ELEMENTARY DIVISOR DOMAINS AND BÉZOUT DOMAINS

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Abstract. It is well-known that an Elementary Divisor domain R is a Bézout domain, and it is a classical open question to determine whether the converse statement is false. In this article, we provide new chains of implications between *R is an Elementary Divisor domain* and *R is Bézout* defined by hyperplane conditions in the general linear group. Motivated by these new chains of implications, we construct, given any commutative ring R, new Bézout rings associated with R.

KEYWORDS Elementary Divisor ring, Hermite ring, Bézout ring, Rings defined by matrix properties, Symmetric matrix, Trace zero matrix.

MSC: 13F10, 15A33, 15A21

1. Introduction

A commutative ring R in which every finitely generated ideal is principal is called a *Bézout ring*. By definition, a noetherian Bézout domain is a principal ideal domain. Examples of non-noetherian Bézout domains can be found for instance in [4], 243-246.

A commutative ring R is called an *Elementary Divisor ring* if every matrix A with coefficients in R admits diagonal reduction, that is, if $A \in M_{m,n}(R)$, then there exist invertible matrices $P \in \mathrm{GL}_m(R)$ and $Q \in \mathrm{GL}_n(R)$ such that $PAQ = D$ with $D = (d_{ij})$ diagonal (i.e., $d_{ij} = 0$ if $i \neq j$) and every d_{ii} is a divisor of $d_{i+1,i+1}$. Note that for a commutative ring R, every diagonal matrix with coefficients in R admits diagonal reduction if and only if R is a Bézout ring ([16], (3.1)).

Kaplansky showed in [13], 5.2, that a Bézout domain is an Elementary Divisor domain if and only if it satisfies:

($*$) For all $a, b, c \in R$ with $(a, b, c) = R$, there exist $p, q \in R$ such that $(pa, pb + qc) = R$.

(See also [8], 6.3.) It is well-known that a principal ideal domain is an Elementary Divisor domain. Consideration of the Elementary Divisor problem for a non-noetherian ring can be found as early as 1915 in Wedderburn [19].

It is an open question dating back at least to Helmer [12] in 1942 to decide whether a Bézout domain is always an Elementary Divisor domain. Gillman and Henriksen gave examples of Bézout rings that are not Elementary Divisor rings in [10]. In 1977, Leavitt and Mosbo in fact stated in [15], Remark 8, that it has been conjectured that there exists a Bézout domain that is not an Elementary Divisor domain (see also Problem 5 in [8], p. 122).

Our contribution to this question is the introduction, in 3.2 and 4.11, of new chains of implications between *R is an Elementary Divisor domain* and *R is Bézout.* Motivated by these new chains of implications, we construct, given any commutative

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ring R which is not Bézout, new Bézout rings associated with R (see 3.5 and 4.10). We have not been able to determine whether these rings are Elementary Divisor rings.

2. Orbits under the action of $\mathrm{GL}_n(R)$

Let R be a commutative ring. Let $M_n(R)$ denote the ring of $(n \times n)$-matrices with coefficients in R, and endowed with the action of $\mathrm{GL}_n(R)$ on the right. Recall that a commutative ring R is called a *Hermite*[1] ring if for each matrix $A \in M_{m,n}(R)$, there exists $U \in \mathrm{GL}_n(R)$ such that $AU = (b_{ij})$ is lower triangular (i.e., $b_{ij} = 0$ whenever $i < j$). In fact, Kaplansky shows that R is Hermite as soon as for each matrix (a, b), there exists $U \in \mathrm{GL}_2(R)$ such that $(a, b)U = (d, 0)$ for some d ([13], 3.5).

Let L_n denote the R-submodule of $M_n(R)$ consisting of all lower triangular matrices. We note that *a domain R is Hermite if and only if for some $n \geq 2$, the orbit of L_n under the right action of $\mathrm{GL}_n(R)$ is equal to $M_n(R)$.* Indeed, it is clear that if R is Hermite, the orbit of L_n is the whole space $M_n(R)$. Suppose now that the orbit of L_n is $M_n(R)$. Let $a, b \in R$ and consider the $(n \times n)$-matrix $A = (a_{ij})$ with $a_{11} = a$, $a_{12} = b$, $a_{ii} = 1$ if $i = 2, \ldots, n$, and all other coefficients equal to 0. Then there exists $U \in \mathrm{GL}_n(R)$ such that AU is lower triangular. When R is a domain, it follows that U has its first two lines of the form $(u_{11}, u_{12}, 0, \ldots, 0)$ and $(u_{21}, u_{22}, 0, \ldots, 0)$. Let U' denote the (2×2) matrix $(u_{ij}, 1 \leq i, j \leq 2)$. Then $U' \in \mathrm{GL}_2(R)$, and $(a, b)U' = (d, 0)$. By Kaplansky's Theorem, R is Hermite.

Let S_n denote the R-submodule of $M_n(R)$ consisting of all symmetric matrices. It is natural to wonder whether there exist rings R such that the orbit of S_n under $\mathrm{GL}_n(R)$ is equal to $M_n(R)$. This led us to the following definitions.

Definition 2.1 Let $n \geq 1$. A ring R *satisfies Condition* $(SU)_n$ (resp. satisfies Condition $(SU')_n$) if, given any $A \in M_n(R)$, there exist a symmetric matrix $S \in M_n(R)$ and an invertible matrix $U \in \mathrm{GL}_n(R)$ (resp. $U \in \mathrm{SL}_n(R)$) such that $A = SU$.

Remark 2.2 It is easy to check that if R satisfies Condition $(SU)_n$ or $(SU')_n$, and I is any proper ideal of R, then R/I also satisfies Condition $(SU)_n$ or $(SU')_n$. It is also true that if $T \subset R$ is a multiplicative subset, then the localization ring $T^{-1}(R)$ satisfies Condition $(SU)_n$ or $(SU')_n$. We note that the Hermite property is also preserved by passage to factor rings or localizations at multiplicative subsets.

