

TORSION AND EXCEPTIONAL UNITS

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APPENDIX A. DATA

Let K/\mathbb{Q} be a number field. Let $N \geq 5$ be prime. We say that E/K is an N -special elliptic curve if $E(K)$ contains a point of order N and the Tamagawa number $c(E/K)$ is not divisible by N . We list below the results of our search for N -special elliptic curves E/K when $[K : \mathbb{Q}] = d$ and $8 \leq d \leq 12$. When $d \leq 7$, the data appears in [4]. Our search mostly used the algorithm 4.8 proposed in [4].

A.1 Number fields of degree 8. We include below the octic fields K where we found at least one N -special elliptic curve E/K with $N = 17, 19$, or 23 . When $N = 17$, we do not include the fields K that contain the quartic field of discriminant 725 since this field appears in the table of quartic fields in [4], 1.4. The notation is as in 1.3 and 7.2 of [4].

N	field K (degree 8 and rank 3)	$L(K)$	$M(K)$	$ex(K)$	$discr(K)$
11(12), 13(2), 17(2)*	$x^8 - 2x^7 + 4x^5 - 4x^4 + 3x^2 - 2x + 1$	16	≥ 12	438	1257728
17(2)	$\mathbb{Q}(\zeta_{15})$	16	≥ 10	440	1265625
11(2), 13, 23	$x^8 - x^7 + 2x^6 - 3x^5 + 3x^4 - 3x^3 + 3x^2 - 2x + 1$	17	≥ 11	432	1282789
11(2), 17, 19	$x^8 - x^7 + x^6 + x^4 - 2x^3 + 3x^2 - 3x + 1$	13	≥ 10	422	1342413
11(2), 17	$x^8 - 2x^7 + x^6 + 2x^5 - 3x^4 - 2x^3 + x^2 + 2x + 1$	13	≥ 8	374	1513728
11(2), 13, 17(2)	$x^8 - 2x^7 + 3x^5 - x^4 - 3x^3 + 2x + 1$	11	≥ 8	354	1578125

N	field K (degree 8 and rank 4)	rk	$ex(K)$	$discr(K)$
17	$x^8 - x^7 - 2x^6 + x^5 + x^3 + 2x^2 - 1$	4	1116	-4297507
11(6), 13, 17, 19	$x^8 - 3x^7 + 5x^6 - 6x^5 + 4x^4 - x^3 - x^2 + x - 1$	4	1062	-4542739
11(2), 13, 17	$x^8 - x^6 - 2x^5 + x^4 + x^2 + 2x - 1$	4	1044	-4663051
11(2), 17	$x^8 - x^7 + 2x^6 - x^5 + x^3 - 2x^2 + 2x - 1$	4	1020	-4785667
13, 19	$x^8 + 3x^6 - x^5 + 2x^4 - 3x^3 - 2x + 1$	4	1008	-4858379
11(4), 13, 19	$x^8 - x^7 - x^6 + x^5 + x^2 - x - 1$	4	966	-5118587
11(2), 17	$x^8 - x^7 + x^6 - 2x^4 + x^3 - x^2 - x + 1$	4	930	-5204491
11(2), 17	$x^8 - 2x^7 + 2x^5 + x^4 - x^3 - 2x^2 + x + 1$	4	936	-5233147
11(2), 13, 17	$x^8 - x^7 - x^6 + 3x^5 - x^4 - 3x^3 + 2x^2 - 1$	4	966	-5272027

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N	field K (degree 8 and rank 4, continued)	rk	$ex(K)$	$discr(K)$
17	$x^8 - x^6 - x^5 + x^4 + 2x^3 - 2x - 1$	4	912	-5346947
11(2), 13, 17	$x^8 - 2x^7 + x^6 + x^5 - 2x^4 + 2x^3 + x^2 - 2x - 1$	4	810	-6483187
11(2), 17	$x^8 - 2x^7 + 2x^6 - 3x^4 + 4x^3 - x^2 - x + 1$	4	726	-7245127
11(2), 17	$x^8 - x^7 - x^6 + 4x^5 - x^4 - 3x^3 + 3x^2 - 1$	4	684	-7751827

The field of prime discriminant -4542739 in the above table is the first field in our data where four different raw form equations have a point (r, s) with r, s exceptional units.

