Let

$$d_G(x_1,\ldots,x_n) := \mathrm{determinant}(M_G) \in \mathbb{Z}[x_1,\ldots,x_n].$$

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The matrix M_G is considered in [14], where the principal ideal of $\mathbb{Z}[x_1,\dots,x_n]$ generated by $d_G(x_1,\dots,x_n)$ is called the n-th *critical ideal* of the graph G. We will call d_G the *critical polynomial* of G. Since the adjacency matrix A'_G associated with a different ordering of the vertices of G is of the form $A'_G = P^{-1}A_GP$ for some permutation matrix P, the polynomial d_G is indeed independent of the choice of the ordering of the vertices of G. The polynomial d_G is also considered in [19], Proposition 1, where it is shown that two simple graphs G and G' on n vertices are isomorphic if and only if there is an ordering of the vertices of G and of the vertices of G' such that the associated polynomials $d_G(x_1,\dots,x_n)$ and $d_{G'}(x_1,\dots,x_n)$ are equal.

When G is a graph, it is well-known that the set $V_{d_G}(1)$ always contains the value 0. Indeed, recall that the *degree* d_i of the vertex v_i in a graph G is the number of edges of G attached to v_i. The *Laplacian* of the graph is obtained by evaluating M_G at $x_i = d_i$ for $i = 1,\dots,n$. The Laplacian of G has determinant 0 since the columns of the Laplacian add to zero, and so are linearly dependent. We show in this article in Theorem 1.1 (i) that the set $V_{d_G}(1)$ can be completely described, and is always equal to $\mathbb{Z}_{\geq 0}$.

Motivated by geometric questions recalled in 1.4, we also investigate in this article the properties of the set $V_{d_G}(2)$. Consider for instance the example of the path A_2 on two vertices. In this case,

$$M_{A_2} = \begin{pmatrix} x_1 & -1 \\ -1 & x_2 \end{pmatrix} \text{ and } d_{A_2}(x_1,x_2) = x_1x_2 - 1.$$

It is clear that $V_{d_{A_2}}(1) = \mathbb{Z}_{\geq 0}$. The complement of $V_{d_{A_2}}(2)$ in $V_{d_{A_2}}(1)$ is infinite since

$$V_{d_{A_2}}(2) = \mathbb{Z}_{\geq 1} \setminus \{p-1 \mid p \text{ prime}\}.$$

To state our results on $V_{d_G}(2)$, we first recall the following concepts. Let $a_1,\dots,a_n \in \mathbb{Z}_{>0}$. Let us denote by $M_G(a_1,\dots,a_n)$ the integer matrix obtained from M_G by evaluating M_G at $x_i = a_i$, $i = 1,\dots,n$. Such matrices play a considerable role in geometry, where they might be in addition endowed with a positivity property. Recall that a symmetric integer matrix $B \in M_n(\mathbb{Z})$ is *positive semi-definite* (resp. *positive definite*) if for every non-zero vector $X \in \mathbb{Z}^n$, we have ${}^tXBX \geq 0$ (resp. ${}^tXBX > 0$). When matrices of the form $M_G(a_1,\dots,a_n)$ are positive definite, they are M-matrices ([33], Definition).

In some geometric contexts, such as when $M_G(a_1,\dots,a_n)$ is obtained as the intersection matrix associated with a finite set of curves on a surface, the following finite abelian group Φ_{M_G} is of interest. Let $M \in M_n(\mathbb{Z})$. Then

(a) If $\det(M) \neq 0$, the *discriminant group* $\Phi_M := \mathbb{Z}^n/\mathrm{Im}(M)$ has order $|\det(M)|$.
(b) More generally, when $\mathrm{rank}(M) = \rho < n$, then $\mathbb{Z}^n/\mathrm{Im}(M)$ is isomorphic to the product of $\mathbb{Z}^{n-\rho}$ by a finite abelian group that we will denote Φ_M. In other words, Φ_M is isomorphic to the torsion subgroup of $\mathbb{Z}^n/\mathrm{Im}(M)$.

For instance, given a connected graph G, consider its Laplacian $L = M_G(d_1,\dots,d_n)$. The kernel of L is generated by the vector ${}^tR = (1,\dots,1)$ and the group Φ_L can be identified with the group $\mathrm{Ker}({}^tR)/\mathrm{Im}(L)$. Its order is the number of spanning trees of the graph G. The group Φ_L is found under various names in the literature (see for instance the introduction to [27], and [25], [4], [5]); in this article, we will call Φ_L the *critical group* of the graph.

The pair (L, R) attached to G can be generalized as follows. An *arithmetical structure* on G (see [25], Theorem 1.4) is a pair (M, R) such that $M = M_G(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in \mathbb{Z}_{\geq 1}$ and such that ${}^tR = (r_1, \dots, r_n)$ is an integer vector with positive coefficients and $\gcd(r_1, \dots, r_n) = 1$ satisfying $MR = 0$. It turns out that then M is positive semi-definite of rank $n - 1$. The associated group Φ_M is isomorphic to $\mathrm{Ker}({}^tR)/\mathrm{Im}(M)$. Such arithmetical structures arise in algebraic geometry, and much is known about the associated group Φ_M (see, e.g., [26]).

Define now a subset $V_G(r) \subseteq V_{d_G}(r)$ as follows: $u \in \mathbb{Z}_{\geq 0}$ belongs to $V_G(r)$ if and only if there exists $a_1, \dots, a_n \in \mathbb{Z}_{\geq r}$ such that:

(i) $u = \det(M_G(a_1, \dots, a_n))$.
(ii) The matrix $M_G(a_1, \dots, a_n)$ is positive definite when $\det(M_G) \neq 0$, and positive semi-definite of rank $n - 1$ when $\det(M_G) = 0$.
(iii) The associated group $\Phi_{M_G(a_1,\dots,a_n)}$ is cyclic.

Recall that a graph H is an *induced subgraph of* G if it can be obtained by removing from G a non-empty set of vertices of G along with all the edges of G attached to any of these vertices. We are now ready to state our main theorems.

Theorem 1.1. (proved in 2.9) *Let G be a connected graph. Then*

(a) $V_G(1) = \mathbb{Z}_{\geq 0}$.
(b) *Suppose that G contains an induced subgraph H such that $1 \in V_H(2)$. Then*
   (i) $V_G(2)$ *contains* $\mathbb{Z}_{>0}$.
   (ii) *If G is obtained from H through a sequence of induced subgraphs $H = H_1 \subset H_2 \subset H_k = G$ such that for each $i = 1, \dots, k - 1$, H_{i+1} is constructed from H_i by adding exactly one vertex of degree at least 2, then $V_G(2) = \mathbb{Z}_{\geq 0}$.*

As noted above, we always have $0 \in V_{d_G}(1)$ because the determinant of the Laplacian L of G is 0. On the other hand, the critical group Φ_L associated with L is not always cyclic. The question of determining the proportion of connected graphs having cyclic critical groups was raised for instance in [27], section 4, and progress on this question for random graphs can be found in [35]. Part (a) of Theorem 1.1 implies in particular that on any graph G, there always exists an arithmetical structure (M, R) whose associated group Φ_M is cyclic (in fact, even trivial, see 2.10).

Recall that a graph is *simple* if there is at most one edge between any two vertices of G. The smallest simple graphs H with $1 \in V_H(2)$ both have 4 vertices: the extended cycle $\mathcal{C}_3^+$ (see 3.1) and the cone $C(A_3)$ on the path A_3 on 3 vertices (see 3.5). Part (b) of Theorem 1.1 allows us to prove the following general theorem.

Theorem 1.2. (proved in 3.7) *Let G be a connected simple graph. Then $V_G(2) \supset \mathbb{Z}_{>0}$, except possibly if G is a tree, a cycle $\mathcal{C}_n$, a complete bipartite graph $\mathcal{K}(p, q)$, a complete graph $\mathcal{K}_n$, $n \leq 13$, an extended cycle $\mathcal{C}_n^+$, $n \leq 7$, or the cone $C(A_3)$.*

Whether the value 1 belongs to $V_G(2)$ when G is a tree was investigated in [8, 9, 10, 11, 12]. Corollary 11 in [10] gives a list of trees H such that if a tree G of diameter at least

4 contains such H, then $1 \in V_G(2)$. The smallest such trees have 8 vertices, starting with the Dynkin diagram E_8.

Recall that the star $\mathcal{S}_n$ on $n \geq 4$ vertices is a tree with a vertex of degree $n-1$. It is shown in [9], Proposition 6, that $1 \notin V_{\mathcal{S}_n}(2)$ when $n \leq 59$. No integer $n \geq 4$ is known such that $1 \in V_{\mathcal{S}_n}(2)$.

Denote by $\mathcal{S}_n^+$ the graph obtained by adjoining a single vertex to the star $\mathcal{S}_n$ and linking it with a single edge to a vertex of degree 1 in $\mathcal{S}_n$. The family $\mathcal{S}_n^+$, $n \geq 4$, is another family of graphs where none of its members are known to have $1 \in V_{\mathcal{S}_n^+}(2)$.

When Part (b) of Theorem 1.1 does not apply, the set $V_G(2)$ seems very difficult to describe precisely. Theorem 4.2 can often be used to prove that, at least, $V_{d_G}(2)$ is *dense* in $\mathbb{Z}_{\geq 0}$ (the definition of dense in this context is recalled in 4.1). We state below an explicit consequence of Theorem 4.2 which complements Theorem 1.2. It would be interesting to determine whether $V_{d_G}(2)$ is always dense in $\mathbb{Z}_{\geq 0}$.

Theorem 1.3. (proved in 4.4) *Recall the list of graphs in Theorem* 1.2 *where it is not known that $V_{d_G}(2)$ contains $\mathbb{Z}_{>0}$. When G is one of the following graphs, the set $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$:*

(a) *G is a Dynkin diagram, an extended Dynkin diagram, a star $\mathcal{S}_n$, or an extended star $\mathcal{S}_n^+$.*
(b) *G is a cycle $\mathcal{C}_n$ or an extended cycle $\mathcal{C}_n^+$.*
(c) *G is the cone $C(A_3)$, the complete graph $\mathcal{K}_n$, the complete bipartite graph $\mathcal{K}(2,n)$ or $\mathcal{K}(3,n)$.*

The only examples of graphs where the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is *known to be infinite* are the *banana graphs*, the graphs on two vertices linked by $a \geq 1$ edges. Thus it is also natural to wonder whether the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is not only of density 0 but in fact is finite for most graphs.

Our study was motivated by considerations from algebraic geometry. In the next remark, we give a brief exposition of how the results of this article pertain to this field of research.

Remark 1.4. Matrices of the form $M_G(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \geq 1$ arise in algebraic geometry when considering a finite collection of curves $\{C_i,\ i = 1, \ldots, n\}$, on a non-singular surface S. Attached to each pair of distinct curves C_i and C_j is an *intersection number* $(C_i \cdot C_j) \geq 0$ which counts (with multiplicities) how many times C_i and C_j intersect (see, e.g., [24], 9.1). The *dual graph* G attached to the configuration of curves is the graph on n vertices $v_1, \ldots, v_n$ such that when $i \neq j$, v_i is linked to v_j by $(C_i \cdot C_j)$ edges.

Each curve C_i on S has a *self-intersection number* $(C_i \cdot C_i)$, and these numbers are known to be strictly negative when the configuration $\cup_{i=1}^n C_i$ occurs as the exceptional divisor of the resolution of a singularity. The *intersection matrix* $((C_i \cdot C_j))_{1 \leq i,j \leq n}$ is then of the form $-M_G(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \geq 1$, and is known to be negative-definite. It is often the case that a *minimal* resolution of singularities leads to a matrix $M_G(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \geq 2$, which explains our interest in understanding the possible values of the determinants of such matrices when $a_1, \ldots, a_n \geq 2$.

Matrices $M_G(a_1,\ldots,a_n)$ which are only positive semi-definite arise from configuration of curves associated with a degeneration of a non-singular curve, and in this case *minimal* special fibers of degenerating curves generally also lead to matrices $M_G(a_1,\ldots,a_n)$ with $a_1,\ldots,a_n \geq 2$.

The collection of curves $\{C_i, i=1,\ldots,n\}$ attached to the resolution of a surface singularity, and its associated intersection matrix, play an important role in understanding the singularity. It is still an open problem to completely characterize the matrices that can occur as intersection matrices associated to $\mathbb{Z}/p\mathbb{Z}$-quotient surface singularities in prime characteristic p. In previous works on such singularities, the author showed that the intersection matrix M associated with the resolution of such quotient singularity can only have determinant equal to a power of p, and that the finite group Φ_M associated to M is killed by p (see [29], 3.18, and [30], 6.3, 7.1, for examples). The results of this article indicate that matrices $M_G(a_1,\ldots,a_n)$ of prime determinant p are plentiful.