Further properties of rings R satisfying Condition $(SU)_n$ or $(SU')_n$ are discussed in the next section and as we shall see, these rings are quite special. There are other interesting R-submodules of $M_n(R)$ for which the above question can be considered. For instance, let $T_n \subset M_n(R)$ be the R-submodule consisting of all matrices having trace zero. We are led to the following definitions.

Definition 2.3 Let $n \geq 1$. A ring R *satisfies Condition* $H_{n,1}$ (resp. satisfies Condition $H'_{n,1}$) if the orbit of T_n under the action of $\mathrm{GL}_n(R)$ (resp. under the action of $\mathrm{SL}_n(R)$) is equal to $M_n(R)$.

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[1] A different notion of Hermite ring is also in use in the literature; See for instance the appendix to section I.4 in [14]. The notion of Hermite ring used here is due to Kaplansky in 1949, as is the notion of Elementary Divisor ring [13]. The terminology *Bézout ring* seems to be slightly more recent. In 1943, Helmer calls such a ring a Prüfer ring [12], but as early as 1956, the terminology of *Prüfer ring* is reserved for rings where all finitely generated ideals are projective [1]. In 1954, Gillman and Henriksen [10] call a Bézout ring a F-ring. In 1960, Chadeyras [2] uses the term *anneau semi-principal ou de Bézout* to refer to a Bézout ring.

Further properties of rings R satisfying Condition $H_{n,1}$ or $H'_{n,1}$ are discussed in the fourth section. In particular, the analogue of 2.2 also holds. When $n = 2$, the conditions $(SU)_2$ and $H_{2,1}$ are equivalent (4.11).

Our choice of notation indicates that the cases of S_n and T_n are different, as it is also possible to consider stronger Conditions $H_{n,s}$ or $H'_{n,s}$ for $1 \leq s \leq n-1$. Indeed, for $s > 0$, endow the product $(M_n(R))^s$ with the diagonal action of $\mathrm{GL}_n(R)$ (that is, for $g \in \mathrm{GL}_n(R)$ and $a := (a_1, \ldots, a_s) \in (M_n(R))^s$, let $a \cdot g := (a_1 g, \ldots, a_s g)$). We further define:

Definition 2.4 Let n and s be positive integers. A ring R *satisfies Condition* $H_{n,s}$ (resp. satisfies Condition $H'_{n,s}$) if the orbit of $(T_n)^s$ under the action of $\mathrm{GL}_n(R)$ (resp. under the action of $\mathrm{SL}_n(R)$) is equal to $(M_n(R))^s$.

As we note in 4.1, no ring satisfies Condition $H_{n,s}$ or $H'_{n,s}$ when $s \geq n$. Several obvious generalizations of the notions introduced above also lead to vacuous classes of rings. For instance, the orbit of $S_n \times S_n$ in $M_n(R) \times M_n(R)$ under the diagonal action of $\mathrm{GL}_n(R)$ is never equal to $M_n(R) \times M_n(R)$. Indeed, let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then the element (B, C) is not in the orbit of $S_2 \times S_2$.

The orbit of $S_n \cap T_n$ is never equal to $M_n(R)$. Indeed, the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ cannot be written as BU with B symmetric and trace 0, and U invertible.

3. Condition $(SU)_n$

Gillman and Henriksen have proved in [9], Theorem 3, that a commutative ring is a Hermite ring if and only if the following condition is satisfied:

(∗∗) for every $a, b \in R$, there exist c, d and g in R such that $a = cg$, $b = dg$, and $(c, d) = R$.

It follows immediately that a Bézout domain is a Hermite domain.

Proposition 3.1. *Let R be any commutative ring. If R satisfies Condition $(SU)_n$ for some $n \geq 2$, then R is a Hermite ring.*

Let $a, b \in R$, and let $A := \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. Then

(i) *If there exists $V := \begin{pmatrix} u & v \\ s & t \end{pmatrix} \in \mathrm{GL}_2(R)$ such that AV is symmetric, then there exists $g \in R$ such that $a = ug$, $b = vg$, and $(u, v) = R$.*

(ii) *If R is a Hermite ring, then there exists $V \in \mathrm{SL}_2(R)$ such that AV is symmetric.*

Proof. (i) The cases where $a = 0$ or $b = 0$ are easy and left to the reader. Assume that $a \neq 0$ and $b \neq 0$. The product AV is symmetric if and only if $av = bu$. The matrix V is invertible if and only if $ut - sv = \epsilon \in R^*$. Then $aut - asv = a\epsilon = u(at - bs)$, and $at - bs$ divides a. Similarly, $v(at - bs) = b$. Therefore, $(at - bs) \subseteq (a, b) \subseteq (at - bs)$, and we find that the ideal (a, b) is principal. We also have $(u, v) = R$, as desired.

Suppose now that R satisfies Condition $(SU)_n$ for some $n \geq 2$. Let $a, b \in R$. Consider the square $(n \times n)$-matrix $A = (a_{ij})$ with all null entries, except for $a_{11} := a$ and $a_{21} := b$. Assume that there exists $V = (v_{ij}) \in \mathrm{GL}_n(R)$ such that AV is symmetric. Then we find that $v_{13} = \cdots = v_{1n} = 0$, and $av_{12} = bv_{11}$. Expanding the determinant of V using the first row, we find that we can write $\det(V) = v_{11}s - v_{12}t \in R^*$ for some $s, t \in R$. We conclude as above with $g = at - bs$.

