Complements to: Special fibers of Néron models and wild ramification

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April 24, 2009

In Proposition 1.8 of our paper [2], we use several results of [B-X]. We quote here from [2]:

Remark 1.3 The proof of [B-X], 4.11 (ii), uses Lemma 4.2 of *loc. cit.*, which is incorrect in the case of perfect residue fields. The authors of [B-X] have informed us that they can provide a different proof of 4.11 (ii) without using 4.2.

This mistake in [B-X] was first noted by Chai in [1], Remark 4.8 (2). Chai then notes that he was informed by Bosch that the mistake does not affect any other subsequent results in [B-X].

Our aim in this note is to carefully go through the proof of Proposition 1.8 in [2] and detail what results of [B-X] we use, so that the careful reader will be convinced that the proof of Proposition 1.8 is complete, and is not affected by the mistake in [B-X]. The comments on our original proof are in italic.

Proposition 1.8 *Let A/K be an abelian variety whose Néron model $\mathcal{A}/\mathcal{O}_K$ has toric rank equal to 0. Then $\Phi(A)$ is killed by $[L:K]^2$.*

Proof: Proposition 2.15 in [Lor2] shows that the prime-to-p part of $\Phi(A)$ is killed by $[L:K]^2$. To prove the general case, we proceed as follows. Consider the subgroups $\Theta_2 \subseteq \Theta_1$ of $\Phi(A)$ introduced on page 480 of [B-X]. Since $t_K = 0$ by hypothesis, we find that $\Theta_1 = \Phi(A)$. It follows from [B-X], 5.9, that Θ_1/Θ_2 is killed by $[L:K]$.

Rather than working with Θ_1 and 5.9, we will explain below how to get the same result using the subgroup Σ_1: this will use 'less' of the paper [B-X], and make it easier to write down all details.

Let $\Psi_{K,L}$ denote the kernel of the natural map $\Phi(A) \to \Phi(A_L)$. Then $[L:K]$ kills $\Psi_{K,L}$ ([ELL], Thm. 1). To conclude the proof of the proposition, it is sufficient to note that the subgroup Θ_2 is contained in $\Psi_{K,L}$. Indeed, consider the rigid analytic uniformization of A/K as in [B-X], S1:

$$\begin{array}{ccccc} & & T & & \\ & & \downarrow & & \\ \Lambda & \longrightarrow & G & \longrightarrow & A \\ & & \downarrow & & \\ & & B & & \end{array}$$

with T/K a torus, B/K an abelian variety with potentially good reduction, and Λ/K a lattice. The group Θ_2 is defined to be the image under the natural map $\Phi(G) \to \Phi(A)$ of the subgroup $\Phi(G)_{tors}$.

The subgroup Σ_1 is defined to be the image of $\Phi(G) \to \Phi(A)$, so that $\Theta_2 \subseteq \Sigma_1$. We will show below that when $t_K = 0$, $\Theta_2 = \Sigma_1$.

The change of base L/K induces natural maps

$$\begin{array}{ccc} \Phi(G) & \to & \Phi(A) \\ \downarrow & & \downarrow \\ \Phi(G_L) & \to & \Phi(A_L) \end{array}$$

It follows from [B-X], 4.11 (see 1.3), that the map $\Phi(T_L) \to \Phi(G_L)$ is an isomorphism (recall that $\Phi(B_L) = (0)$). Thus, $\Phi(G_L)$ is free since $\Phi(T_L)$ is. Hence, the image of $\Phi(G)_{tors}$ in $\Phi(G_L)$ is trivial.

Let us detail our use of 4.11 above. We use it on the exact sequence $0 \to T_L \to G_L \to B_L \to 0$. Then T_L is split, and we can use 4.11 (ii) with that hypothesis to obtain that $\Phi(T_L) \to \Phi(G_L)$ is surjective. The injectivity is obtained using 4.11 (i), whose proof shows with no additional hypotheses that $\Phi(T_L) \to \Phi(G_L)$ is injective on the free parts. But here $\Phi(T_L)$ is free.

Let us now show that when $t_K = 0$, then $\Theta_2 = \Sigma_1$. We will show in fact that $\Phi(G)$ is torsion, so equal to $\Phi(G)_{tors}$. This is immediate from 4.11 (i), which shows that $\Phi(T) \to \Phi(G)$ has finite kernel and finite cokernel. When $t_K = 0$, we find that $\Phi(T)$ is torsion. Note that the proof of 4.11 (i) does invoke 4.2, but only the proven part of 4.2 in the case where the torus splits over an unramified extension: it uses 4.2 on the maximal split subtorus $T_{K,I}$.

Finally, we need to show that $\Phi(A)/\Sigma_1$ is killed by $[L : K]$. This is obtained in [B-X] from 5.5 (i), where it is shown that $\Phi(A)/\Sigma_1$ injects into $H^1(I, M_K)$, M_K being what we denoted by Λ in this proof. The proof of 5.5 in [B-X] states that this is immediate using 4.12. For completeness, with our notation, 4.12 states that we have an exact sequence

$$0 \to \Phi(\Lambda) \to \Phi(G) \to \Phi(A) \to H^1(I, \Lambda).$$

To prove 4.12, Bosch and Xarles use 4.9, which does not use the incorrect part of 4.2. This shows that $\Phi(G) \to \Phi(A) \to H^1(I, \Lambda)$ is exact, and this is all we need.

Corollary. Let A/K be any abelian variety. Then $\Theta_2 \subseteq \Psi_{K,L}$.

Proof. Follows immediately from the proof of 1.8.

Remark. When $\ell \neq p$, it is likely that the ℓ-parts of Θ_2 and $\Psi_{K,L}$ coincide (see [Lor2], 3.22, for some evidence). We do not have an example where Θ_2 and $\Psi_{K,L}$ have different p-parts.

References

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[1] C.-L. Chai, *Néron models for semiabelian varieties: congruence and change of base field*, Asian J. Math. *4* (2000), no. 4, 715–736.

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