Motivated by the problem of classifying resolutions of cyclic quotient singularities, it is natural to wonder, given a graph G, whether, for all but finitely many primes p, there always exists a set of diagonal elements $a_1,\ldots,a_n \geq 2$ (depending on p) such that $M_G(a_1,\ldots,a_n)$ has determinant p. The answer to this question would be positive if it were possible to show, more generally, that the complement of the set $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ is finite.

Remark 1.5. It is a classical problem in number theory to study the integer values taken by an integer polynomial $f(x_1,\ldots,x_n)$. When G is a graph, the polynomial $d_G(x_1,\ldots,x_n)$ is a polynomial of degree n in n variables consisting only of *squarefree* monomials. A famous polynomial in number theory, $f := x_1^n+\cdots+x_n^n$, is also of degree n in n variables but is indeed as far as having squarefree monomials as possible. The problem of determining the set $V_f(1)$ in this case is related to the classical Waring's problem. When $n=2$, the set $V_f(1)$ has positive density in $\mathbb{Z}_{\geq 0}$, but does not contain any integer that is congruent to 3 modulo 4. When $n=3$, the set $V_f(1)$ is infinite, but does not contain any integer that is congruent to 4 or 5 modulo 9. It is an open problem in this case to determine whether the set $V_f(1)$ has positive density (see [16] for positive evidence towards this question).

Let us mention here another analogous question in number theory where the set $V_g(1)$ in this case misses only finitely many values, but where it is still an open question to completely determine $V_g(1)$. The polynomial $g := xy+yz+zx$ consists only of squarefree monomials. When $i>0$ is any integer, the equation $xy+yz+zx=i$ always has solutions in positive integers except for at most 19 values of i ([6] Theorem 1.1). The first 18 such values are known explicitly and are in the interval $[1,462]$. If the Generalized Riemann Hypothesis is assumed, the complement of the set $V_g(1)$ in $\mathbb{Z}_{>0}$ consists exactly of these 18 known values.

This article exhibits many graphs G where the set $V_{d_G}(2)$ misses some positive values, but computations nevertheless suggest that it contains all positive values except for finitely many (see, e.g., 3.1, 5.4, 6.2). It would be interesting to determine if these polynomials $d_G(x_1, \dots, x_n)$ indeed have this property. The easiest example of such polynomial is $d_{A_3}(x, y, z) = xyz - x - z$, associated with the path A_3 on 3 vertices. Computations suggest that the complement of the set $V_{A_3}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the set $[0, 1, 2, 3, 5, 6, 9, 11, 14, 15, 35, 105, 510]$ (see Proposition 6.3).

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2. First Main Theorem

Let $M \in M_n(\mathbb{Z})$. We will use the following standard notation. Let M_{ij} denote the submatrix obtained by removing from M its i-th row and its j-th column. Let M^* denote the *comatrix* of M, with $MM^* = (M^*)M = \det(M)\mathrm{Id}_n$. By definition, $(M^*)_{ij} = (-1)^{i+j}\det(M_{ji})$. The matrix M is a *positive* matrix if all the entries of M are positive.

The group Φ_M is isomorphic by definition to the torsion subgroup of $\mathbb{Z}^n/\mathrm{Im}(M)$. If $0 < \mathrm{rank}(M) = \rho < n$, then there exist two matrices P and Q in $\mathrm{GL}_n(\mathbb{Z})$ such that PMQ is a diagonal matrix of the form $\mathrm{Diag}(0, \dots, 0, f_1, \dots, f_\rho)$, with $f_1 \mid f_2 \mid \cdots \mid f_\rho$. This diagonal matrix is called the Smith Normal Form of M. The group Φ_M is isomorphic to $\prod_{i=1}^{\rho} \mathbb{Z}/f_i\mathbb{Z}$, and thus Φ_M is cyclic if and only if $\rho = 1$ or $f_{\rho-1} = 1$.

2.1. Let G be a connected graph on n vertices. The matrices $M = M_G(a_1, \dots, a_n) \in M_n(\mathbb{Z})$ with $a_1, \dots, a_n \geq 1$ have several very useful properties when they are positive semi-definite.

(a) Assume that $\det(M) \neq 0$ and that M is positive definite. Then the inverse M^{-1} of M is a positive matrix.
(b) Assume that $\det(M) = 0$ and that M is positive semi-definite of rank $n-1$. Then there exists a unique vector R in $\mathbb{Z}^n_{>0}$ with coprime coefficients and such that $MR = 0$. We have $M^* = |\Phi_M| R({}^tR)$.
(c) Assume that $M_G(a_1, \dots, a_n)$ is as in (a) or (b). For any non-zero vector $(b_1, \dots, b_n) \in \mathbb{Z}^n_{\geq 0}$, the matrix $M_G(a_1 + b_1, \dots, a_n + b_n)$ is positive definite, and

$$\det M_G(a_1, \dots, a_n) < \det M_G(a_1 + b_1, \dots, a_n + b_n).$$

Property (a) is F15, page 180 of [33]. Property (b) follows from Proposition 1.1 and Theorem 1.4 in [25]. Property (c) is A3, page 179 of [33], when $M_G(a_1, \dots, a_n)$ is positive definite. If it is only positive semi-definite, show first that $M_G(a_1, \dots, a_i + 1, \dots, a_n)$ is definite positive for any i, and apply A3 to these n positive definite matrices.

Remark 2.2. When $M = M_G(a_1, \dots, a_n)$ is positive definite as in (a), there exists a unique positive vector R minimal with the property that MR is positive ([3], page 132). This vector is called the *fundamental vector* of the matrix M. The quantity $({}^tR)MR$ is an important numerical invariant associated with M. When $M = M_G(a_1, \dots, a_n)$ is positive semi-definite of rank $n-1$, numerical invariants associated with the arithmetical structure (M, R) described in (b) are discussed in [28], 2.1 and 4.1.

Remark 2.3. Let $w \in \mathbb{Z}_{\geq 0}$. It is known that there are only *finitely many* points $(a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 1}$ such that $M_G(a_1, \dots, a_n)$ is positive semi-definite and has determinant w. This statement is proved in [23], Theorem 1, when $w > 0$ and the matrix is positive definite, and in [25], 1.6, when $w = 0$ and the matrix is positive semi-definite of rank $n-1$.

Counting explicitly the number of distinct arithmetical structures on certain graphs is addressed for instance in [2], [7], and [15]. Counting the number of solutions to $d_G(x_1, \dots, x_n) = 1$ when G is the path A_n is found in [23].

Our next proposition shows that the existence of an arithmetical structure on G implies that infinitely many values in $V_{d_G}(1)$ are known explicitly. We denote by κ the *number of spanning trees* of a graph G.

Proposition 2.4. *Let G be a connected graph on n vertices.*

(a) *Suppose that every vertex in G has degree at least d. Then $V_{d_G}(d)$ contains all positive multiples of κ. Moreover, for each $\ell > 0$, there exists a positive definite matrix $M_G(a_1, \dots, a_n)$ with $a_i \geq d$ such that $d_G(a_1, \dots, a_n) = \ell\kappa$.*

(b) *More generally, let (M, R) be any arithmetical structure on G. Write ${}^tR = (r_1, \dots, r_n)$ with $\gcd(r_1, \dots, r_n) = 1$, $M = \mathrm{Diag}(a_1, \dots, a_n) - A_G$, and let Φ_M denote the associated group. Let a_{min} denote the minimum of the integers $a_1, \dots, a_n$. Then $V_{d_G}(a_{min})$ contains every integer of the form $\ell|\Phi_M|r_i^2$ for any integer $\ell \geq 0$ and any $i = 1, \dots, n$.*

Proof. (a) Recall that $M_G(d_1, d_2, \dots, d_n)$ is the Laplacian of G. It is well-known that the determinant of any principal submatrix of size $n-1$ of the Laplacian is equal to κ. Consider the matrix $M_G(t, d_2, \dots, d_n) \in \mathbb{Z}[t]$. Its determinant is $\kappa(t - d_1)$. Indeed, it is clear that this determinant is a linear polynomial in t. The coefficient of t is κ, and $t = d_1$ must be a root of the polynomial.

For every value $\ell + d_1 > d_1 \geq d$, we have $M_G(\ell + d_1, d_2, \dots, d_n)$ positive definite since $M_G(d_1, d_2, \dots, d_n)$ is positive semi-definite of rank $n-1$ (2.1 (c)).

(b) Recall that for an arithmetical structure (M, R), we have $M^* = |\Phi|R({}^tR)$ (2.1 (b)). Consider the matrix $M_G(t, a_2, \dots, a_n)$. Its determinant is $|\Phi_M|r_1^2(t - a_1)$. For every value $\ell + a_1 > a_1 \geq a_{min}$, we have $M_G(a_1 + \ell, a_2, \dots, a_n)$ positive definite of determinant $|\Phi_M|r_1^2\ell$. □

Remark 2.5. Let $M_G(a_1, \dots, a_n)$ be a positive definite matrix with $a_i \geq 2$ such that $d_G(a_1, \dots, a_n) = \ell\kappa$, as in Proposition 2.4 (a) with $d = 2$. It is not always possible to find such matrix such that its associated group Φ_{M_G} is cyclic. Indeed, in the case of the cycle $\mathcal{C}_2$ on $n = 2$ vertices, which has $\kappa = 2$, the matrix

$$M_{\mathcal{C}_2}(x, y) = \begin{pmatrix} x & -2 \\ -2 & y \end{pmatrix}$$

has determinant $xy - 4$. When $xy - 4 = 2\ell$, and $2\ell + 4$ is a power of 2 and $x, y \geq 2$, this equation has only solutions with both x and y even. In this case, ℓ is even, and the associated group $\Phi = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ is not cyclic.

In particular, $G = \mathcal{C}_2$ is an example where the complement of $V_G(2)$ in $V_{d_G}(2)$ is infinite, since this complement contains every integer of the form 2ℓ with $\ell = 2^m - 2$.

Corollary 2.6. *Let G be a tree. Then $V_{d_G}(1) = \mathbb{Z}_{\geq 0}$.*

Proof. The corollary follows immediately from Proposition 2.4 (a) since $\kappa = 1$ when G is a tree. □

The following lemma is needed in the proof of our next proposition.

Lemma 2.7. *Let N denote a $n-1 \times n-1$ square matrix with coefficients in a commutative ring A. Let M denote the following $n \times n$ matrix in $A[t, t_2, \ldots, t_n]$:*

$$M = \begin{pmatrix} t & t_2 & \cdots & t_n \\ t_2 & & & \\ \vdots & & N & \\ t_n & & & \end{pmatrix}.$$

Let ${}^tT := (t_2, \ldots, t_n)$. Then

$$\det(M) = \det(N)t - ({}^tT)(N^*)T.$$

Proof. Recall that by definition, $(N^*)_{ij} = (-1)^{i+j} \det(N_{ji})$. The lemma follows by expanding $\det(M)$ using the first row of M. □

Given a vertex v of G, let G_v denote the subgraph of G obtained by removing from G the vertex v and all the edges attached to v.

Proposition 2.8. *Let G be a connected graph. Let v be a vertex such that G_v is connected and $1 \in V_{G_v}(r)$. Then*

(a) *$V_G(r) \supseteq \mathbb{Z}_{>0}$ when $r = 1$ or 2. In general, $V_G(r) \supseteq \mathbb{Z}_{\geq r-1}$.*
(b) *There exists on G an arithmetical structure such that the associated group Φ is trivial and, hence, cyclic. In particular, $0 \in V_G(1)$.*
(c) *If the degree of v is at least 2 and $1 \in V_{G_v}(2)$, then $0 \in V_G(2)$. More precisely, there exists on G an arithmetical structure (M, R) with $M = M_G(a_1, \ldots, a_n)$ and $a_1, \ldots, a_n \geq 2$ such that the associated group Φ_M is trivial.*

Proof. Without loss of generality, we may assume that $v = v_1$. By hypothesis we can find $a_2, \ldots, a_n \geq r$ such that the matrix $N := M_{G_v}(a_2, \ldots, a_n)$ has determinant 1 and is positive definite. Consider then the determinant of the matrix $M_G(t, a_2, \ldots, a_n)$, which has N in its lower right corner. Since N is positive definite by hypothesis, we find that N^* is a positive matrix. Hence, Lemma 2.7 applied to $M_G(t, a_2, \ldots, a_n)$ shows that

$$\det M_G(t, a_2, \ldots, a_n) = t - a$$

with $a > 0$.