(ii) Let $a, b \in R$. Assume that there exist $c, d, g \in R$ such that $a = gc$ and $b = gd$, and that there exist $s, t \in R$ such that $cs + dt = 1$. We can write

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & d \\ -t & s \end{pmatrix} = \begin{pmatrix} ac & ab/g \\ ab/g & bd \end{pmatrix}.$$

□

Proposition 3.2. *Let R be any commutative ring. Consider the following properties:*
a) *R is an Elementary Divisor ring.*
b) *R satisfies Condition $(SU')_n$ for all $n \geq 2$.*
c) *R satisfies Condition $(SU)_n$ for all $n \geq 2$.*
d) *R is a Hermite ring.*
Then a) $\Longrightarrow$ b) $\Longrightarrow$ c) $\Longrightarrow$ d).

Proof. *a) $\Longrightarrow$ b).* Let $A \in M_n(R)$. Choose $P, Q \in \mathrm{GL}_n(R)$ such that $PAQ = D$ is a diagonal matrix. Let $\epsilon := \det(P)\det(Q)^{-1}$. Let E denote any invertible diagonal matrix with determinant ϵ. Then $PAQE = DE$ is still symmetric since D is diagonal. We find that

$$AQE(P^{-1})^t = P^{-1}DE(P^{-1})^t$$

is symmetric, with $\det(QE(P^{-1})^t) = 1$. It is obvious that *b) $\Longrightarrow$ c)*. The last implication *c) $\Longrightarrow$ d)* follows from 3.1. □

It is completely obvious from the previous proposition that if R is an Elementary Divisor domain and satisfies Condition $(SU)_n$, then it is also satisfies Condition $(SU)_{n-1}$. We can strengthen this assertion as follows.

Proposition 3.3. *Let R be a commutative domain which satisfies Condition $(SU)_n$ for some $n \geq 3$. Then R is a Bézout domain, and satisfies Condition $(SU)_{n-1}$.*

Proof. Proposition 3.1 shows that the domain R is Bézout. Let $A \in M_{n-1}(R)$. Since R is Bézout, it is possible to find two invertible matrices $P, Q \in \mathrm{GL}_{n-1}(R)$ such that PAQ consists in its upper left corner of a square $(r \times r)$-matrix A' with $r = \mathrm{rank}(A)$ and $\det(A') \neq 0$, and such that all other coefficients of PAQ are zeros. Indeed, since R is a domain, we can define the rank of A to be its rank when A is viewed as a matrix with coefficients in the field of fractions K of R. Suppose that the columns $A_1, \ldots, A_{n-1}$ of A are linearly dependent over K (i.e., that $\mathrm{rank}(A) < n-1$). Since R is a Bézout domain, we can then find $a_1, \ldots, a_{n-1} \in A$ such that $\sum a_i A_i = 0$ and $(a_1, \ldots, a_{n-1}) = A$. Then there exists a matrix $Y \in \mathrm{GL}_{n-1}(R)$ such that the last column of Y has entries $a_1, \ldots, a_{n-1}$ (see, e.g., [13], 3.7). It follows that the matrix AY has its last column equal to the zero-vector. We proceed similarly with the rows of AY, to find an invertible matrix $X \in \mathrm{GL}_{n-1}(R)$ such that XAY consists of a square $(n-2 \times n-2)$-matrix $A^{(1)}$ in the top left corner, and zeros everywhere else. If $\mathrm{rank}(A^{(1)}) < n-2$, we repeat the process with $A^{(1)}$, and so on.

Let $B \in M_n(R)$ be the matrix with A' in the upper left corner, and with all other entries zeros. By hypothesis, there exists $U \in \mathrm{GL}_n(R)$ such that BU is symmetric. Clearly, the last $n - \mathrm{rank}(A)$ rows of BU consists only in zeros. Since the matrix BU is symmetric, its last $n - \mathrm{rank}(A)$ columns also consists only in zeros. Let W denote any vector in $R^{\mathrm{rank}(A)}$ obtained from one of the $n - \mathrm{rank}(A)$ last columns of

U by removing from the column its last $n - \mathrm{rank}(A)$ coefficients. Then $A'W = 0$. Since $\det(A') \neq 0$, we find that $W = 0$. Let V denote the square $\mathrm{rank}(A)$-matrix in the upper left corner of U, and let V' denote the square $(n - \mathrm{rank}(A))$-matrix in the lower right corner of U. Then $\det(U) = \det(V)\det(V')$. Hence, V is invertible, and we have $A'V$ symmetric.

Consider now the square matrix T of size $(n-1)$ consisting of two blocks: V in the upper left corner, and an identity matrix of the appropriate size in the lower right corner. The matrix T is invertible. By construction, $PAQT$ is symmetric. Then $AQT(P^{-1})^t$ is also symmetric, with $QT(P^{-1})^t$ invertible. □

Remark 3.4 A key step in the above proof in general cannot be performed if the ring R is not a domain, even when R is a principal ideal ring. Indeed, let $R := k[\epsilon]/(\epsilon^2)$, with k any field. The diagonal matrix $D := \mathrm{diag}(\epsilon, \epsilon)$ has determinant 0, and has two linearly dependent columns. But it is not possible to find $U \in \mathrm{GL}_2(R)$ such that DU has a null bottom row.

If R satisfies Condition $(SU)_n$ and R has the property that every unit $r \in R^*$ is an n-th power in R, then R also satisfies Condition $(SU')_n$. Indeed, if $A = SU$ with S symmetric and $\det(U) \in R^*$, write $\det(U) = \epsilon^n$, and $D := \mathrm{diag}(\epsilon, \dots, \epsilon)$. Then $A = (SD)(D^{-1}U)$ with SD symmetric, and $D^{-1}U \in \mathrm{SL}_n(R)$.

It is natural to ask whether any of the implications in our last propositions can be reversed in general. We can also ask whether a commutative Bézout domain which satisfies Condition $(SU')_n$ also satisfies Condition $(SU')_{n-1}$.