N	field K (degree 8 and rank 5)	rk	$ex(K)$	$discr(K)$
11(2), 13, 17	$x^8 - 5x^6 - x^5 + 7x^4 + 4x^3 - 2x^2 - 4x - 1$	5	2736	16324589
11(6), 19	$x^8 - 3x^6 - x^3 + 3x^2 - 1$	5	2784	16526789
11(6), 17	$x^8 - 2x^7 + x^5 - x^3 + x^2 + 2x - 1$	5	2652	17129069
11(4), 13(3), 17	$x^8 - 2x^7 + x^6 + x^5 - 4x^4 + 3x^3 - 2x + 1$	5	2520	17949581
11(4)*, 17(2)**	$x^8 - 3x^6 + 3x^4 - 3x^2 + 1$	5	2430	19360000
11(4), 13, 17	$x^8 - x^7 + 3x^5 - 3x^4 + 2x^2 - 2x - 1$	5	2268	20493125
11(2), 23	$x^8 - 2x^7 + x^6 + 3x^5 - 4x^4 + x^3 + 2x^2 - 2x - 1$	5	2160	22131469
17	$x^8 - x^5 + x^4 + 3x^3 - 3x^2 - x + 1$	5	2022	23511413
11(2), 17	$x^8 - 3x^7 + 3x^6 - x^5 - 3x^4 + 6x^3 - 5x^2 + 4x - 1$	5	1932	25338125
11(4), 17	$x^8 - x^7 - x^6 - 2x^4 + 6x^3 - 5x + 1$	5	1836	26393125
17	$x^8 + x^6 - 2x^4 - 2x^2 + 1$	5	1770	28047616
13, 19 (2)	$x^8 - 2x^7 - x^6 + 8x^5 - 8x^4 - x^3 + 7x^2 - 6x + 1$	5	1692	30348397
19	$x^8 - x^7 - 2x^6 + 5x^5 - 2x^4 - 4x^3 + 4x^2 - x - 1$	5	1422	38169253
13, 17	$x^8 - 3x^7 + 4x^6 - 4x^5 + 3x^3 - 5x^2 + 4x - 1$	5	1182	45393125

We did not find examples with $N \geq 17$ when the rank of K is 6 or 7 (and K does not contain the quartic field of discriminant 725). We include the following examples for information only. The analytic rank of $X_1(11)$ over $\mathbb{Q}(\zeta_{17})^+$ is 5.

N	field K (degree 8)	rk	$ex(K)$	$discr(K)$
13(2)**	$x^8 - 4x^7 + 14x^5 - 8x^4 - 12x^3 + 7x^2 + 2x - 1$	7	15804	282300416
11(24)	$\mathbb{Q}(\zeta_{17})^+$	7	11700	$17^7 = 410338673$
11(2), 13**	$x^8 - 8x^6 - 2x^5 + 19x^4 + 7x^3 - 13x^2 - 4x + 1$	7	8700	707295133

In view of this data, it is natural to wonder whether there are only finitely many octic fields with an N -special elliptic curve E/K with $N \geq 17$ and which do not contain the quartic field of discriminant 725 when a 17-special curve exists.

A.2 Number fields of degree 9.

We include below the nonic fields K where we found at least one N -special elliptic curve E/K with $N = 17, 19, 23, 29, \text{ or } 31$.