(a) Let $a_1 > a$. Since N is positive definite and $\det M_G(a_1, a_2, \ldots, a_n) > 0$, we find that the matrix $M_G(a_1, a_2, \ldots, a_n)$ is positive definite. In addition, its associated group Φ is cyclic. Indeed, $M_G(a_1, a_2, \ldots, a_n)$ contains the square submatrix N of size $n-1 \times n-1$, which has determinant 1. It is well-known that the group Φ is cyclic if and only if the greatest common divisor of all the minors of size $n-1$ is equal to 1.

Suppose that $\max(r, a+1) = r$. Then $V_G(r) \supseteq \mathbb{Z}_{\geq r-a}$. Suppose now that $\max(r, a+1) = a+1$. Then $V_G(r) \supseteq \mathbb{Z}_{\geq 1}$. This proves (a).

(b) Consider now the matrix $M := M_G(a, a_2, \ldots, a_n)$, of determinant 0. We claim that this matrix is positive semi-definite. We prove this claim by exhibiting a positive vector R, of the form ${}^tR = (1, {}^tR_0)$, with $MR = 0$. Recall that the matrix M has the form

$$M = \begin{pmatrix} a & -s_2 & \cdots & -s_n \\ -s_2 & & & \\ \vdots & & N & \\ -s_n & & & \end{pmatrix},$$

for some non-negative integers $s_2, \ldots, s_n$. Since the graph G is connected, one at least of the integers $s_2, \ldots, s_n$ must be positive. Write ${}^tS = (s_2, \ldots, s_n)$. Since $\det(N) = 1$, we can find an integer vector R_0 such that $-S + NR_0 = 0$. Since S is a positive vector and N^* is a positive matrix, we find that R_0 is a positive vector. Lemma 2.7 shows that $\det(N)a = ({}^tS)N^*S$. It follows that $MR = 0$, as desired. Since $\det(N) = 1$ and the first coefficient of R is 1, we find that the group Φ associated with M is trivial and, hence, cyclic. This shows that $0 \in V_G(1)$.

(c) Consider again the structure (M, R) introduced in (b). It is clear that when $1 \in V_{G_v}(2)$ and $a \geq 2$, then $0 \in V_G(2)$. The integer a is obtained as $a = ({}^tS)N^*S$. The hypothesis that the degree of v is at least 2 implies that if v is linked to only one vertex w of G_v, then the number of edges between v and w is at least 2. Thus, since the matrix N^* is positive, we must have $a = ({}^tS)N^*S \geq 2$. □

2.9. *Proof of Theorem 1.1.*

In Part (a), the graph G contains an induced subgraph H on two vertices linked by $a \geq 1$ edges. The polynomial $d_H(x, y) = xy - a^2$ takes all non-negative values when $x, y \geq 1$. In particular, the value 1 is taken with $x = a^2 + 1$ and $y = 1$ and so, $1 \in V_H(1)$.

Let now H be any induced subgraph of G. Let $w_1, \ldots, w_k$ denote the vertices of G that do not belong to H. For $j \leq k$, let H_j denote the induced subgraph of G on the vertices of H and $\{w_1, \ldots, w_j\}$. We have $H \subset H_1 \subset \cdots \subset H_k = G$.

Assume that $1 \in V_H(r)$, with $r = 1$ or 2. Then Proposition 2.8 (a) can be applied successively to each pair $H \subset H_1$, $H_1 \subset H_2$, $\ldots$, $H_{k-2} \subset H_{k-1}$ to show that $1 \in V_{H_{k-1}}(r)$.

When $1 \in V_{H_{k-1}}(1)$, we use Proposition 2.8 (a) and (b) to conclude that $V_{H_k}(1) = \mathbb{Z}_{\geq 0}$. This proves Theorem 1.1 (a).

When $1 \in V_{H_{k-1}}(2)$, we use Proposition 2.8 (a) to conclude that $V_{H_k}(2) \supseteq \mathbb{Z}_{>0}$. We then use Proposition 2.8 (c) to conclude the proof of Theorem 1.1 (b). □

Corollary 2.10. *Let G be a connected graph. Then there exists on G an arithmetical structure (M, R) such that its associated group Φ_M is trivial.*

Proof. In the proof of Theorem 1.1 (a) above in 2.9, we find that the graph G is such that $1 \in V_{H_{k-1}}(1)$. Proposition 2.8 (b) applied to $H_{k-1} \subset G$ immediately implies the corollary. □

Remark 2.11. While $0 \in V_G(1)$, it may happen that $0 \notin V_G(2)$. Indeed, $0 \notin V_{d_G}(2)$ when G is a Dynkin diagram (Proposition 5.3). Three additional such examples, the extended cycles $\mathcal{C}_n^+$ for $n = 2, 3, 5$, are found in Remark 3.4, and it would be interesting to classify the graphs where $0 \notin V_{d_G}(2)$.

It is also possible to have $0 \in V_{d_G}(2)$ but $0 \notin V_G(2)$. Indeed, this happens for instance for the extended Dynkin diagram $\tilde{D}_n$ with n even: This graph has only one arithmetical structure, with matrix $M_{\tilde{D}_n}(2, \ldots, 2)$, and its associated group Φ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when n is even.

One may wonder whether it is possible to have $0 \in V_G(2)$, but no arithmetical structure $M_G(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \geq 2$ such that the associated group Φ is trivial. One such example might be $\mathcal{C}_6^+$ (see Remark 3.4). Another example might be the bipartite graph $\mathcal{K}(4, 4)$.

Remark 2.12. Modified appropriately, the proof of Theorem 1.1 does produce some information on the set $V_G(3)$ when $1 \in V_H(3)$. We do not investigate the properties of the sets $V_G(r)$ any further in this article when $r \geq 3$.

3. Second Main Theorem

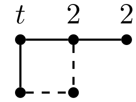
Let us denote by $\mathcal{C}_n^+$ the graph on $n + 1$ vertices obtained by attaching a single vertex using a single edge to the cycle $\mathcal{C}_n$ on n vertices. Such graph is sometimes called a *pan*, and is a type of *tadpole* graph.

Example 3.1. Let $G = \mathcal{C}_n^+$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is always finite. In the table below, we provide a set L which computations indicate contains the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$.

n	L
2	[0] (see 6.5)
3	[0, 2, 14, 20, 26, 38, 44, 68, 254]
4	[2, 3, 7, 10, 19, 39, 79, 154]
5	[0, 2, 8, 12, 18]
6	empty
7	[6, 66, 94]

With the choice of labeling as below, we find that $\det M_G(t, 2, \ldots, 2) = (n+1)t-(3n+2)$, so that choosing $t = 3$ produces $1 \in V_G(2)$ when $G = \mathcal{C}_n^+$.

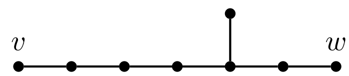
t 2 2



2 2

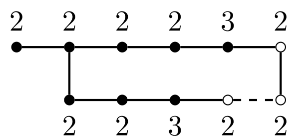
Proposition 3.2. *Let $G = \mathcal{C}_n^+$. Then $1 \in V_G(2)$, and if $n \geq 8$, then $V_G(2) = \mathbb{Z}_{\geq 0}$.*

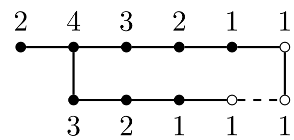
Proof. The computations in Example 3.1 show that $1 \in V_G(2)$ when $n \leq 7$. The tree E_8



has $1 \in V_{E_8}(2)$ (see 5.3). The graph $G = \mathcal{C}_8^+$ is obtained by attaching a new vertex v_0 to both v and w with one edge. To get all graphs $\mathcal{C}_n^+$ with $n > 8$, we first lengthen the chain at v, so that the resulting graph has $n-1$ vertices, and then add a vertex v_0 as above. Theorem 1.1 (b) then implies that $V_G(2) \supseteq \mathbb{Z}_{>0}$. To prove that $0 \in V_G(2)$, we use the following lemma. $\square$

Lemma 3.3. *Let $G = \mathcal{C}_n^+$ with $n \geq 7$. Then G has the following arithmetical structure (M, R). There are $k \geq 0$ white vertices in the graphs below. The vertices of the left graph are adorned with the corresponding coefficient of the diagonal of M, and the vertices of the right graph are adorned with the corresponding coefficient of R:*



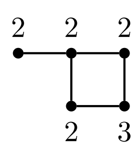


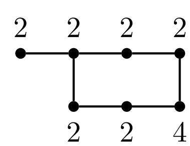
The associated group Φ is cyclic of order $2k+5$.

Proof. We leave it to the reader to verify that (M, R) is an arithmetical structure. Consider the submatrix M' of M obtained by removing the row and column corresponding to the unique vertex v of degree 3. Its determinant is $16(2k+5)$. This shows that $|\Phi| = 2k+5$ since the coefficient of R corresponding to v is 4 and $M^* = |\Phi| R({}^tR)$ (see 2.1 (b)).

To show that Φ is cyclic, it suffices to compute the determinant of a well-chosen $n-2 \times n-2$ submatrix, and show that it is coprime to $|\Phi|$. For this, one can use the submatrix of M' where the row and column corresponding to a vertex of degree 2 adjacent to v have been removed. Its determinant is $2(12k+29)$. We leave the details to the reader. $\square$

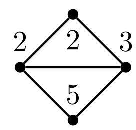
Remark 3.4. When $n = 4$ and 6, the following are arithmetical structures on $G = \mathcal{C}_n^+$, with groups Φ of order 1 and 3 (the vertices are adorned with the corresponding elements on the diagonal of the matrix).





It is likely that $0 \notin V_{d_G}(2)$ when $n = 3, 5$. When $n = 2$ this statement is proved in Proposition 6.5. Preliminary computations did not find any arithmetical structure (M, R) on $\mathcal{C}_4^+$ and $\mathcal{C}_6^+$ where all coefficients of the diagonal of M are at least 2, other than the ones given above.

Example 3.5. Consider the cone $G = C(A_3)$ on the path A_3 (sometimes called the *diamond* graph). Computations suggest that the complement of $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ is finite, and contained in the set $L := [5, 17, 29, 71, 77, 101, 137, 551]$. Note that $\det M_G(2, 2, 5, 3) = 1$, and the graph G below is adorned with the corresponding coefficients of the diagonal.



Theorem 3.6. *Let G be a connected simple graph which is not isomorphic to either $\mathcal{C}_n^+$, $n \geq 3$, or to the cone $C(A_3)$, and does not contain an induced subgraph H isomorphic to either $\mathcal{C}_n^+$, $n \geq 3$, or $C(A_3)$. Then G is either a tree, or a cycle $\mathcal{C}_n$, $n \geq 3$, or a complete graph $\mathcal{K}_n$, $n \geq 4$, or a complete bipartite graph $\mathcal{K}(p, q)$, $p, q \geq 2$.*

Proof. Assume that G is neither a tree nor a cycle. Let m denote the length of the shortest cycle in G. Since we assume that G is simple and not a tree, $m \geq 3$. Consider such a cycle $\mathcal{C}$ of length m in G, with consecutive vertices $w_1, \dots, w_m$. For $i = 1, \dots, m-1$, the vertex w_i is linked to the vertex w_{i+1} by exactly one edge, and w_m is linked to w_1 be one edge. This cycle has to be an induced subgraph of G. Indeed, if there existed an edge between two vertices of $\mathcal{C}$ that are not consecutive, the graph G would have a cycle of length smaller than m. We show below that the case $m \geq 5$ is impossible, that when $m = 4$, the graph G is of the form $\mathcal{K}(p, q)$, and that when $m = 3$, the graph G is of the form $\mathcal{K}_n$.

Since $G \neq \mathcal{C}$, let w be a vertex of G not contained in $\mathcal{C}$, but connected by an edge to a vertex of $\mathcal{C}$. Without loss of generality, we can assume that w is connected to w_1. If w is not connected to any other vertices of $\mathcal{C}$, then G is equal to $\mathcal{C}_m^+$, or contains $\mathcal{C}_m^+$ as induced subgraph, contradicting our hypothesis. Let us then assume that w is also connected by an edge to a vertex w_i with $i > 1$. This is not possible if $m \geq 5$. Indeed, if $m \geq 5$, then G would contain a cycle of length smaller than m.