Example 3.5 Proposition 3.2 suggests the following construction of new Bézout rings. Let R be any commutative ring and fix $n > 1$. Let $X = (x_{ij})_{1 \leq i,j \leq n}$ denote the square $n \times n$-matrix in the indeterminates $x_{ij}, 1 \leq i,j \leq n$. For each matrix $A \in M_n(R)$, consider the subset $I(A)$ of $R[x_{11}, \dots, x_{nn}]$ consisting of $\det(X) - 1$ and of the $(n^2 - n)/2$ polynomial equations obtained by imposing the condition that the matrix AX is symmetric. Let $< I(A) >$ denote the ideal of $R[x_{11}, \dots, x_{nn}]$ generated by the elements of $I(A)$. We claim that $< I(A) > \neq R[x_{11}, \dots, x_{nn}]$. Indeed, choose a maximal ideal M of R, and consider the field $K := R/M$. If $1 \in < I(A) >$, then 1 is also contained in the ideal of $K[x_{11}, \dots, x_{nn}]$ generated by the images of the elements of $I(A)$ modulo M. This is not possible since K is a principal ideal domain, and Proposition 3.2 shows then that K satisfies Condition $(SU')_n$.

Consider the set $\mathcal{I}$ of all subsets $I(A)$, $A \in M_n(R)$, such that there exists no homomorphism of R-algebras between $R[x_{11}, \dots, x_{nn}]/ < I(A) >$ and R (i.e., such that there exists no matrix $Y \in \mathrm{SL}_n(R)$ with AY symmetric). For each subset $I = I(A) \in \mathcal{I}$, we let $\mathbf{x}^I$ denote the set of n^2 variables labeled $x^I_{11}, \dots, x^I_{nn}$, and we denote by $(\mathbf{x}^I)$ the matrix (x^I_{ij}). We now let $I(A, \mathbf{x}^I)$ be the subset of $R[\mathbf{x}^I]$ consisting of $\det((\mathbf{x}^I)) - 1$ and of the $(n^2 - n)/2$ polynomial equations obtained by imposing the condition that the matrix $A(\mathbf{x}^I)$ is symmetric. It is not difficult to check that the ideal $< I(A, \mathbf{x}^I), I \in \mathcal{I} >$ is a proper ideal of the polynomial ring $R[\mathbf{x}^I, I \in \mathcal{I}]$. Indeed, if $1 \in < I(A, \mathbf{x}^I), I \in \mathcal{I} >$, then there exist finitely matrices $A_1, \dots, A_s$ such that $1 \in < I(A_i, \mathbf{x}^{I(A_i)}), i = 1, \dots, s >= R[\mathbf{x}^{I(A_i)}, i = 1, \dots, s]$. Reducing modulo the ideal generated by a maximal ideal M of R leads as above to a contradiction. We define the quotient ring

$$s_n(R) := R[\mathbf{x}^I, I \in \mathcal{I}]/ < I(A, \mathbf{x}^I), I \in \mathcal{I} > .$$

Note that if R satisfies Condition $(SU')_n$, then $\mathcal{I} = \emptyset$ and, in particular, $s_n(R) = R$. It is clear that we have a natural morphism of R-algebras $R \to s_n(R)$. By construction, given any matrix $B \in M_n(R)$, there exists $U \in \mathrm{SL}_n(s_n(R))$ such that BU is symmetric. Indeed, it suffices to take $U :=$ (class of$(x_{ij}^{I(B)})$ in $s_n(R)$).

Let $s_n^{(1)}(R) := s_n(R)$, and for each $i \in \mathbb{N}$, we set $s_n^{(i)}(R) := s_n(s_n^{(i-1)}(R))$. Finally, we let

$$\mathcal{S}_n(R) := \varinjlim_i s_n^{(i)}(R).$$

Let $C \in M_n(\mathcal{S}_n(R))$. Then the finitely many coefficients of C all lie in a single ring $s_n^{(i)}(R)$ for some $i > 0$. By construction, there exist $U := (u_{ij}) \in \mathrm{SL}_n(s_n^{(i)}(R))$ such that CU is symmetric. It follows that $\mathcal{S}_n(R)$ satisfies Condition $(SU')_n$.

Given any prime ideal P of $\mathcal{S}_n(R)$, the quotient $\mathcal{S}_n(R)/P$ satisfies Condition $(SU')_n$ and, thus, is a Bézout domain (3.3). It is natural to wonder whether one could show for a well-chosen ring R that one such domain is not an Elementary Divisor domain, for instance by showing that $\mathcal{S}_n(R)/P$ does not satisfy Condition $(SU')_{n+1}$.

4. Hyperplane Conditions

Let R be any commutative ring. Let $f \in R[x_{11}, \dots, x_{nn}]$ be any polynomial in the indeterminates $x_{ij}, 1 \leq i, j \leq n$. Denote by $Z_f(R)$ the set of solutions to the equation $f = 0$ in R^{n^2}. (The notation $Z_f(R)$ stands for the *zeroes* of f in R^{n^2}.)

Lemma 4.1. *Let R be a commutative ring, and let n and s be positive integers. The following are equivalent:*

(a) *R satisfies Condition $H_{n,s}$.*
(b) *Given any system of s linear homogeneous polynomials $h_i \in R[x_{11}, \dots, x_{nn}]$, $i = 1, \dots, s$, we have*

$$\mathrm{GL}_n(R) \cap (\bigcap_{i=1}^{s} Z_{h_i}(R)) \neq \emptyset.$$

Moreover, R satisfies Condition $H'_{n,s}$ if and only if (b) holds with $\mathrm{GL}_n(R)$ replaced by $\mathrm{SL}_n(R)$. No ring R satisfies Condition $H_{n,s}$ or $H'_{n,s}$ when $s \geq n$.

Proof. Let $h(x_{11}, \dots, x_{nn}) = \sum a_{ij} x_{ij}$ be a linear homogeneous polynomial. Let A denote the associated matrix $(a_{ij}) \in M_n(R)$. Let $X := (X_{ij})$ be any matrix. The equivalence follows immediately from the fact that the trace of the matrix AX^t is equal to $h(X_{11}, \dots, X_{nn})$.

