We give below some details regarding an assertion used in Corollary 6.7 of the paper [4].

Assertion: *Let K be a complete discrete valuation field with ring of integers $\mathcal{O}_K$ and residue field k. Assume that k is algebraically closed of characteristic $p > 0$. Let E_K/K be an elliptic curve, $E/\mathcal{O}_K$ its minimal regular model, and let T denote the type of E_k. Let $m = p^n \geq 1$ if the type is additive, and let $m > 0$ be an arbitrary positive integer otherwise. Then the group $H^1(K, E_K)$ contains an element of order m.*

The case where m is coprime to p is explained for instance on page 142 of [2], where it is indicated that the set of elements killed by m in $H^1(K, E_K)$ is isomorphic to the dual of the group of points in $E_K(K)$ of order m which reduce in the connected component of zero of the Néron model $E/\mathcal{O}_K$. When the reduction is semi-stable, there exists k-rational elements of any order m coprime to p in the connected component of zero of the special fiber, and such elements can be lifted to K-rational torsion points of order m, due to the fact that the kernel of the reduction does not contain any elements of order coprime to p.

Let us consider now the case where m is a power of p. As we indicate on page 497 of [4], second paragraph, when k is algebraically closed, the group $H^1(K, E_K)$ is divisible by p. Thus, our claim is proved as soon as we exhibit a non-trivial element of order p in $H^1(K, E_K)$.

Recall that Shafarevich's pairing is a perfect pairing

$$H^1(K, E_K) \times \pi_1(E_K) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

This is proved in [1] and [2]. Here $\pi_1(E_K)$ is a group associated with the Greenberg realization of E_K using the work of Serre in [5], as we recall below. Using Shafarevich's pairing, we are reduced to exhibiting a group homomorphism $\pi_1(E_K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ of order p. We now recall just enough of the construction of $\pi_1(E_K)$ to be able to exhibit such an element.

Associated with the Néron model $E/\mathcal{O}_K$ is a system of smooth group schemes $\mathcal{G}_i(E)/k$, $i \in \mathbb{N}^*$, with morphisms of group schemes $\mathcal{G}_{i+1}(E) \to \mathcal{G}_i(E)$. The group scheme $\mathcal{G}_i(E)/k$ is called the Greenberg realization of level i ([3], page 276). The group $\mathcal{G}_1(E)/k$ is the connected component of zero of the special fiber of E over k.

One defines (see, e.g., [2], 2.1, page 143) a pro-etale group scheme $\pi_i(E_K)$ as the value of the i-th derived functor of a functor π_0 on the object $\{\mathcal{G}_j(E)/k\}_{j\in\mathbb{N}^*}$. If $E^0/\mathcal{O}_K$ denotes the complement in E of the connected components of the special fiber of E which do not contain the identity, then $\pi_1(E_K)$ can be computed as the value of π_1 on the object $\{\mathcal{G}_n(E^0)/k\}_{n\in\mathbb{N}^*}$ ([2], Lemma 2, page 144).

Each individual group scheme $\mathcal{G}_j(E^0)/k$ is also an object on which one can evaluate the functor π_i. The natural map $\mathcal{G}_{n+1}(E^0) \to \mathcal{G}_n(E^0)$ has a kernel V_n which is smooth, connected, and unipotent ([1], 4.1.1 (3)). Since it is connected, $\pi_0(V_n) = (0)$, and the long exact sequence of derived functors gives a surjection $\pi_1(\mathcal{G}_{n+1}(E^0)) \to \pi_1(\mathcal{G}_n(E^0))$. A similar argument shows that for any $n \geq 1$, one has a natural surjective morphism

$$\pi_1(E_K) \to \pi_1(\mathcal{G}_n(E^0)/k).$$

Thus, we can exhibit a group homomorphism $\pi_1(E_K) \to \mathbb{Q}/\mathbb{Z}$ of order p simply by exhibiting first a group homomorphism $\pi_1(\mathcal{G}_n(E^0)/k) \to \mathbb{Q}/\mathbb{Z}$ of order p for some n. It will suffice to consider the cases $n = 1$ and $n = 2$.

We now need to use results of Serre from [5]. Bertapelle notes on page 143 of [2], penultimate paragraph, that when X/k is a smooth commutative group scheme of finite type, the group $\pi_1(X)$ that she defines on page 143 is the same as the value of the first derived functor in [5], 5.3, obtained by evaluating on the perfection of X. Thus we are now able to use the results of [5] applied to the groups $\pi_1(\mathcal{G}_j(E^0)/k)$.

The group $\pi_1(\mathcal{G}_1(E^0)/k)$ is either equal to $\pi_1(\mathbb{G}_a/k)$, or $\pi_1(\mathbb{G}_m/k)$, or $\pi_1(B/k)$ where B/k is an elliptic curve. In [5], Corollaire, page 53, the group $\pi_1(\mathbb{G}_a/k)$ is shown to be isomorphic to $\mathrm{Hom}(k, \mathbb{Q}/\mathbb{Z})$. Therefore, in this case there exists a homomorphism $\pi_1(\mathbb{G}_a/k) \to \mathbb{Q}/\mathbb{Z}$ of order p. One finds in [5], Corollaire 3, page 45, that the p-primary component of $\pi_1(B/k)$ is isomorphic to $\mathbb{Z}_p^s$, where $s = 0$ or 1. Thus we cannot immediately obtained the desired homomorphism of degree p in this case. We now show that there exists instead a homomorphism $\pi_1(\mathcal{G}_2(E^0)/k) \to \mathbb{Q}/\mathbb{Z}$ of order p when the reduction is semi-stable.

Recall that if $i \geq 2$, then $\pi_i(\mathbb{G}_m/k) = (0)$ and $\pi_i(B/k) = (0)$ ([5], Corollaires 2 and 3 on page 45). We also have ([5], Proposition 7, page 38) functorial isomorphisms for all $i > 0$:

$$\mathrm{Ext}^{\mathrm{i}}(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(\pi_i(G), \mathbb{Q}/\mathbb{Z}).$$

Hence,

$$\mathrm{Ext}^2(\mathbb{G}_m/k, \mathbb{Q}/\mathbb{Z}) = (0) = \mathrm{Ext}^2(B/k, \mathbb{Q}/\mathbb{Z}).$$

Consider now the natural exact sequence of group schemes over k:

$$(0) \longrightarrow \mathbb{G}_a \longrightarrow \mathcal{G}_2(E^0) \longrightarrow \mathcal{G}_1(E^0) \longrightarrow (0).$$

It follows from the fact that the groups Ext^2 are trivial that we obtain an exact sequence

$$(0) \longleftarrow \mathrm{Ext}^1(\mathbb{G}_a, \mathbb{Q}/\mathbb{Z}) \longleftarrow \mathrm{Ext}^1(\mathcal{G}_2(E^0), \mathbb{Q}/\mathbb{Z})$$

We conclude using the fact that $\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{Q}/\mathbb{Z})$ is isomorphic to $\mathbb{G}_a$ ([5], Proposition 3, page 52).

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