Assume now that $m = 4$, with the cycle $\mathcal{C}$ having vertices w_1, w_2, w_3, w_4, and with a vertex w of G not on $\mathcal{C}$ connected to w_1. In order for G not to have cycles of length 3, the vertex w can only be connected to w_3. Let w' be any other vertex of G connected to a vertex of $\mathcal{C}$. We claim that w' is then only connected to both w_1 and w_3. Indeed, assume that w' is connected to w_2. Then it must be connected to w_4, otherwise G contains a cycle of length 3 or has $\mathcal{C}_4^+$ as induced subgraph. But now we again find a contradiction by considering the cycle $\{w_2, w', w_4, w_3\}$ with w attached to w_3. This is an induced subgraph isomorphic to $\mathcal{C}_4^+$, contradicting our hypothesis. We have shown so far that G contains a bipartite graph $\mathcal{K}(2, q)$ for some $q \geq 3$, with $\{w_1, w_3\}$ being the first set of 2 vertices in the partition, and $\{w_2, w_4, w, w', \dots\}$ the second set of $q \geq 3$ vertices.

Consider now a maximal subgraph of G of the form $\mathcal{K}(r, s)$, with $r \geq 2$ and $s \geq q$. By maximal we mean that G does not contain a subgraph of the form $\mathcal{K}(r+1, s)$ or $\mathcal{K}(r, s+1)$. We claim then that $G = \mathcal{K}(r, s)$. For convenience, let us denote by $\{u_1, \dots, u_r\}$ and $\{t_1, \dots, t_s\}$ the vertices of the bipartite graph $\mathcal{K}(r, s)$, so that in $\mathcal{K}(r, s)$, there are no edges between vertices in $\{u_1, \dots, u_r\}$ and no edges between vertices in $\{t_1, \dots, t_s\}$.

Suppose that $G \neq \mathcal{K}(r, s)$. There cannot exist an edge of G that links two vertices of $\mathcal{K}(r, s)$ that is not already an edge of $\mathcal{K}(r, s)$, since otherwise the graph G would contain a cycle of length 3. Thus there exists a vertex v of G that is not a vertex of $\mathcal{K}(r,s)$, and is linked by at least one edge to a vertex of $\mathcal{K}(r,s)$. Without loss of generality, we can assume that v is linked to u_1. If v is not linked to any other vertex of $\mathcal{K}(r,s)$, then G contains an induced subgraph of the form $\mathcal{C}_4^+$, which is a contradiction. If v is linked to any of the vertices $\{t_1,\dots,t_s\}$, then G contains a cycle of length 3, again a contradiction. Suppose now that v is linked to u_2, but that for some $i \leq r$, v is not linked to u_i. Then $\{v,u_1,t_1,u_2\}$ are the vertices of a 4-cycle, and adding u_i to it gives an induced subgraph of G of the form $\mathcal{C}_4^+$, again a contradiction. Thus we find that G contains a graph of the form $\mathcal{K}(r,s+1)$, and this is not possible by maximality of the graph $\mathcal{K}(r,s)$. Therefore, $G=\mathcal{K}(r,s)$.

Let us consider now the case $m=3$, with the cycle $\mathcal{C}$ having vertices w_1,w_2,w_3, and with a vertex w of G not on $\mathcal{C}$ connected to w_1. Since G is not equal to $\mathcal{C}_3^+$, and does not contain $\mathcal{C}_3^+$ as induced subgraph, w is connected to a second vertex of $\mathcal{C}$, say, without loss of generality, w_2. If w is not connected to w_3, then G contains the cone $C(A_3)$ as induced subgraph, contradicting our hypothesis. Hence, w is also connected to w_3, and so G contains $\mathcal{K}_4$ as induced subgraph.

Consider now a subgraph of G that is of the form $\mathcal{K}_r$ for some $r \geq 4$ and which is maximal, in the sense that G does not contain a subgraph isomorphic to $\mathcal{K}_{r+1}$. We claim then that $G=\mathcal{K}_r$.

Assume that $G \neq \mathcal{K}_r$. Since G is simple, there must then exist a vertex v of G that is not a vertex of $\mathcal{K}_r$. For convenience, let us denote by $\{u_1,\dots,u_r\}$ the vertices of $\mathcal{K}_r$, and assume that v is linked to u_1. If v is not linked to any other vertex of $\mathcal{K}_r$, then G contains an induced subgraph of the form $\mathcal{C}_3^+$, which is a contradiction. Suppose then that v is linked to u_2, and that there exists some u_i which is not linked to v. Then $\{v,u_1,u_2,u_i\}$ are the vertices of an induced subgraph of G of the form $C(A_3)$, again a contradiction. Suppose then that v is linked in G to all vertices of $\mathcal{K}_r$. This is impossible since G would then contain a subgraph of the form $\mathcal{K}_{r+1}$, contradicting the maximality of $\mathcal{K}_r$. Hence, $G=\mathcal{K}_r$. □

3.7. *Proof of Theorem 1.2.* Let G be a connected simple graph that is neither a tree, a cycle, a complete bipartite graph $\mathcal{K}(p,q)$, a complete graph $\mathcal{K}_n$, nor $\mathcal{C}_n^+$, $n \geq 3$, or the cone $C(A_3)$. Theorem 3.6 shows then that G has to contain an induced subgraph H of the form $C(A_3)$ or $\mathcal{C}_n^+$ for some $n \geq 3$. All these graphs H are such that $1 \in V_H(2)$ (see Proposition 3.2 and Example 3.5). We can thus apply Theorem 1.1 (b) to obtain that $V_G(2)$ contains $\mathbb{Z}_{>0}$ for such G.

When $n \geq 14$, the graph $\mathcal{K}_n$ has the property that $V_{\mathcal{K}_n}(2)$ contains $\mathbb{Z}_{>0}$ (see Corollary 8.3 (b)). When $n \geq 8$, the graph $\mathcal{C}_n^+$ has the property that $V_{\mathcal{C}_n^+}(2)$ contains $\mathbb{Z}_{>0}$ (see Proposition 3.2). □

Let $\mathcal{K}_n^+$ denote the graph obtained from the complete graph $\mathcal{K}_n$ on n vertices by adding one vertex and linking it to $\mathcal{K}_n$ by exactly one edge. Such graph is a type of *lollipop* graph. Recall that the *wheel* $\mathcal{W}_n$ is the cone on the cycle $\mathcal{C}_n$. In particular, $\mathcal{W}_3$ is the complete graph $\mathcal{K}_4$ on 4 vertices.

Corollary 3.8. *Let $n \geq 4$. We have $V_G(2)=\mathbb{Z}_{\geq 0}$ when G is one of the following graphs:*

(a) $G = \mathcal{K}_n^+$.
(b) *G is the cone $C(A_n)$ on the path A_n on n vertices.*
(c) *G is the wheel $\mathcal{W}_n$.*
(d) *G is the graph on $n \geq 5$ vertices obtained from the cycle $\mathcal{C}_n$ by adding a new edge linking two vertices which are not already connected in the cycle.*

Proof. We can apply Theorem 1.2 to each graph in the statement of the corollary to obtain that $V_G(2) \supset \mathbb{Z}_{>0}$. In each case, we can further strengthen this result by showing that $0 \in V_G(2)$ as follows.

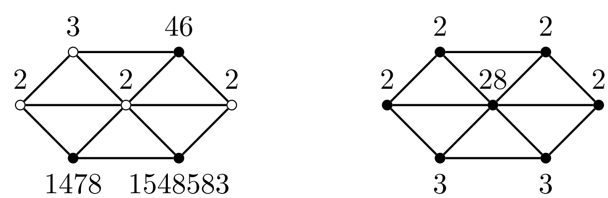
(a) Since $\mathcal{K}_n$ is the cone on $\mathcal{K}_{n-1}$, Theorem 1.1 (b) (ii) can be used to show that $0 \in V_G(2)$.

(b) and (d) To show that $0 \in V_G(2)$, we take the usual Laplacian of G, and note that its associated critical group is always cyclic ([27], Corollary 6.7).

(c) The wheel $\mathcal{W}_n$ is the cone on the cycle $\mathcal{C}_n$. By removing a vertex v on the wheel that belongs to the original cycle, we obtain an induced subgraph isomorphic to the cone $C(A_{n-1})$. We have shown above that $1 \in V_{C(A_{n-1})}(2)$ when $n-1 \geq 3$. It follows then immediately from Theorem 1.1 (b) (ii) that $V_G(2) = \mathbb{Z}_{\geq 0}$. □

Example 3.9. Let $G = \mathcal{W}_n$. The critical group Φ associated to the Laplacian of $\mathcal{W}_n$ is not cyclic (see [5], 9.2). Starting with the extended cycle $\mathcal{C}_3^+$, Theorem 1.1 (b) lets us construct on G a different arithmetical structure (M, R) whose associated group Φ is trivial.

We note below on the example of $\mathcal{W}_6$ that the coefficients of the diagonal of M quickly become very large with this construction. In the wheel $\mathcal{W}_6$ on the left below, the vertices of the original $\mathcal{C}_3^+$ are indicated in white. We then constructed $\mathcal{W}_6$ by adding in sequence three vertices, whose corresponding coefficients on the diagonal are 46, 1478, and 1548583.



We describe in Lemma 3.10 a different arithmetical structure on $\mathcal{W}_{2k}$, where Φ is cyclic of order $6k-1$, and where the coefficients on the diagonal do not grow as fast. We have illustrated the case $k = 3$ on the right above.

Arithmetical structures on the wheel $\mathcal{W}_6$ are plentiful. For instance, looking only among structures with $M_G(a_1, \ldots, a_7)$ having $a_i \in [2, 30]$ for all $i = 1, \ldots, 7$, we found structures with 198 different group orders for their associated group Φ. Among the orders found, 94 are squarefree, so that the corresponding groups are cyclic.

Lemma 3.10. *Order the vertices of $\mathcal{W}_{2k}$ as follows. The first vertex in our enumeration is u, the unique vertex of degree $2k$. Then we let $v_1, \ldots, v_k, w_k, w_{k-1}, \ldots, w_1$ denote the consecutive vertices on the cycle, all of degree 3. The transpose of the vector R is $(1, r_1, \ldots, r_k, s_k, \ldots, s_1)$, with $r_i = s_i$ for all $i = 1, \ldots, k$. We set $s_k = k$,*

$s_{k-1} = k + (k-1)$, and so on, until we get to $s_1 = k + \cdots + 2 + 1 = k(k+1)/2$. Note that $a := 2\sum_{i=1}^{k} s_i = 2\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/3$. The diagonal of the matrix M of the structure is $(a, 2, \ldots, 2, 3, 3, 2, \ldots, 2)$. Then (M, R) is an arithmetical structure on $\mathcal{W}_{2k}$ whose associated group Φ is cyclic of order $6k-1$.

Proof. It is easy to check that $MR = 0$. The computation of Φ can be done as follows. First, since the first coefficient of R is 1, we can find an integer linear combination of the columns to add to the first column so that the resulting first column is the zero-column. The same operations on the lines produces a new matrix whose first line is the zero-line and whose first column is the zero-column. Let M' denote the bottom right $2k \times 2k$ square submatrix of this matrix. The group Φ is obtained by doing a row and column reduction of the matrix M'. In particular, the determinant of this matrix gives us $|\Phi|$. It follows that $|\Phi| = d_{\mathcal{C}_{2k}}(2\ldots, 2, 3, 3, 2, \ldots, 2) = 6k-1$. Choose now a row and column of M' where the diagonal element is 3. To show that the group is cyclic, we consider the submatrix M'' of M' obtained by removing the chosen row and column. The matrix M'' has determinant $4k-1$. Since $6k-1$ and $4k-1$ are always coprime, we find that Φ is cyclic. We leave the details to the reader. $\square$

Remark 3.11. We note here an arithmetical structure on $\mathcal{W}_{2k+1}$ very similar to the structure described in Lemma 3.10, with the exception that this new structure has its group Φ isomorphic to $(\mathbb{Z}/(2k+1)\mathbb{Z})^2$.