Consider now the polynomials $h_i := x_{1,i}$ for $i = 1, \dots, n$. It is clear that for this choice of n polynomials, $\mathrm{GL}_n(R) \cap (\bigcap_{i=1}^n Z_{h_i}(R)) = \emptyset$. Thus, no ring R can satisfy Condition $H_{n,s}$ when $s \geq n$. □

Remark 4.2 (See 4.7.) We note that if R satisfies Condition $H_{n,s}$, and I is any proper ideal of R, then R/I also satisfies Condition $H_{n,s}$. It is also true that if $T \subset R$ is a multiplicative subset, then the localization ring $T^{-1}(R)$ satisfies Condition $H_{n,s}$.

Our motivation for introducing Condition $H_{n,s}$ is the following lemma.

Lemma 4.3. *Let R be a commutative ring satisfying Condition $H_{n,n-1}$ for some $n \geq 2$. Then R is a Hermite ring.*

Proof. Let $a, b \in R$. Condition $H_{n,n-1}$ implies the existence of $V = (v_{ij}) \in \mathrm{GL}_n(R)$ satisfying the following $n-1$ hyperplane conditions: $v_{13} = \cdots = v_{1n} = 0$, and $av_{12} = bv_{11}$. Expanding the determinant of V using the first row, we find that we can write $\det(V) = v_{11}s - v_{12}t \in R^*$ for some $s, t \in R$. We conclude as in the proof of 3.1 (i) that $g := (as - bt)\det(V)^{-1}$ is such that $gv_{11} = a$ and $gv_{12} = b$, with $(v_{11}, v_{12}) = R$. □

In analogy with Proposition 3.2, we may wonder whether an Elementary Divisor ring, or even a Hermite ring, satisfies Condition $H'_{n,n-1}$ for all $n \geq 2$. Our results on this question are Proposition 4.4 below, and Proposition 4.8, which shows that an Elementary Divisor ring satisfies Condition $H'_{n,1}$ for all $n \geq 2$.

Proposition 4.4. *Any field K satisfies Condition $H_{n,n-1}$ for all $n \geq 2$.*

Proof. We thank J. Fresnel for making us aware of [7], Exer. 2.3.16, p. 112, which details a proof of the proposition under the assumption that K is infinite. The suggested proof in fact shows that the proposition holds if $|K| \geq r+1$. The key to 4.4 is the following statement, proved under the assumption that $|K| \geq r+1$ in [6], and in general in [17]: *If W is a subspace of the K-vector space $M_n(K)$ and $\dim(W) > rn$, then W contains an element of rank bigger than r.*

Indeed, let $h_i \in K[x_{11}, \ldots, x_{nn}]$, $i = 1, \ldots, s$, be any system of s linear homogeneous polynomials. Then the set $(\bigcap_{i=1}^{s} Z_{h_i}(K))$ is in fact a subspace of $M_n(K)$ of dimension at least $n^2 - s$. If this vector space does not contain any element of $\mathrm{GL}_n(K)$, then all its elements have rank at most $n-1$, and its dimension would be at most $n(n-1)$. This is a contradiction since $n(n-1) < n^2 - s$ when $s = n-1$. □

Proposition 4.5. *Let $n > s > 0$ be integers. Let R be any commutative ring. Let P be a prime ideal of R, with localization R_P. Suppose that there exists $k > 0$ such that the R_P/PR_P-vector space $(PR_P)^k/(PR_P)^{k+1}$ has dimension greater than $n-s$. Then R does not satisfy Condition $H_{n,s}$.*

Assume now that R is noetherian and that it satisfies Condition $H_{n,s}$. Then the Krull dimension of R is at most 1, and every maximal ideal M of R is such that MR_M can be generated by at most $n-s$ elements. Moreover, every maximal ideal M of R can be generated by at most $n-s+1$ elements.

Proof. Let us assume that R satisfies Condition $H_{n,s}$. Then R_P also satisfies Condition $H_{n,s}$. By hypothesis, there exist $r > n-s$ and elements $a_1, \ldots, a_r$ of $(PR_P)^k \subset R_P$ whose images in $(PR_P)^k/(PR_P)^{k+1}$ are linearly independent. Consider the following s linear homogeneous polynomials in $R_P[x_{11}, \ldots, x_{nn}]$:

$$a_1x_{11} + a_2x_{12} + \cdots + a_{n-s+1}x_{1,n-s+1},\ x_{1,n-s+2},\ \ldots,\ x_{1,n}.$$

Using Condition $H_{n,s}$, there exists a matrix $U = (u_{ij}) \in \mathrm{GL}_n(R_P)$ such that $a_1u_{11} + a_2u_{12} + \cdots + a_{n-s+1}u_{1,n-s+1} = 0$, and $u_{1,n-s+2} = \cdots = u_{1,n} = 0$. Expanding the determinant of U along the first row, we find that there exist $b_i \in R_P$, $i = 1, \ldots, n-s+1$, such that $b_1u_{11} + b_2u_{12} + \cdots + b_{n-s+1}u_{1,n-s+1}$ is a unit in R_P. In particular, there exists at least one u_{1j} with $j \leq n-s+1$ which does not belong to PR_P. It follows that $a_1u_{11} + a_2u_{12} + \cdots + a_{n-s+1}u_{1,n-s+1} = 0$ produces a non-trivial linear relation between the images of $a_1, \ldots, a_r$ in the R_P/PR_P-vector space $(PR_P)^k/(PR_P)^{k+1}$, and this is a contradiction.

