Math 2250
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First Lecture

Homework (please staple your homework if you turn in more than one page.)

Due Wednesday in class; 8, in section 4.5, translation only, as we did an example in class).
Due Thursday in class; sec 1.1, 1 - 21 odd, 55, 62 a. (this is review for you).

Let us start by discussing what Calculus can do for you. First, a little bit a review. What is a function?

Example $f(x) = \sin(x)$, domain of $f(x)$: all real numbers (also written as $-\infty < x < \infty$).

This is an example of a function given by a mathematical formula. But in real life, you will first encounter functions given in English sentences:
Example variable: t = time, in hours.
function: $f(t)$ = temperature in the room at time t, in ° Celcius.

We want to monitor the temperature over a 2-day period starting at 10 am today. The best way to write the domain of our function $f(t)$ in this case is to choose $t = 0$ at 10 a.m. today; the domain is then $0 \leq t \leq 48$. But choosing when $t = 0$ is our choice. For instance, had we chosen to set $t = 0$ at 9 a.m. today, the domain would be $1 \leq t \leq 49$.

Example t = time, in seconds.
$f(t)$ = distance from the space shuttle to the center of the earth, in km.
What is wrong with the above description of $f(t)$? The mathematical notation indicates that $f(t)$ is a function of t, but no time appear on the right hand side of the equal sign!) So let us restate the function as:
$f(t)$ = distance from the space shuttle to the center of the earth at time t, in km.
This is a 5-day mission, with take off tomorrow at 5 a.m. We choose to set $t = 0$ to be at 5 am tomorrow. So:
domain: $0 \leq t \leq 5 \cdot 24 \cdot 60 \cdot 60$ (since we measure time in seconds).
(Having set when $t = 0$, and assuming that it is now 10 am, we find that the time now is $t = -19 \cdot 3600$ seconds.)

Real Life Problems

In science, or in any "real life" problem, using mathematical tools always involve a 2-step process:

1. Translate the "real life" problem given in "English sentences" into a mathematical expression.
2. Use mathematical tools to help solve the given problem.

An optimization problem (also max/min problem) is a story problem (a "real life" problem) where we want to optimize something:

e.g.: maximize: profit, weight on a beam before breaking point, or
minimize: cost, amount of materials needed to build something, time to travel somewhere.

Example: A farmer has 200 yards of fence to be used in constructing three sides of a rectangular pen. An existing long straight wall will be used for the 4th side. What dimensions of the pen will maximize the area of the pen? What is the maximal area of such a pen?

Solution: (1) Translation into a pure math problem

First, draw a picture and label the various quantities involved.
Quantity to be maximized: area of the pen. Let us denote it in mathematical notation by the letter A.
Clearly, $A = xy$, so the area A seems to be a function of 2 variables. In this course we will try to stick with functions of one variable. Functions of several variables are studied in third semester calculus.
But note that the problem says that we have only 200 yards of fence: So $x + y + x = 200$. This relation[1] between x and y allows us to solve for y in terms of x (or for x in terms of y if we needed to, or chose to).

[1]The statement of the problem would only allow us at first to state that $2x + y \leq 200$, since it is not stated in the problem that the farmer has to use all the fence that he has. On the other hand, we deduce logically that since we are asked to maximize the area, the farmer will in the end need to use all the fence available, and thus it is correct to assume that $2x + y = 200$.

$$x + y + x = 200 \Leftrightarrow 2x + y = 200 \Leftrightarrow y = 200 - 2x.$$

Hence: $A = xy = x(200 - 2x)$.

We are now ready to translate this problem into a "pure math" problem.

Variable: $x =$ side of the pen. *wrong*
$x =$ length of the side of the pen. *better, but still not precise enough*
$x =$ length of one of the two sides of the pen that touches the wall.
Almost right: there is one important piece of information that you still need to state: if I tell you that the side has length 2, you would immediately ask: 2 what? You need to choose units, and to state your choice of units. In this problem, the only units of length used are "yards", so it is natural for us to choose "yards" as our units.
$x =$ length of one of the two sides of the pen that touches the wall (in yards). *correct statement*

Function that matters in this problem:
$A(x) =$ area of the pen.
wrong: on the left hand side $A(x)$ is a mathematical notation for a function of x. On the right hand side, no x appear.
$A(x) =$ area of the pen when the sides of the pen that touch the wall have length x.
Much better, but still, you are missing an important piece of information, the units. Since we chose "yards" as units of length, it is natural for us to choose square yards as units of surface.
$A(x) =$ area of the pen when the sides of the pen that touch the wall have length x (in square yards).
A very common mistake: instead of $A(x) =$ area of the pen..., some students will state $A(x) =$ maximum area of the pen... The area is a function that depends on x, while the maximum area is only a single number, namely, the maximum value taken by the function $A(x)$.

Mathematical expression for $A(x)$: $A(x) = x(200 - 2x)$.

Domain: In this problem, the variable x is what we control: we choose the length of the side of the pen. Once this choice is made, we can compute the area of the pen. The domain of the function "area" is the set of all the possible values of the variable x that "make sense":

a) x is the length of the side, so we clearly can state that we want $x \geq 0$.

b) Similarly, we want $y \geq 0$. But $y = 200 - 2x \geq 0$. So $200 \geq 2x$ and hence $100 \geq x$.

Hence, the values that "make sense" in this problem are $0 \leq x \leq 100$. So, the domain to be considered in this problem: x in $[0, 100]$. We have now translated our initial real life problem into a purely mathematical problem:

Pure Math Problem: We now want to forget any reference to 'real life'. The question *What is the maximal area of such a pen?* can be phrased in purely mathematical terms:
Find the maximal value taken by the function $A(x) = x(200 - 2x)$ on the interval $[0, 100]$.
The question *What dimensions of the pen will maximize the area of the pen?* can be solved after solving the following pure math problem:
Find the value of x in the interval $[0, 100]$, say $x = c$, such that $A(c)$ is the maximum value taken by the function $A(x) = x(200 - 2x)$ on the interval $[0, 100]$. Once c is obtained (c is the length of the sides that touch the wall in the pen of maximal area, the other 'dimension', the length of the side parallel to the wall, is obtained using the relation $2x + y = 200$.
Be careful, the answer to the first question above is a number of square yards, while the answer to the second question is a number of yards.

As you may have noted, the translation that we did above did not use any mathematical tools that you have not yet encountered. However, to completely solve the purely mathematical problem stated above, we need to use new mathematical tools, and I will spend several weeks of this semester teaching you these tools.
Every time that I assign a min/max problem as a homework problem and tell you to only translate it into a pure math problem, I expect you to proceed as above. Namely, you should state in English the variable that you chose, which function matters in the problem, and the mathematical expression that you found for that function. Moreover, you should state the domain to consider, and the pure math problem that needs to be solved, as we did in the above example. At least one min/max problem will appear on the first two tests and on the final.