Review of John Tate's 1966 Bourbaki Seminar

Math Reviews has not yet published a review of one of the most important papers in number theory, Tate's 1966 Bourbaki seminar. I sent in March 2005 the unsolicited 'free' review below to the editor of Math Reviews, who decided not to publish it.

Tate's paper reviewed below first appeared in:

MR0205779 (34 #5605) *Séminaire Bourbaki: Vol. 1965/1966, Exposés 295–312. (French) W. A. Benjamin, Inc., New York-Amsterdam 1966 viii+293 pp.*

and was soon afterwards reprinted in:

MR0241437 (39 #2777) *Dix exposés sur la cohomologie des schémas. (French) Advanced Studies in Pure Mathematics, Vol. 3 North-Holland Publishing Co., Amsterdam; Masson & Cie, Editeur, Paris 1968 vi+386 pp.*

In 1995, the Bourbaki seminar volume was reprinted.

MR1610977 *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki, Vol. 9, Exp. No. 306, 415–440, Soc. Math. France, Paris, 1995.*

Neither of these three occurences of the paper by Tate has been reviewed by Math Reviews (as of May 2005).

REVIEW

The author introduces the full Birch and Swinnerton-Dyer conjecture for abelian varieties over a number field (Conjectures A and B), including the precise form of the leading term of the L-function. He then provides a sketch of the proof that the truth of Conjecture B depends only on the K-isogeny class of A/K. A more detailed proof later appeared in I.7.3 (see also III.9.8) of

MR0881804 (88e:14028) *Milne, J. S.(1-MI) Arithmetic duality theorems. Perspectives in Mathematics, 1. Academic Press, Inc., Boston, MA, 1986. x+421 pp.*

The author then notes that an analogue conjecture for abelian varieties over function fields can be stated. Motivated by this latter conjecture, he introduces a conjecture for a surface X over a finite field k of characteristic p. Let $P_2(t)$ denote the characteristic polynomial of Frobenius acting on the etale cohomology group $H^2(\overline{X}, \mathbb{Q}_\ell)$. The multiplicity of the root $t = |k|^{-1}$ of this polynomial is conjectured by Tate to be the rank of the Néron-Severi group of the surface. In a refinement of this conjecture jointly introduced with Michael Artin, the leading term of $P_2(t)$ is expressed in terms of global invariants attached to X (Artin-Tate Conjecture C). Given a surface X/k with a morphism $f : X \to V$ to a smooth projective curve V/k such that the generic fiber of f is smooth and geometrically connected, Artin and Tate predict in a last conjecture (Conjecture d) that Conjecture B for the jacobian of the generic fiber of f is equivalent to Conjecture C for the surface X.

In the second part of this paper, the author reports on joint results with Artin on Conjecture C. In particular, they prove that the Brauer group $\mathrm{Br}(X)$(non-p) is endowed with a canonical skew-symmetric form whose kernel consists exactly of the divisible elements. They prove also that if the ℓ-part of $\mathrm{Br}(X)$ is finite for some prime $\ell \neq p$ then Conjecture C is true up to powers of p. This statement was generalized by Milne

MR0414558 (54 #2659) *Milne, J. S., On a conjecture of Artin and Tate. Ann. of Math. (2) 102 (1975), no. 3, 517–533.*,

who completed the proof of the statement: if the ℓ-part of $\mathrm{Br}(X)$ is finite for some prime ℓ, then Conjecture C of Artin and Tate holds.

The analogous statement that if the ℓ-part of the Shafarevich-Tate group Ш(A) of an abelian variety over a function field is finite, then Conjecture B holds, is proved in

MR2000469 (2004h:11058) *Kato, Kazuya; Trihan, Fabien, On the conjectures of Birch and Swinnerton-Dyer in characteristic* $p > 0$. *Invent. Math. 153 (2003), no. 3, 537–592.*

These two results imply that conjecture d) holds, as noted in

Liu, Qing; Lorenzini, Dino; Raynaud, Michel, On the Brauer group of a surface, Invent. Math. March 2005.

The precise relationship between the Shafarevich-Tate group of the jacobian of the generic fiber of $f : X \to V$ and $\mathrm{Br}(X)$ is described in 4.3 of

MR2092767 *Liu, Qing; Lorenzini, Dino; Raynaud, Michel, Néron models, Lie algebras, and reduction of curves of genus one. Invent. Math. 157 (2004), no. 3, 455–518*

completing the result of

MR0528839 (80h:14010) *Gordon, W. J. Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer. Compositio Math. 38 (1979), no. 2, 163–199.*