Assume now that R is noetherian. To prove that $\dim(R) \leq 1$, it suffices to show that for any maximal ideal M of R, $\dim(R_M) \leq 1$. Since R_M is a noetherian local ring, the function $f(k) := \dim_{R_M/MR_M}((MR_M)^k/(MR_M)^{k+1})$ is given for k large enough by the values of a polynomial $g(k)$ of degree equal to $(\dim(R_M) - 1)$. In particular, if $\dim(R_M) > 1$, there always exists a value k such that $f(k) > n - s$. This implies by our earlier considerations that Condition $H_{n,s}$ cannot be satisfied, and this is a contradiction. Assume now that $\dim(R_M) \leq 1$, and that MR_M can be minimally generated by r elements $a_1, \dots, a_r$. Then the images of $a_1, \dots, a_r$ in $MR_M/(MR_M)^2$ are linearly independent. It follows that $r \leq n - s$. The statement regarding the number of generators of M follows from a strengthening of a theorem of Cohen, as in [11], Theorem 3, and the remark on page 383. □

Let R be any commutative ring. Let $X_n := ((x_{ij}))_{1 \leq i,j \leq n}$ denote the square matrix in the indeterminates $x_{ij}, 1 \leq i, j \leq n$. Set

$$d_n := \det(X_n) \in R[x_{11}, \dots, x_{nn}].$$

For $\mu \in R$, denote by $Z_{d_n - \mu}(R)$ the set of solutions to the equation $d_n - \mu = 0$ in R^{n^2}. Clearly, $\mathrm{SL}_n(R) = Z_{d_n - 1}(R)$.

Definition 4.6 Let n and s be positive integers. We say that a commutative ring R *satisfies Condition $J_{n,s}$* if, given any s linear homogeneous polynomials $h_i(x_{11}, \dots, x_{nn})$, $i = 1, \dots, s$, and $\nu_1, \dots, \nu_s \in R$ such that $\cap_{i=1}^s Z_{h_i - \nu_i}(R) \neq \emptyset$, then for all $\mu \in R$, we have

$$Z_{d_n - \mu}(R) \cap (\cap_{i=1}^s Z_{h_i - \nu_i}(R)) \neq \emptyset.$$

In other words, stratify $M_n(R)$ using the determinant, so that

$$M_n(R) = \sqcup_{\mu \in R} Z_{d_n - \mu}(R).$$

When R satisfies Condition $J_{n,s}$, any linear subvariety $\cap_{i=1}^s Z_{h_i - \nu_i}(R)$ in $R^{n^2} = M_n(R)$ which is not empty meets every stratum of the stratification. As with Condition $H_{n,s}$, no ring R satisfies Condition $J_{n,s}$ with $s \geq n$.

Lemma 4.7. *Let R be a commutative ring which satisfies Condition $J_{n,s}$.*

(a) *Let I be any proper ideal of R. Then R/I also satisfies Condition $J_{n,s}$.*
(b) *Let $T \subset R$ be a multiplicative subset. Then the localization ring $T^{-1}(R)$ also satisfies Condition $J_{n,s}$.*

Proof. (a) Let $\overline{\mu} \in R/I$. Let $\overline{h}_i \in (R/I)[x_{11}, \dots, x_{nn}]$ and $\overline{\nu}_i \in R/I$, $i = 1, \dots, s$, be such that $\cap_{i=1}^s Z_{\overline{h}_i - \overline{\nu}_i}(R/I) \neq \emptyset$. Choose a point $(\overline{r}_{11}, \dots, \overline{r}_{nn})$ in this intersection. Choose a lift $(r_{11}, \dots, r_{nn}) \in R$ of $(\overline{r}_{11}, \dots, \overline{r}_{nn})$, and choose a lift $h_i \in R[x_{11}, \dots, x_{nn}]$ of $\overline{h}_i$ for each $i = 1, \dots, s$. Set $\nu_i := h_i(r_{11}, \dots, r_{nn})$. Then $(r_{11}, \dots, r_{nn})$ belongs to $\cap_{i=1}^s Z_{h_i - \nu_i}(R)$. Choose a lift $\mu \in R$ of $\overline{\mu}$. Apply Condition $J_{n,s}$ on R to find $U = (u_{ij})$ of determinant μ contained in $\cap_{i=1}^s Z_{h_i - \nu_i}(R)$. Then the class of U is $(R/I)^{n^2}$ is the desired element in $Z_{d_n - \overline{\mu}}(R/I) \cap (\cap_{i=1}^s Z_{\overline{h}_i - \overline{\nu}_i}(R/I))$.

(b) Without loss of generality, we may assume that we are given $h_i \in R[x_{11}, \dots, x_{nn}]$ and $\nu_i \in R$ such that $\cap_{i=1}^s Z_{h_i - \nu_i}(T^{-1}(R))$ contains a point $(r_{11}/t, \dots, r_{nn}/t)$. Let $\mu \in T^{-1}(R)$, which we write as $\mu = \mu_0/t_0$, with $\mu_0 \in R$ and $t_0 \in T$. Then $\cap_{i=1}^s Z_{h_i - t_0 t \nu_i}(R)$ contains $(t_0 r_{11}, \dots, t_0 r_{nn})$. Using Condition $J_{n,s}$ on R, we find $U = (u_{ij})$ of determinant $\mu_0 t^n t_0^{n-1}$ contained in $\cap_{i=1}^s Z_{h_i - t_0 t \nu_i}(R)$. Then $(u_{ij}/t_0 t)$ has determinant μ_0/t_0 and is contained in $\cap_{i=1}^s Z_{h_i - \nu_i}(T^{-1}(R))$, as desired. □

The key ideas in the proof of the following proposition are due to Robert Varley.

Proposition 4.8. *Let R be an Elementary Divisor ring. Then R satisfies Condition $J_{n,1}$ for all $n > 1$.*

Proof. Fix $h(x_{11}, \dots, x_{nn}) = \sum a_{ij} x_{ij} \in R[x_{11}, \dots, x_{nn}]$, and $\nu, \mu \in R$. Assume that $Z_{h-\nu}(R) \neq \emptyset$. Then any generator of the ideal $(a_{11}, \dots, a_{nn})$ divides ν. Write $B := (a_{ij}) \in M_n(R)$, and denote by A the transpose of B. We need to show the existence of $U \in M_n(R)$ such that $\det(U) = \mu$ and such that AU has trace $\mathrm{Tr}(AU) = \nu$.