Order the vertices of $\mathcal{W}_{2k+1}$ as follows. As before, the first vertex in our enumeration is u, the unique vertex of degree $2k+1$. Then we let $v_1, \ldots, v_k, u', w_k, w_{k-1}, \ldots, w_1$ denote the consecutive vertices on the cycle, all of degree 3. The transpose of the vector R is $(1, r_1, \ldots, r_k, 1, s_k, \ldots, s_1)$, with $r_i = s_i$ for all $i = 1, \ldots, k$. We set $s_k = 1 + k$, $s_{k-1} = 1 + k + (k-1)$, and so on, until we get to $s_1 = 1 + k + \cdots + 2 + 1 = 1 + k(k+1)/2$. Note that $a := 1 + 2\sum_{i=1}^{k} s_i = 1 + 2k + 2\sum_{i=1}^{k} i^2 = 1 + 2k + k(k+1)(2k+1)/3$. The diagonal of the matrix M of the structure is $(a, 2, \ldots, 2, 2k+3, 2, \ldots, 2)$. It is easy to check that $MR = 0$. We can obtain in a similar way that $|\Phi| = d_{\mathcal{C}_{2k+1}}(2\ldots, 2, 2k+3, 2, \ldots, 2) = (2k+1)^2$.

4. Third Main Theorem

4.1. Let us now recall the following definitions needed for the proof of Theorem 4.2. Let $S \subset \mathbb{Z}_{\geq 0}$ be any subset. Let $S(\ell) := \{0, 1, 2, \ldots, \ell\} \cap S$ and $s(\ell) := |S(\ell)|$. Recall that the *lower density* $\underline{d}(S)$ of S is defined as

$$\underline{d}(S) := \liminf_{\ell \to \infty} \frac{s(\ell)}{\ell}.$$

Similarly, the upper density $\overline{d}(S)$ of S is given by $\overline{d}(S) := \limsup_{\ell\to\infty} \frac{s(\ell)}{\ell}$. If both $\underline{d}(S)$ and $\overline{d}(S)$ exist and are equal, the *natural density* $d(S)$ of S is defined as $d(S) := \underline{d}(S)$.

We say that S is *dense in* $\mathbb{Z}_{\geq 0}$ if $d(S) = 1$. When S is dense, its complement in $\mathbb{Z}_{\geq 0}$ has density 0. Any finite set has density 0. Let $a, b \in \mathbb{Z}_{\geq 0}$, $a \neq 0$. A set of the form $\{am + b \mid m \geq 0\}$ is called an *arithmetic progression* and has density $1/a$.

For later use, we note here the following facts. Consider a set S of positive integers which contains a union

$$U := \bigcup_{i=1}^{k}\left(\bigcup_{j=1}^{r_i}\{a_i t + b_{ij} \mid t \geq 0\}\right)$$

of arithmetic progressions. Then the lower density $\underline{d}(S)$ of S satisfies $\underline{d}(S) \geq d(U)$. When the a_i are pairwise coprime, and for each i, the r_i arithmetic progressions are distinct, we find that

$$d(U) = 1 - \prod_{i=1}^{k}\left(1 - \frac{r_i}{a_i}\right).$$

Theorem 4.2. *Let G be a connected graph. Let $v \in G$ be a vertex such that the subgraph G_v has one of the following properties:*

(a) *The complement of $V_{d_{G_v}}(2)$ in $\mathbb{Z}_{\geq 0}$ is finite.*
(b) *Up to possibly reordering the vertices of G_v, there exist $a_2, \dots, a_{n-1} \in \mathbb{Z}_{\geq 2}$ such that $d_{G_v}(t, a_2, \dots, a_{n-1}) = \alpha t - \beta$ with $\alpha \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{Z}$, and $\gcd(\alpha, \beta) = 1$.*
(c) *$V_{d_{G_v}}(2)$ contains an infinite subset of pairwise coprime values $\{u_1, u_2, \dots\}$ such that $\lim_{j\to\infty} \prod_{i=1}^{j}(1 - 1/u_i) = 0$.*

Then $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$.

Proof. We claim that Assumption (a) implies Assumption (c). Indeed, since the complement of $V_{d_{G_v}}(2)$ is finite, it must contain all but finitely many prime numbers. Listing the prime numbers in the complement as $\{p_i\}_{i=1}^{\infty}$, we find that $\lim_{j\to\infty} \sum_{i=1}^{j} 1/p_i$ diverges, since by Euler's theorem the sum of the reciprocals of all primes diverges ([22], page 21), and our set of primes misses only finitely many primes by hypothesis. Lemma 5.1.1 in [1] can be used to deduce that $\lim_{j\to\infty} \prod_{i=1}^{j}(1 - 1/p_i) = 0$, as desired.

We claim that Assumption (b) also implies Assumption (c). Indeed, our assumption that $\gcd(\alpha, \beta) = 1$ allows us to use Dirichlet's Theorem on primes in arithmetic progression: The set $\mathcal{P}$ of primes in the arithmetic progression $\{\alpha t - \beta \mid t \in \mathbb{Z}_{\geq 0}\}$ is infinite because the Dirichlet density of the primes in $\mathcal{P}$ is equal to $1/\alpha$. This in turn implies that $\sum_{p\in\mathcal{P}} 1/p$ is infinite (see [22], page 251). As before, we conclude using Lemma 5.1.1 in [1] that $\lim_{p\in\mathcal{P}} \prod_{p}(1 - 1/p) = 0$, as desired.

Let us now prove the theorem when Assumption (c) holds. Without loss of generality, we can assume that $v = v_1$. For each value u_i, choose $a_{i,2}, \dots, a_{i,n} \geq 2$ such that $d_{G_v}(a_{i,2}, \dots, a_{i,n}) = u_i$. Consider the polynomial $d_G(t, a_{i,2}, \dots, a_{i,n})$. By hypothesis, the matrix $M_G(t, a_{i,2}, \dots, a_{i,n})$ has a lower right $n-1 \times n-1$ submatrix $M_{G_v}(a_{i,2}, \dots, a_{i,n})$ of determinant u_i. Thus $d_G(t, a_{i,2}, \dots, a_{i,n}) = u_i t - w_i$ for some $w_i \in \mathbb{Z}$.

Let U_i denote the set of positive values taken by $u_i t - w_i$ when $t \geq 2$ if $w_i \leq 0$, and when $t \geq 2 + w_i/u_i$ if $w_i \geq 0$. It is clear that, since up to finitely many values the set U_i is an arithmetic progression, the density of U_i is $1/u_i$. By construction, $U_i \subseteq V_{d_G}(2)$. Since the elements of the set $\{u_1, u_2, \dots\}$ are pairwise coprime, we find that the union $\cup_{i=1}^{j} U_i$ has lower density equal to $1 - \prod_{i=1}^{j}(1 - 1/u_i)$. Clearly, $V_{d_G}(2)$ contains $\cup_{i=1}^{\infty} U_i$, and taking now the limiting value as $j \to \infty$, we find that $\underline{d}(V_{d_G}(2)) \geq 1$. It follows that $d(V_{d_G}(2)) = 1$ and $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$. $\square$

In our next corollary, the symbols A_n and D_n denote Dynkin diagrams, whose definition is recalled in 5.1.

Corollary 4.3. *Suppose that a graph G on $n+1$ vertices contains an induced subgraph H on n vertices of the form G_v such that H is either A_n, $\mathcal{C}_n$, D_n, $\mathcal{S}_n$, $\mathcal{K}_n$, or $\mathcal{K}(2, n-2)$. Then $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$.*

Proof. We exhibit for each graph H a choice of diagonal elements $(t, a_2, \dots, a_n)$ with $a_i \geq 2$ such that $d_H(t, a_2, \dots, a_n) = \alpha t - \beta$ with $\gcd(\alpha, \beta) = 1$. We can then use Theorem 4.2 (b) to conclude that $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$.

(i) $H = A_n$. Assume that v_1 is a vertex of degree 1. We find that

$$d_H(t, 2, \dots, 2) = nt - (n-1).$$

Clearly $\gcd(n, n-1) = 1$.

(ii) $H = \mathcal{C}_n$. Label the vertices consecutively along the cycle. Recall that n is the number of spanning trees of H, and it follows from the proof of Proposition 2.4 (a) that $d_H(t, 2, \dots, 2, 2) = n(t-2)$. We find that

$$d_H(t, 2, \dots, 2, 3) = n(t-2) + (n-1)t - (n-2) = (2n-1)t - (3n-2).$$

Again, we find that $\gcd(2n-1, 3n-2) = 1$.

(iii) $H = D_n$. This star-shaped graph has a single vertex of degree 3, and three terminal chains, each ending with a vertex of degree 1. Assume that v_1 is the vertex of degree 1 on the chain of length $n-3$ in H, and that v_n is another vertex of degree 1. We find that

$$d_H(t, 2, \dots, 2, 2, 3) = (n+3)t - (n+2),$$

and clearly $\gcd(n+3, n+2) = 1$. To compute the coefficient of t in this expression, note that it equals $d_{H_{v_1}}(2, \dots, 2, 3)$, which is $4 + (n-1)$, with $4 = d_{D_{n-1}}(2, \dots, 2)$ and $n - 1 = d_{A_{n-2}}(2, \dots, 2)$. The constant coefficient is similarly obtained as $4 + (n-2)$.

(iv) $H = \mathcal{S}_n$. Assume that the vertex v_1 has degree $n-1$. We use $(t, a_2, \dots, a_n)$ with $a_2 = 2$ and $a_{i+1} = i$-th prime number. We find that

$$d_H(t, a_2, \dots, a_n) = (a_2 \cdots a_n)t - (a_2 \cdots a_n)(\frac{1}{a_2} + \cdots + \frac{1}{a_n}).$$

Since $a_2, \dots, a_n$ are distinct primes, we find that $(a_2 \cdots a_n)(\frac{1}{a_2} + \cdots + \frac{1}{a_n})$ is coprime to $(a_2 \cdots a_n)$.

(v) $H = \mathcal{K}_n$. Set $a_i + 1 = b_i$ and choose $a_2 + 1 = 3$, and $a_i + 1 = i$-th prime number. We find using (8.2) that

$$\begin{aligned} d_H(t, a_2, \dots, a_n) = & \ (t+1)(b_2 \cdots b_n) - (t+1)(b_2 \cdots b_n)(\textstyle\sum_{i=2}^n \frac{1}{b_i}) - (b_2 \cdots b_n) \\ = & \ t(b_2 \cdots b_n)(1 - \textstyle\sum_{i=2}^n \frac{1}{b_i}) - (b_2 \cdots b_n)(\textstyle\sum_{i=2}^n \frac{1}{b_i}). \end{aligned}$$

Since $b_2, \dots, b_n$ are distinct primes, $(b_2 \cdots b_n)$ is coprime to $(b_2 \cdots b_n)(\frac{1}{b_2} + \cdots + \frac{1}{b_n})$.

(vi) $H = \mathcal{K}(2, n-2)$. We compute the determinant of

$$M = M_H(t, x, y_1, \ldots, y_{n-2}) = \begin{pmatrix} t & 0 & -1 & \ldots & -1 \\ 0 & x & -1 & \ldots & -1 \\ -1 & -1 & y_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ -1 & -1 & 0 & & y_{n-2} \end{pmatrix}.$$

Subtracting the second row from the first, and the resulting second column from the first, we obtain

$$\begin{pmatrix} t+x & -x & 0 & \ldots & 0 \\ -x & x & -1 & \ldots & -1 \\ 0 & -1 & y_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & -1 & 0 & & y_{n-2} \end{pmatrix}.$$

Expanding the determinant along the first row gives

$$\det(M) = \left(xy_1 \cdots y_{n-2} - y_1 \cdots y_{n-2}(\sum_{i=1}^{n-2} \frac{1}{y_i}) \right) t - xy_1 \cdots y_{n-2}(\sum_{i=1}^{n-2} \frac{1}{y_i}).$$

We choose $y_1, \ldots, y_{n-2}$ to be the first $n-2$ prime numbers. We choose x to be coprime to $y_1 \cdots y_{n-2}(\sum_{i=1}^{n-2} \frac{1}{y_i})$. $\square$

4.4. *Proof of Theorem 1.3.* We proceed with a case-by-case verification.

(1) Let G be a Dynkin diagram, an extended Dynkin diagram, or the cone $C(A_3)$. Then G contains an induced subgraph of the form $G_v = A_n$ for some n. Corollary 4.3 applies.

(2) Let G be a complete graph $\mathcal{K}_n$, or an extended complete graph $\mathcal{K}_n^+$. Then G contains an induced subgraph of the form $G_v = \mathcal{K}_{n-1}$ (resp. $\mathcal{K}_n$). Corollary 4.3 applies. Let G be a cycle $\mathcal{C}_n$, an extended cycle $\mathcal{C}_n^+$, or the wheel $\mathcal{W}_n$. Then G contains an induced subgraph of the form $G_v = A_n$ (resp. A_{n+1}, resp. $\mathcal{C}_n$). Corollary 4.3 applies.