Let P and Q in $\mathrm{GL}_n(R)$ be such that $PAQ = \mathrm{diag}(d_1, \dots, d_n)$ and d_i divides d_{i+1} for all $i = 1, \dots, n-1$. Then $(a_{11}, \dots, a_{nn}) = (d_1)$. Multiply both sides of $PAQ = \mathrm{diag}(d_1, \dots, d_n)$ on the right by $D := \mathrm{diag}(1, \dots, 1, \mu \det(P)^{-1} \det(Q)^{-1})$. Write $\nu = d_1 s$ with $s \in R$, and add s times the first column of $PAQD$ to its last column. Permute the first row with the last row. If n is odd, permute the first and second column, then the third and forth column, etc, to obtain a matrix with $(0, \dots, 0, \nu)$ on the diagonal. If n is even, permute the second and third column, then the forth and fifth column, etc, to again obtain a matrix with $(0, \dots, 0, \nu)$ on the diagonal. We have thus proved the existence of P' and Q' in $\mathrm{GL}_n(R)$ such that $P'PAQDQ'$ is a matrix with $(0, \dots, 0, \nu)$ on the diagonal, and $\det(P'PDQQ') = \pm\mu$. Multiplying both sides by $\mathrm{diag}(-1, 1, \dots, 1)$ if necessary, we may assume that $\det(P'PDQQ') = \mu$. By construction, $\mathrm{Tr}((P'P)(AQDQ')) = \nu$, so that $\mathrm{Tr}(AQDQ'P'P) = \nu$. We can therefore choose $U := QDQ'P'P$ to satisfy the conditions of the proposition. ☐

Remark 4.9 It is natural to wonder whether an Elementary Divisor ring satisfies Condition $J_{n,n-1}$ for all[2] $n > 1$. Here we note that without any assumptions on the commutative ring R, it is true that a $(n \times n)$-matrix with $n-1$ prescribed entries can always be completed into a matrix in $M_n(R)$ of determinant μ, for any $\mu \in R$. Said more precisely, choose the polynomials h_ℓ to be distinct monomials, say $h_\ell := x_{i_\ell j_\ell}$ for $\ell = 1, \dots, n-1$, and let $\nu_1, \dots, \nu_{n-1} \in R$. Then for any $\mu \in R$, $Z_{d_n-\mu}(R) \cap (\cap_{i=1}^{n-1} Z_{h_i-\nu_i}(R)) \neq \emptyset$. (To prove this fact, it suffices to show that one can reduce to the case where all prescribed entries are above the main diagonal. In such a case, we set all but one element on the diagonal to be 1, and the remaining one to be μ. All other coefficients are set to 0.) When R is a field, it is also possible in addition to prescribe the characteristic polynomial of the matrix ([18], Theorem 3).

Let us also note the following known related result. Assume that $R = \mathbb{Z}$, and let $r \geq 1$. Pick polynomials $h_\ell \in \mathbb{Z}[x_{ij}, i \neq j, 1 \leq i, j \leq n]$, $\ell = 1, \dots, r$, and integers $\nu_1, \dots, \nu_r$. If $\mathcal{H} := \cap_{\ell=1}^r Z_{h_\ell-\nu_\ell}(\mathbb{Z}) \neq \emptyset$, then either $Z_{d_n-1}(\mathbb{Z}) \cap \mathcal{H} \neq \emptyset$, or $Z_{d_n+1}(\mathbb{Z}) \cap \mathcal{H} \neq \emptyset$ ([5], Theorem 1).

Assume that $R = \mathbb{Z}$. The set $Z_{d_n-\mu}(\mathbb{Z}) \cap (\cap_{i=1}^s Z_{h_i-\nu_i}(\mathbb{Z}))$ appearing in Condition $J_{n,n-1}$ is nothing but the set of integer points on the affine algebraic variety defined by the ideal $(d_n - \mu, h_i - \nu_i, i = 1, \dots, n-1)$. When $n = 3$, this variety can be defined over $\mathbb{Q}$ by a single polynomial of degree 3 in 7 variables. Many results in the literature pertain to the existence of infinitely many integer points on a hypersurface of degree 3 (see, e.g., [3], Introduction), but none of these results seem to be applicable to Condition $J_{3,2}$.

Example 4.10 We now use Lemma 4.3 to construct examples of new Bézout rings which satisfy Condition $H'_{n,n-1}$ for some $n > 1$. Let R be any commutative ring, and fix $n > 1$. For ease of notation, let us note here that the coefficients of a set of $n-1$

[2] T. Shifrin and R. Varley have proved that a field satisfies Condition $H'_{n,n-1}$ for all $n > 1$. J. Fresnel has shown that an Euclidean domain satisfies Condition $J_{n,n-1}$ for all $n > 1$.

homogeneous linear polynomials in $R[x_{11},\dots,x_{nn}]$ determine a $n^2\times(n-1)$ matrix A with entries in R. Conversely, such a matrix A determines $n-1$ linear homogeneous polynomials, namely the $n-1$ entries of the matrix $(x_{11},\dots,x_{nn})A$. Let $X=(x_{ij})$ denote the square $n\times n$-matrix in the indeterminates x_{ij}, $1\le i,j\le n$.