(3) Let G be a star $\mathcal{S}_n$, an extended star $\mathcal{S}_n^+$, or the complete bipartite $\mathcal{K}(2, n)$ or $\mathcal{K}(3, n)$. Then G contains an induced subgraph of the form $G_v = \mathcal{S}_{n-1}$ (resp. $\mathcal{S}_n$, resp. $\mathcal{S}_{n+1}$, resp. $\mathcal{K}(2, n)$). Corollary 4.3 applies.

It is natural to wonder whether a vertex v such that Hypothesis (b) in Theorem 4.2 holds might exist for all graphs. We have not been able to answer this question beyond the following extension result.

Lemma 4.5. *Suppose that G is a graph on n vertices such that for some choice of $a_2, \ldots, a_n \geq 2$, we have $d_G(t, a_2, \ldots, a_n) = \alpha t - \beta$ with $\gcd(\alpha, \beta) = 1$. Let G^+ denote the graph obtained by attaching a new vertex v_0 with $e \geq 1$ edges to the vertex v_1. Let $q \geq 1$ denote any integer coprime to $e\alpha\beta$. Then $d_{G^+}(q, t, a_2, \ldots, a_n)$ is of the form $\alpha' t - \beta'$ with $\gcd(\alpha', \beta') = 1$.*

Proof. An explicit computation shows that $d_{G^+}(q, t, a_2, \ldots, a_n) = q(\alpha t - \beta) - e^2\alpha = q\alpha t - (q\beta + e^2\alpha)$. It follows from our hypotheses that $q\alpha$ is coprime to $(q\beta + e^2\alpha)$. $\square$

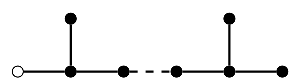
Remark 4.6. The connected simple graphs on four vertices consist of the path A_4, the star D_4, the two graphs $\mathcal{C}_3^+$ and $\mathcal{C}_4$ with Betti number equal to 1, the cone on A_3, and the complete graph $\mathcal{K}_4$. Theorem 1.3 and Corollary 4.3 show that for such graph G, the set $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geq 0}$. Computations suggest that except possibly for $\mathcal{K}_4$, the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ might be finite.

5. Dynkin diagrams

5.1. Recall the following terminology.

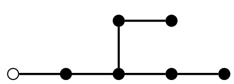
A_n, $n \geq 2$: the chain on n vertices.

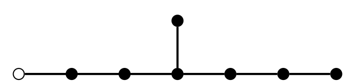
$\tilde{D}_n$, $n \geq 4$: a chain on $n-1$ vertices, with two additional vertices, attached to the two vertices of degree 2 of the chain that are linked to a vertex of degree 1 (when $n = 4$, there is only one such vertex, and in this case both new vertices are attached to this vertex). The graph D_n, $n \geq 4$, is obtained from $\tilde{D}_n$ by removing one of the two additional vertices. Such vertex is indicated in white below.

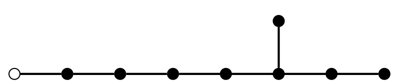


$\tilde{D}_n$

$\tilde{E}_n$, $n = 6, 7, 8$: a tree on $n + 1$ vertices described below. Removing one vertex from $\tilde{E}_n$ produces the tree E_n on n vertices. The vertex that needs to be removed is marked in white below.

$\tilde{E}_6$ 

$\tilde{E}_7$ 

$\tilde{E}_8$ 

The graphs A_n ($n \geq 2$), D_n ($n \geq 4$), and E_n, $n = 6, 7, 8$, are called *Dynkin diagrams* and have n vertices. The graphs $\tilde{D}_n$, $n \geq 4$, and $\tilde{E}_n$, $n = 6, 7, 8$, are called *extended Dynkin diagrams* or *affine Dynkin diagrams*, and have $n + 1$ vertices. In the context of elliptic curves, they are called *Kodaira types*, and are denoted by I_n^*, $n \geq 0$, and $\mathrm{IV}^*, \mathrm{III}^*$, and II^*, respectively (see for instance [34], page 46). The notation I_n refers in this context to the cycle $\mathcal{C}_n$.

The following proposition is well-known. Part (a) is found in [18], Lemma 3.1, with proof and a reference to [21], Satz page 219. Part (b) is stated in [21], page 228.

Proposition 5.2. *Let G be a connected graph on n vertices.*

(a) *If $M_G(2, \ldots, 2)$ is positive definite, then G is either A_n with $n \geq 2$, D_n with $n \geq 4$, or E_n, $n = 6, 7, 8$.*

(b) *If $M_G(2, \ldots, 2)$ is positive semi-definite and* $\det M_G(2, \ldots, 2) = 0$, *then G is either $\mathcal{C}_n$ with $n \geq 2$, $\tilde{D}_n$ with $n \geq 4$, or $\tilde{E}_n$, $n = 6, 7, 8$.*

When G is a Dynkin diagram, computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is finite, except when $G = A_2$. We present below some data for the Dynkin diagrams D_n and E_n. The data for the chains A_n is presented in Example 6.2.

Proposition 5.3. (a) *If $G = A_n$, $n \geq 2$, then $V_{d_G}(2) \subseteq \mathbb{Z}_{\geq n+1}$.*
(b) *If $G = D_n$, $n \geq 4$, then $V_{d_G}(2) \subseteq \mathbb{Z}_{\geq 4}$.*
(c) *If $G = E_6$, then $V_{d_G}(2) \subseteq \mathbb{Z}_{\geq 3}$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set*

$$L_{E_6} := \begin{array}{c}[0,1,2,4,5,6,8,10,12,14,16,17,20,24,26,28,30,32,34,38,\\ 44,46,48,56,60,64,74,80,88,92,98,132,158,170].\end{array}$$

(d) *If $G = E_7$, then $V_{d_G}(2) \subseteq \mathbb{Z}_{\geq 2}$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set*

$$L_{E_7} := [0,1,3,4,7,12,15,25,28].$$

(e) *If $G = E_8$, then $V_{d_G}(2) \subseteq \mathbb{Z}_{>0}$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set*

$$L_{E_8} := \begin{array}{c}[0,2,3,4,6,8,10,11,14,16,18,22,23,24,28,34,38,40,\\ 46,58,60,62,88,94,134,178].\end{array}$$

Proof. Since each of the graphs $G = A_n$, D_n, and E_n, is such that $M_G(2,\ldots,2)$ is positive definite (see 5.2 (a)), we find from 2.1 (c) that the smallest value taken by $d_G(x_1,\ldots,x_n)$ when $x_i \geq 2$ is $d_G(2,\ldots,2)$. It is classical that $d_{A_n}(2,\ldots,2) = n+1$, $d_{D_n}(2,\ldots,2) = 4$, $d_{E_6}(2,\ldots,2) = 3$, $d_{E_7}(2,\ldots,2) = 2$, and $d_{E_8}(2,\ldots,2) = 1$. $\square$

Remark 5.4. (a) Let $G = D_4$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{D_4} := \begin{array}{c}[0,1,2,3,5,6,7,9,10,11,13,14,17,18,19,21,23,25,26,30,31,34,35,37,38,41,\\ 45,47,49,53,58,61,65,66,67,74,77,79,83,86,91,93,97,101,103,109,110,\\ 114,115,121,125,126,129,130,131,143,145,153,167,173,178,181,187,199,\\ 206,210,223,229,247,251,258,265,301,325,343,391,417,426,437,451,\\ 517,593,595,606,633,637,649,671,763,823,859,871,937,977,\\ 1027,1087,1330,1517,1661,4477,4585,5273].\end{array}$$

(b) Let $G = D_5$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{D_5} := [0,1,2,3,5,6,7,10,11,13,15,21,22,30,31,37,43,46,55,58,75,91,102,165,330].$$

(c) Let $G = D_6$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{D_6} := \begin{array}{c}[0,1,2,3,5,6,7,9,11,13,14,15,17,18,23,25,27,29,33,35,38,\\ 45,47,49,50,53,69,71,78,95,97,105,133,203,245].\end{array}$$

(d) Let $G = D_7$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{D_7} := \begin{array}{c} [0,1,2,3,5,6,7,9,10,13,14,15,17,19,22,23,26,27,30, \\ 33,38,42,43,49,55,57,62,78,79,110]. \end{array}$$

(e) Let $G = D_8$. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{D_8} := \begin{array}{c} [0,1,2,3,5,6,7,9,10,11,13,14,15,17,18,19,21,22,25,26,29,30, \\ 31,33,35,37,41,43,46,49,50,54,55,58,59,61,63,65,71,73,90, \\ 91,94,101,105,118,121,138,169,183,205,250]. \end{array}$$

Let now G be an extended Dynkin diagram. The data below for $\tilde{E}_n$, $n = 6,7,8$, suggests that the complement of the set $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ might be finite. The available data for $G = \tilde{D}_n$, $n = 5,6,7$, seems to support the same assertion for these graphs. The data in the case of $G = \tilde{D}_4$, the star on 5 vertices, is less clear.

Let G be the cycle $\mathcal{C}_n$, $n \geq 3$. The available data when $n = 4,5,6$ also seems to suggest that the complement of the set $V_G(2)$ is finite. The case of $G = \mathcal{C}_3$ is less clear. The data is further discussed in Examples 7.1 and 7.3.

Proposition 5.5. (a) *If $G = \mathcal{C}_n$, $n \geq 2$, then $V_{d_G}(2) \subseteq \{0\} \sqcup \mathbb{Z}_{\geq n}$, and $V_{d_G}(2)$ contains all positive multiples of n.*

(b) *If $G = \tilde{D}_n$, $n \geq 4$, then $V_{d_G}(2) \subseteq \{0\} \sqcup \mathbb{Z}_{\geq 4}$. Moreover, $V_{d_G}(2)$ contains all positive multiples of 4.*

(c) *If $G = \tilde{E}_6$, then $V_{d_G}(2) \subseteq \{0\} \sqcup \mathbb{Z}_{\geq 3}$. Moreover, $V_{d_G}(2)$ contains all positive multiples of 3. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set*

$$L_{\tilde{E}_6} := \begin{array}{c} [1,2,4,5,7,8,11,13,14,16,19,20,22,23,26,29,32,34,35,37,41,44,46,49, \\ 53,56,58,62,71,74,82,89,95,104,106,118,128,137,140,167,172,184,188, \\ 212,218,271,287,302,386]. \end{array}$$

(d) *If $G = \tilde{E}_7$, then $V_{d_G}(2) \subseteq \{0\} \sqcup \mathbb{Z}_{\geq 2}$. Moreover, $V_{d_G}(2)$ contains all even positive integers. Computations indicate that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set*

$$L_{\tilde{E}_7} := \begin{array}{c} [1,3,5,9,11,13,15,19,21,23,25,29,33,43,45,49, \\ 51,59,75,81,115,121,141,145,159,189]. \end{array}$$

(e) *If $G = \tilde{E}_8$, then $V_G(2) = \mathbb{Z}_{\geq 0}$.*

Proof. Since each of the graphs $G = \mathcal{C}_n$, $\tilde{D}_n$, and $\tilde{E}_n$, is such that $M_G(2,\dots,2)$ is definite semi-positive, the smallest value taken by $d_G(x_1,\dots,x_n)$ when $x_i \geq 2$ is $d_G(2,\dots,2) = 0$.

(a) When $G = \mathcal{C}_n$, the matrix $M_G(2,\dots,2)$ is the usual Laplacian. Recall that $d_G(t,2,\dots,2) = n(t-2)$. We conclude from Proposition 2.4 (b) that every multiple of n is in the set $V_{d_G}(2)$ and that n is the smallest non-zero value in $V_{d_G}(2)$. Since the critical group of $\mathcal{C}_n$ is known to be cyclic, we find that $0 \in V_G(2)$.

(b) When $G = \tilde{D}_n$, the matrix $M_G(2,\dots,2)$ is the matrix associated with an arithmetical structure with $|\Phi| = 4$. The group is cyclic of order 4 when n odd, and it is $(\mathbb{Z}/2\mathbb{Z})^2$ when n is even. The matrix $M_G(2,\dots,2)$ is the unique arithmetical structure on G of the form $M_G(a_1,\dots,a_n)$ with $a_1,\dots,a_n \geq 2$. Thus $0 \in V_G(2)$ when n is odd, and $0 \notin V_G(2)$ when n is even. Proposition 2.4 (b) shows that every multiple of 4 is in $V_{d_G}(2)$

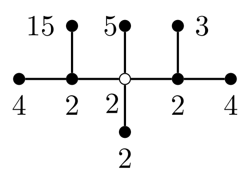
(c) and (d) When $G = \tilde{E}_n$, $n = 6$ (resp. 7), the matrix $M_G(2,\dots,2)$ is the matrix associated with an arithmetical structure with $|\Phi| = 3$ (resp. 2). Proposition 2.4 (b) can be applied again.