For each matrix $A\in M_{n^2,n-1}(R)$, consider the subset $I(A)$ of $R[x_{11},\dots,x_{nn}]$ consisting of $\det(X)-1$ and of the $n-1$ homogeneous linear polynomials obtained from A. Let $<I(A)>$ denote the ideal of $R[x_{11},\dots,x_{nn}]$ generated by $I(A)$. We claim that $<I(A)>\neq R[x_{11},\dots,x_{nn}]$. Indeed, choose a maximal ideal M of R, and let $K:=R/M$. Let $I_M=\{\det(X)-1,h_1,\dots,h_{n-1}\}$ denote the subset of $K[x_{11},\dots,x_{nn}]$ consisting of the images modulo M of the elements of $I(A)$. Proposition 4.4 shows that the intersection $\mathrm{GL}_n(K)\cap(\cap_{i=1}^{n-1}Z_{h_i}(K))$ is not empty. Let C be a matrix in this intersection, and let $\det(C)=c$. It follows that over the field $L:=K(\sqrt[n]{c})$, the matrix $\frac{1}{\sqrt[n]{c}}C$ belongs to $\mathrm{SL}_n(L)\cap(\cap_{i=1}^{n-1}Z_{h_i}(L))$. Therefore, the ideal $<I_M>$ is a proper ideal of $K[x_{11},\dots,x_{nn}]$, and $<I(A)>\neq R[x_{11},\dots,x_{nn}]$.

Consider the set $\mathcal{I}$ of all subsets $I(A)$, $A\in M_{n^2,n-1}(R)$, such that there exists no homomorphism of R-algebras between $R[x_{11},\dots,x_{nn}]/<I(A)>$ and R. For each subset $I=I(A)\in\mathcal{I}$, let $\mathbf{x}^I$ denote the set of n^2 variables labeled $x^I_{11},\dots,x^I_{nn}$, and let $(\mathbf{x}^I)$ denote the associated square matrix. Let $I(A,\mathbf{x}^I)$ be the subset of $R[\mathbf{x}^I]$ consisting of $\det((\mathbf{x}^I))-1$ and of the $n-1$ homogeneous linear polynomials obtained from A. It is not difficult to check that the ideal $<I(A,\mathbf{x}^I),I\in\mathcal{I}>$ is a proper ideal of $R[\mathbf{x}^I,I\in\mathcal{I}]$, so we can define the quotient ring

$$h_n(R):=R[\mathbf{x}^I,I\in\mathcal{I}]/<I(A,\mathbf{x}^I),I\in\mathcal{I}>.$$

Note that if R satisfies Condition $H'_{n,n-1}$, then $\mathcal{I}=\emptyset$, and $h_n(R)=R$. It is clear that we have a natural morphism of R-algebras $R\to h_n(R)$. By construction, given any matrix $B\in M_{n^2,n-1}(R)$, there exists $U\in\mathrm{SL}_n(h_n(R))$ which also belongs to the zero-sets with coefficients in $h_n(R)$ of the $n-1$ homogeneous polynomials defined by B. Indeed, simply take $U:=(\text{class of } x^{I(B)}_{ij} \text{ in } h_n(R))_{1\le i,j\le n}$.

Let $h^{(1)}_n(R):=h_n(R)$, and for each $i\in\mathbb{N}$, we set $h^{(i)}_n(R):=h_n(h^{(i-1)}_n(R))$. Finally, we let

$$\mathcal{H}_n(R):=\varinjlim_i h^{(i)}_n(R).$$

Let $C\in M_{n^2,n-1}(\mathcal{H}_n(R))$. Then the finitely many coefficients of C all lie in a single ring $h^{(i)}_n(R)$ for some $i>0$. By construction, there exist $U:=(u_{ij})\in\mathrm{SL}_n(h^{(i)}_n(R))$ which also belongs to the zero-sets with coefficients in $\mathcal{H}_n(R)$ of the $n-1$ homogeneous polynomials defined by C. It follows that $\mathcal{H}_n(R)$ satisfies Condition $H'_{n,n-1}$. Thus, it satisfies Condition $H_{n,n-1}$ and 4.3 implies that R is a Hermite ring.

Given any prime ideal P of $\mathcal{H}_n(R)$, the quotient $\mathcal{H}_n(R)/P$ is also a $H'_{n,n-1}$-domain and, thus, a Bézout domain (4.3). It is natural to wonder whether one could show for a well-chosen ring R that one such domain is not an Elementary Divisor domain, for instance by showing that $\mathcal{H}_n(R)/P$ does not satisfy Condition $H_{n+1,1}$ and use 4.8.

In the simplest case where $n=2$, the relationships between the conditions introduced in this paper can be summarized as follows.

Proposition 4.11. *Let R be any commutative ring. Consider the following properties:*

a) *R is an Elementary Divisor ring.*

b) *R satisfies Condition* $J_{2,1}$.
c') *R satisfies Condition* $H'_{2,1}$.
d') *R satisfies Condition* $(SU')_2$.
c) *R satisfies Condition* $H_{2,1}$.
d) *R satisfies Condition* $(SU)_2$.
e) *R is a Hermite ring.*

Then a) $\Longrightarrow$ *b)* $\Longrightarrow$ *c')* $\Longleftrightarrow$ *d')* $\Longrightarrow$ *c)* $\Longleftrightarrow$ *d)* $\Longrightarrow$ *e).*

Proof. The implication $a) \Longrightarrow b)$ is proved in Proposition 4.8. The implications $b) \Longrightarrow c')$, $c') \Longrightarrow c)$, and $d') \Longrightarrow d)$, are obvious. The implication $d) \Longrightarrow e)$ is proved in 3.1.

Proof of $c') \Longleftrightarrow d')$ and $c) \Longleftrightarrow d)$. Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Consider the polynomial $h := cX - aY + dU - bV$. Condition $H'_{2,1}$ implies that $\mathrm{SL}_2(R) \cap Z_h(R) \neq \emptyset$. Hence, we can find $x, y, u, v \in R$ such that $xv - yu = 1$ and such that

$$A \begin{pmatrix} x & y \\ u & v \end{pmatrix} =: S$$

with S symmetric, since the condition $h(x, y, u, v) = cx - ay + du - bv = 0$ implies that $ay + bv = cx + du$. This shows that $c') \Longrightarrow d')$. The proof of $c) \Longrightarrow d)$ is similar. The proofs of the converses are also similar and left to the reader. □

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