(d) Since $1 \in V_{E_8}(2)$, Theorem 1.1 (b) implies that $V_{\tilde{E}_8}(2) = \mathbb{Z}_{\geq 0}$. □

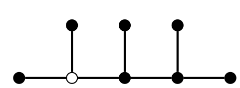
Remark 5.6. In view of Proposition 5.2, it is natural to wonder whether it is possible to list all the graphs G such that $M_G(3,2,\dots,2)$ is either positive definite or positive semi-definite. Examples of such graphs which are not Dynkin diagrams and where $M_G(3,2,\dots,2)$ is positive definite can be found among the rational triple points described in [3], page 135. We note below two constructions that use the classical extended Dynkin diagrams and produce infinite families of graphs G where $M_G(3,2,\dots,2)$ is positive semi-definite with determinant 0. A more sporadic example on 5 vertices is presented in Remark 3.4. One may wonder whether, in the positive semi-definite case with determinant 0, all such graphs have to appear in the list given in [32].

Let H be an extended Dynkin diagram on $n+1$ vertices, or a cycle. Denote by (M_H, R_H) its associated arithmetical structure. Thus, $M_H = M_H(2,\dots,2)$. Fix a vertex v of H, and without loss of generality, assume that v is the first vertex of H. If the coefficient of R_H associated with v is 1 or 2, we construct a graph G on $n+3$ vertices by attaching two new vertices to v, each by a single edge. Label the vertices of G by $w, w', v_1, \dots, v_{n+1}$. We claim that $M_G(2,2,3,2,\dots,2)$ has determinant 0. To show this, note the following. If $R_H = (2,\dots)$, then consider the vector ${}^tR_G := (1,1,{}^tR_H)$. It is easy to check that $M_G(2,2,3,2,\dots,2)R_G = 0$. If $R_H = (1,\dots)$, then consider the vector ${}^tR_G := (1,1,2({}^tR_H))$. It is easy to check that in this case also, $M_G(2,2,3,2,\dots,2)R_G = 0$.

We illustrate this construction with two examples. First on the left below, we use $H = \tilde{D}_6$ to obtain a graph G_1 on 9 vertices. The old vertex of $\tilde{D}_6$ chosen for the construction is indicated in white.



G_1



G_2

The graph G_1 is denoted (15) in Table 1, page 521 in [8]. It is minimal in the sense that $1 \in V_{G_1}(2)$ but $1 \notin V_T(2)$ if T is any subtree of G_1. Computations indicate that $V_{d_{G_1}}(2) = \mathbb{Z}_{\geq 0}$. We have labeled the graph with a set of coefficients on the diagonal that give $\det M_{G_1}(a_1,\dots,a_9) = 1$.

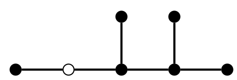
The graph G_2 above on the right has 8 vertices and is obtained from the construction with $H = \tilde{D}_5$. The old vertex of $\tilde{D}_5$ chosen for the construction is indicated in white. The graph G_2 is a subgraph of G_1. Computations indicate that the complement of $V_{d_{G_2}}(2)$ in

$\mathbb{Z}_{\geq 0}$ is contained in

$$L_{G_2} = \begin{array}{c}[1,5,23,25,31,53,61,71,73,145,163,199,\\ 211,229,275,289,365,379,383,421,451,493,799,1153].\end{array}.$$

A different construction is as follows. Start with an extended Dynkin diagram $H = \tilde{D}_n$ with $n \geq 4$. Fix a pair of vertices of degree 1 attached to the same vertex in H. Without loss of generality, we can assume that these vertices of degree 1 are v_1 and v_2, attached to a vertex v_3. Consider the graph G obtained by linking a new vertex w to v_1 by a single edge. Order the vertices of G as $w, v_1, v_2, v_3, \dots$. Then we claim that $M_G(2,2,3,2,\dots,2)$ has determinant 0. The vector ${}^tR_G := (2,4,2,6,\dots,6,3,3)$ is such that $M_G R_G = 0$. Here we have ordered the vertices of H such that the last two vertices again have degree 1, and the ante-penultimate vertex has degree 3 in H. When $n = 4$, note that all vertices of degree 1 are linked to the same vertex of degree 4. The construction produces the graph S_5^+ in this case (see 7.4).

We illustrate this second construction with the example of $H = \tilde{D}_5$, obtaining a graph on 7 vertices. The old vertex of $\tilde{D}_5$ chosen for the construction is indicated in white:

G_3 

The graph G_3 is a subgraph of the previous examples. Computations indicate that the complement of $V_{d_{G_3}}(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in $L_{G_3} = [1, 21, 25, 37, 75]$.

6. The graphs A_n and small variations

We discuss in this section the paths A_n, $n \geq 3$, and some graphs $A_3(e, f)$ on three vertices generalizing A_3. Let us note here again that when $G = A_n$, then $0 \notin V_{d_G}(2)$ since the matrix $M_G(2,\dots,2)$ is positive definite. We also note that the group Φ associated to any matrix of the form $M_G(a_1,\dots,a_n)$ with $a_1,\dots,a_n \geq 2$ is always cyclic (see [29], Lemma 3.13). Thus we have $V_G(2) = V_{d_G}(2)$ when $G = A_n$.

Theorem 4.2 shows that the complement of $V_{A_n}(2)$ in $\mathbb{Z}_{\geq 0}$ is dense. The data below in Example 6.2 suggests that this complement might always be finite. The smallest values in the complement can be explicitly described using the next lemma, and there are at least $4n - 6$ such values.

Lemma 6.1. *Let $G = A_n$, $n \geq 3$. The set $V_G(2)$ starts with the following values:*

$$[n+1, 2n+1, 3n-1, 3n+1, 4n-5, 4n, 4n+1, 5n-11, 5n+1, \dots].$$

Proof. We have $d_{A_n}(t,2,\dots,2) = n(t-1)+1$. This shows that $V_G(2)$ contains $n+1, 2n+1, 3n+1, \dots$. We have $d_{A_n}(2,\dots,2,3,2) = 3n-1$ and $d_{A_n}(3,2,\dots,2,3) = 4n$. Moreover, $d_{A_n} = (2,2,\dots,3,2,2) = 4n-5$. More generally, placing 3 on the i-th column from the end (with $2i \leq n$): the determinant is $n+1 + i(n-i+1) = (i+1)n - (i^2 - i - 1)$. With $i = 4$, we obtain $5n - 11$. ∎

Example 6.2. (a) Let $G = A_3$. Computations indicate that the complement of $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{A_3} := [0, 1, 2, 3, 5, 6, 9, 11, 14, 15, 35, 105, 510].$$

(b) Let $G = A_4$. Computations indicate that the complement of $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{A_4} := \begin{array}{c} [0, 1, 2, 3, 4, 6, 7, 8, 10, 12, 14, 15, 20, 22, 24, 26, 28, 38, 40, 42, 48, 52, 68, \\ 104, 132, 150, 188, 314]. \end{array}$$

(c) Let $G = A_5$. Computations indicate that the complement of $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{A_5} := [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 17, 18, 19, 27, 28, 34, 40, 52, 63, 88].$$

(d) Let $G = A_6$. Computations indicate that the complement of $V_G(2)$ in $\mathbb{Z}_{\geq 0}$ is contained in the following set

$$L_{A_6} := \begin{array}{l} [0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 22, 23, 26, 29, 30, 32, 36, 38, \\ 42, 44, 48, 52, 54, 56, 62, 70, 80, 81, 86, 96, 102, 108, 110, 122, 126, 140, 180, 236]. \end{array}$$

It is natural to wonder whether $d_{A_3}(x, y, z) = xyz - x - z$ takes every non-negative value when $x, y, z \geq 2$, except for the values listed in L_{A_3}. We have not been able to answer this question beyond the following remarks.

Proposition 6.3. *The set $V_{A_3}(2)$ contains:*

(a) *All even positive integers n, except possibly those of the form $n = 2^m - 2$ with m odd or $m = 2, 4$.*
(b) *All odd positive integers n such that $n + 2$ is not prime.*
(c) *All odd positive integers congruent to 1 modulo 4 and congruent to 2 or 8 modulo 9.*

Proof. We claim that the complement of $V_{A_3}(2)$ in $\mathbb{Z}_{\geq 0}$ consists only of integers n such that (i) either $n + 2$ is prime or $n + 2$ is a power of 2, and (ii) such that $n + 4$ is prime, or $n + 4$ is not divisible by a prime $p \geq 7$ congruent to 3 modulo 4.

Indeed, writing $xyz - x - z = n$, we find that when $z = 2$, we have $x(2y - 1) = n + 2$. This can be solved with $x, y \geq 2$ when $n + 2$ is not prime, and not a power of 2. When $z = 4$, we have $x(4y - 1) = n + 4$. This can be solved with $x, y \geq 2$ when $n + 4$ is not prime and when at least one divisor of $n + 4$ is congruent to 3 modulo 4 and greater than 3.

Suppose now that $m > 4$ is even. Set $x = 4$, $z = 6$, and $y = (2^{m-3} + 1)/3 \geq 2$. It is clear that $xyz - x - z = 2^m - 2$. When $m \geq 4$ is even, $2^{m-3} + 1$ is always divisible by 3.

Suppose now that n is congruent to 1 modulo 4 and congruent to 2 or 8 modulo 9. Then in particular n is not divisible by 3. Hence, one of $n + 2$ or $n + 4$ has to be divisible by 3. Suppose that $n + 2$ is divisible by 3. Then n is in $V_{A_3}(2)$ since $n + 2$ is neither prime nor a power of 2. Suppose now that $n + 4$ is divisible by 3. Then n is in $V_{A_3}(2)$ unless $(n + 4)/3$ is only divisible by primes congruent to 1 modulo 4 and by a power of 3. Our hypothesis implies that $n + 4$ is congruent to 3 or 6 modulo 9 and so is exactly divisible by 3. Since the power of 3 is odd in $n + 4$, then n has to be congruent to 3 modulo 4 when

$(n+4)/3$ is only divisible by primes congruent to 1 modulo 4. Since we assume that n is congruent to 1 modulo 4, we find that $n \in V_{A_3}(2)$. $\square$

Remark 6.4. The first values of $2^{2m+1}-2$ are $0, 6, 30, 126, 510, 2046, 8190, \ldots$. The values $0, 6$, and 510 are not achieved by $xyz - x - z$ with $x, y, z \geq 2$. The values 30 and 126 are achieved exactly once, with $(x, y, z) = (3, 4, 3)$ and $(12, 2, 6)$, respectively.

Consider more generally the polynomial $f(x, y, z) = xyz - ax - bz$ with $a, b \in \mathbb{Z}_{\geq 1}$. It is clear that $V_f(1) = \mathbb{Z}_{\geq 0}$ since $f(1, N + a + b, 1) = N$ for any integer N.

If an integer p divides a, then $V_f(2)$ contains all non-negative multiples of p. Indeed, assume that p divides N. Set $z = p$, $y = a/p + 1$, and $x = N/p + b \geq 2$. Then $xyz - ax - bz = N$.

Proposition 6.5. *Let $f(x, y, z) = xyz - ax - bz$ with $a \geq b \geq 1$. We have $V_f(2) \supseteq \mathbb{Z}_{>0}$ in the following cases:*

(a) *$a + 1$ or $b + 1$ is not prime.*

(b) *a is divisible by 4 and $b = 1$.*

Moreover, $0 \in V_f(2)$ when $(a, b) \neq (1, 1), (2, 1)$, and $(4, 1)$.

Proof. (a) Without loss of generality, we can assume that $b + 1$ is not prime. Consider the equation $xyz - ax - bz = N$, which we rewrite as $(xy - b)z = N + ax$. Under our hypothesis, the equation $xy - b = 1$ can be solved with $x, y \geq 2$, and so setting $z = N + ax \geq 2$ shows that $N \in V_f(2)$.

(b) Assume now that $b = 1$. If N is even, we use the case $p = 2$ just above to conclude that since 2 divides a, $xyz - ax - z$ takes all possible even values when $x, y, z \geq 2$. If N is odd and $a = 4c$, set $y = 2$ and $z = 2c + 1 \geq 2$. Since z is then odd and N is odd, we can set $x = (N + z)/2 \geq 2$, to get $xyz - 4cx - z = N$.

The equation $xyz - ax - bz = 0$ always has the solution $(x, y, z) = (b, 2, a)$, so that $0 \in V_f(1)$, and when $a, b > 1$, $0 \in V_f(2)$. Suppose now that $b = 1$, and let us show that we can solve $xyz - ax - z = 0$ with $x, y, z \geq 2$ if and only if $a \neq 1, 2, 4$. If we can solve this equation, then $(xy - 1)z = ax$ and x must divide z. Write $z = cx$. We need to solve $(xy - 1)c = a$, so c divide a. This equation can be solved if we can find integers $x, y \geq 2$ such that $xy = 1 + a/c$. We claim that given any integer $m > 1$, $m \neq 2, 4$, we can find a divisor d of m such that $d + 1$ is not prime. This is clear if m is divisible by an odd prime. When m is a power of 2, this is true as soon as 8 divides m. We apply this claim to a, and we find c which divides a such that $1 + a/c$ is not prime, unless $a = 1, 2, 4$, as desired. $\square$

Example 6.6. Computations indicate that in the case of $f = xyz - 2x - z$, the complement in $\mathbb{Z}_{\geq 0}$ of the set $V_f(2)$ might be reduced to $\{0, 1, 3, 7, 15\}$.

Remark 6.7. Fix $e_1, \ldots, e_{n-1} \geq 1$. Consider the graph $A_n(e_1, \ldots, e_{n-1})$ with multiple edges obtained as follows. Given n vertices $v_1, \ldots, v_n$, link v_i to v_{i+1} by e_i edges, for each $i = 1, \ldots, n-1$.

The matrix $M_{A_n(e_1, \ldots, e_{n-1})}(x_1, \ldots, x_n)$ is a tridiagonal matrix, and such matrices occur in the following context. Recall that two symmetric matrices M and N in $M_n(\mathbb{Z})$ are *congruent* if there exists $T \in \mathrm{GL}_n(\mathbb{Z})$ such that $N = ({}^tT)MT$. Newman ([31], Theorem 1) showed that any positive definite matrix $M \in M_n(\mathbb{Z})$ is congruent to a tridiagonal matrix with certain specified properties. When $\det(M) = 1$, the tridiagonal matrix is of the form $M_{A_n(1,\ldots,1,e_{n-1})}(a_1, \ldots, a_n)$. Such matrices are further studied in [17] and [20].

In the case $n = 3$, the graph $G = A_3(e, f)$ has matrix $M_{A_3(e,f)}(x, y, z)$ with determinant $d_G(x, y, z) = xyz - f^2x - e^2z$. As we saw in Proposition 6.5, the determination of $V_{d_G}(2)$ is not immediate when both $e^2 + 1$ and $f^2 + 1$ are prime.

Let $G = A_3(2, 1)$, with $d_G(x, y, z) = xyz - x - 4z$, and $d_G(3, 2, 2) = 1$. The graph G is the graph with the fewest vertices and edges such that $1 \in V_G(2)$. See Examples 3.1 and 3.5 for examples on 4 vertices. Proposition 6.5 shows that $V_{d_G}(2) = \mathbb{Z}_{>0}$.

Let $G = A_3(2, 2)$, with $d_G(x, y, z) = xyz - 4x - 4z$. The set $V_{d_G}(2)$ contains all even positive integers, but it is not immediately clear from computations that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is finite. The situation is similar for $V_h(2)$ when $h := xyz - 2x - 2z$. The integer $N = 538641$ belongs to the complement of both $V_{d_G}(2)$ and $V_h(2)$.

7. Further Examples

Example 7.1. Consider the cycle $G = \mathcal{C}_3$ on $n = 3$ vertices. Then

$$d_G(x, y, z) = xyz - x - y - z - 2.$$

It is not computationally clear in this example that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ is finite. This complement is likely to contain 2201 values in the interval $[1, 2 \cdot 10^6]$.

Theorem 1.1 shows that $V_{d_G}(1) = \mathbb{Z}_{\geq 0}$. The recent preprint [13] studies the set $V_f(1)$ of values taken by the related polynomial $f(x, y, z) = xyz + x + y + z$. The authors suggest on page 3 that the complement of $V_f(1)$ in $\mathbb{Z}_{\geq 0}$ is infinite.

Example 7.2. Consider the graph G on $n = 3$ vertices with matrix

$$M_G = \begin{pmatrix} y & -2 & -1 \\ -2 & z & -1 \\ -1 & -1 & x \end{pmatrix} \text{ and } d_G(x, y, z) = xyz - 4x - y - z - 4.$$

We have $d_G(2, 3, 3) = 0$ and $d_G(2, 5, 2) = 1$. Computations indicate that the set of values missed by d_G is very small and might consists only of $[8, 56, 248]$.

Example 7.3. Consider the cycle G on $n = 4$ vertices. Computations produced a set of 325 positive integers up to 10^6 which contains the complement of $V_{d_G}(2) \cap [1, 10^6]$. The four largest values in that set are $86899, 184549, 204997$, and 858811. The paucity of large values in this set suggests that the set of missing values might be finite.

Consider now the cycle G on $n = 5$ vertices. Computations produced a set of 123 values up to 10^5 which contains the complement of $V_{d_G}(2)$ in $[1, 10^5]$. The largest two values in this set are 5422 and 12489. This data suggests that the set of missing values might be finite in this case also.

Example 7.4. The stars $\mathcal{S}_n$ on n vertices are such that $V_{d_{\mathcal{S}_n}}(2)$ is dense in $\mathbb{Z}_{\geq 0}$ (see Theorem 1.3). The case of $\mathcal{S}_4$ (the Dynkin diagram D_4) is discussed in Example 5.4, where computations indicate that the complement of $\mathcal{S}_4$ in $\mathbb{Z}_{\geq 0}$ is finite. In the case of the star $\mathcal{S}_5$ (which is equal to the extended Dynkin diagram $\tilde{D}_4$), computations leave open the possibility that the complement of $V_{d_{\mathcal{S}_5}}(2)$ in $\mathbb{Z}_{\geq 0}$ might not be finite.

The extended stars $\mathcal{S}_n^+$, defined in the introduction, are also such that $V_{d_{\mathcal{S}_n^+}}(2)$ is dense in $\mathbb{Z}_{\geq 0}$. The graph $\mathcal{S}_4^+$ is equal to the Dynkin diagram D_5, and Example 5.4 suggests that the complement of $V_{d_{\mathcal{S}_4^+}}(2)$ in $\mathbb{Z}_{\geq 0}$ is finite. On the other hand, computations leave open the possibility that the complement of $V_{d_{\mathcal{S}_5^+}}(2)$ in $\mathbb{Z}_{\geq 0}$ might not be finite. Indeed, the complement in this case seems substantial, and its intersection with $[1, 10^5]$ might contain as many as 6000 elements.

Example 7.5. Consider the complete bipartite graph $G = \mathcal{K}(2, 3)$, with $n = 5$. Computations show that d_G might fail to represents a set $\mathcal{C}$ of 693 values in $[1, \ldots, 10^5]$.

8. Complete Graphs

Proposition 8.1. *Let $G = \mathcal{K}_n$, the complete graph on $n \geq 2$ vertices. Then $1 \in V_G(2)$ if and only if the equation*

$$\sum_{i=1}^{n} \frac{1}{y_i} + \frac{1}{y_1 \cdots y_n} = 1 \tag{8.1}$$

can be solved with positive integers $y_1, \ldots, y_n \geq 3$.

Proof. Consider the square matrix

$$M' := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & M_G(x_1, \ldots, x_n) & \\ 0 & & & \end{pmatrix}$$

This matrix clearly has the same determinant as $M_G(x_1, \ldots, x_n)$, and is row and column equivalent to the matrix

$$M'' := \begin{pmatrix} 1 & -1 & \ldots & -1 \\ -1 & x_1 + 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -1 & 0 & 0 & x_n + 1 \end{pmatrix}.$$

Indeed, letting

$$T := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ -1 & 1 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 1 \end{pmatrix},$$

we find that $M'' = TM'({}^tT)$. Note that if H denote the star on $n+1$ vertices $w_0, \ldots, w_n$, then the matrix M'' is $M_H(1, x_1 + 1, \ldots, x_n + 1)$. It is clear that M is positive definite if and only if M', and hence M'', is positive definite. Expanding the determinant of M'' using its first row, we obtain that

$$\det(M_G(x_1,\ldots,x_n)) = \prod_{j=1}^{n}(x_j+1) - \sum_{i=1}^{n}\frac{\prod_{j=1}^{n}(x_j+1)}{x_i+1}. \tag{8.2}$$

Assume now that $1 \in V_G(2)$. Then we can find integers $x_1,\ldots,x_n \geq 2$ such that, setting $\ell := \prod_{j=1}^{n}(x_j+1)$, we have

$$1 = \ell - \sum_{i=1}^{n}\ell/(x_i+1).$$

Setting $y_i = x_i+1$, so that $\ell = \prod_{i=1}^{n} y_i$, we find that $1/\ell = 1 - \sum_{i=1}^{n} 1/y_i$ with $y_i \geq 3$ for all i, as desired.

Reciprocally, assume that there exist $y_1,\ldots,y_n \geq 3$ such that $1 = \prod_{j=1}^{n} y_j - \sum_{i=1}^{n}(\prod_{j=1}^{n} y_j)/y_i$. Setting $x_i = y_i - 1$, we obtain $x_1,\ldots,x_n \geq 2$ such that $\det(M_G(x_1,\ldots,x_n)) = 1$. To show that $1 \in V_G(2)$, it remains to show that $M_G(x_1,\ldots,x_n)$ is positive definite. For this, it suffices, as noted above, to argue that M'' is positive definite. This is clear since $\det(M'') = 1 > 0$ in our case, and all the diagonal elements except for the one in the top left corner are positive. □

Remark 8.2. In a solution $(y_1,\ldots,y_n)$ to Equation (8.1), the integers y_i are pairwise coprime. In particular, they are all distinct. When n is odd, any solution $(y_1,\ldots,y_n)$ to Equation (8.1) must have at least one y_i even.

When $n=13$, one finds in [11], page 8, a solution to Equation (8.1) in pairwise coprime integers obtained by Girgensohn with $3 = y_1 < 4 < 5 < 7 < 29 < \cdots < y_n$. The integer y_{13} has 172 digits. This solution has exactly one even entry, $y_2 = 4$. Note that the same solution is given in [10], page 393, but in that article the given solution has typos.

Once we have a solution $(y_1,\ldots,y_n)$ to Equation (8.1), it is easy to extend it to a solution $(y_1,\ldots,y_n,x)$ satisfying

$$\sum_{i=1}^{n}\frac{1}{y_i} + \frac{1}{x} + \frac{1}{y_1\cdots y_n x} = 1,$$

by setting $x = (y_1\cdots y_n)+1$.

Corollary 8.3. *Let $G = \mathcal{K}_n$ be the complete graph on $n \geq 2$ vertices.*

(a) *If $n = 13$, then $1 \in V_G(2)$.*
(b) *If $n \geq 14$, then $V_G(2) = \mathbb{Z}_{\geq 0}$.*
(c) *If $n \leq 7$, then $1 \notin V_G(2)$.*

Proof. (a) In view of Proposition 8.1, it suffice to show that the equation (8.1) can be solved with $y_1,\ldots,y_n \geq 3$. When $n=13$, we use the solution provided in [11], page 8, and mentioned in the previous remark.

(b) The graph $\mathcal{K}_{n+1}$ is the cone on the graph $\mathcal{K}_n$. Therefore, Theorem 1.1 (b) shows that if $1 \in V_{\mathcal{K}_n}(2)$, then $V_{\mathcal{K}_{n+1}}(2) = \mathbb{Z}_{\geq 0}$.

(c) It is mentioned in [11], page 8, that all solutions $y_1 \leq \cdots \leq y_n$ to Equation (8.1) with $n \leq 9$ have $y_1 = 2$. We have not been able to retrieve the list of known solutions for $n = 8, 9$ to verify this claim. The list of solutions with $n \leq 7$ is provided in [9], page 50 and Appendix. The claim (c) then follows from Proposition 8.1. $\square$

Remark 8.4. Let $G = \mathcal{K}_n$ be a complete graph with $3 \leq n \leq 13$. Corollary 4.3 shows that the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geq 0}$ has density 0. It is natural to wonder whether the complement is finite. We discussed already the case $n = 3$ in 7.1. Computations in the case $n = 4$ also seem to indicate that the complement in this case might be quite substantial and maybe is not finite.

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