N	field K (degree 9 and rank 4)	rk	$ex(K)$	$discr(K)$
11(2), 23	$x^9 - 2x^8 + 2x^7 - 3x^6 + 4x^5 - 5x^4 + 5x^3 - 5x^2 + 3x - 1$	4	1266	32031161
11(8), 23*	$x^9 - 2x^8 + 2x^7 - 2x^5 + 2x^4 - x + 1$	4	1206	33860761
11(6), 13, 19	$x^9 - x^8 - x^7 + 2x^6 - 2x^4 + x^2 - 1$	4	1194	34349041
11(2), 17	$x^9 - x^8 + x^7 + x^6 - x^4 + 2x^3 - 3x^2 + 2x - 1$	4	1164	34590113
11(4), 17	$x^9 - x^8 + 3x^6 - 3x^5 + 3x^3 - 2x^2 - x + 1$	4	1158	35678113
11(2), 17	$x^9 - x^8 + x^6 - x^4 + x^3 + x^2 - 2x + 1$	4	1134	36722413
13, 19	$x^9 - 2x^8 + x^7 - x^6 + x^5 + x^4 + x^2 - 2x + 1$	4	1116	37009129
11(2), 17	$x^9 - x^8 - x^6 - x^5 + 2x^4 + x^2 + x - 1$	4	1104	37354501
11(4), 13(2), 19	$x^9 - x^8 + 3x^7 - 3x^6 + 4x^5 - 4x^4 + 2x^3 - x^2 - x + 1$	4	1092	37732753
13, 17	$x^9 - 2x^8 + 4x^7 - 6x^6 + 7x^5 - 7x^4 + 7x^3 - 5x^2 + 3x - 1$	4	1068	38118173
11(4), 17	$x^9 - x^8 - x^7 + x^6 + x^5 - x^4 - 2x^3 + 2x + 1$	4	1098	38600453
17	$x^9 - 2x^8 + 3x^7 - 4x^6 + 5x^5 - 5x^4 + 4x^3 - 3x^2 + x - 1$	4	1038	39724513
13, 17	$x^9 - 2x^8 + 2x^7 + x^6 - 3x^5 + 2x^4 + x^3 - 2x^2 + 1$	4	1038	40756753
11(4), 13, 17	$x^9 - x^8 - x^7 + 2x^6 + x^5 - 3x^4 + x^3 + 2x^2 - 2x + 1$	4	996	42156761
11(2), 13, 17	$x^9 - x^8 + x^7 + 2x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$	4	990	42173713
11(4), 13, 17	$x^9 - x^8 - x^7 + 2x^3 + 2x^2 - x - 1$	4	996	42818653
11(4), 17	$x^9 + 2x^7 - 2x^6 + 2x^5 - 3x^4 + 2x^3 - 3x^2 + x - 1$	4	960	43552633
13, 17	$x^9 - x^8 + x^7 - 3x^6 + x^5 - x^4 + 3x^3 + x - 1$	4	978	43798753
11(2), 17	$x^9 - x^8 + 2x^7 - x^6 - x^5 + 3x^4 - 4x^3 + 4x^2 - 3x + 1$	4	924	45175393
11(2), 13, 17	$x^9 - 3x^7 - 2x^6 + 3x^5 + 3x^4 - x^2 - x - 1$	4	930	46002241
11(2), 13, 17	$x^9 - 2x^7 - x^6 + x^5 + 2x^4 + x^3 - x^2 - x - 1$	4	948	46094281
17	$x^9 - 2x^8 + 3x^7 - 3x^6 + 4x^5 - 4x^4 + x^3 + 3x^2 - 3x + 1$	4	906	47679193
11(2), 17	$x^9 - 3x^7 + 5x^5 - 3x^3 - x^2 + x + 1$	4	810	54333913
13, 17	$x^9 - x^8 - x^7 + 2x^6 - 3x^4 + 2x^3 + 3x^2 - x - 1$	4	846	54855733
17	$x^9 - x^8 - 2x^7 + x^6 + 3x^5 + x^4 - 3x^3 - x^2 + x + 1$	4	756	60357889
11(2), 17	$x^9 - 2x^8 + 4x^7 - 5x^6 + 4x^5 - 2x^4 - x^3 + 3x^2 - 2x + 1$	4	570	91446973

We searched an additional 7300 fields of degree 9 and rank 4 with discriminant larger than 91446973 in the Table [3] and did not find any additional occurrence of elliptic curves with a point of prime order $N \geq 17$.

N	field K (degree 9 and rank 5)	rk	$ex(K)$	$discr(K)$
19	$x^9 - x^8 - 4x^7 + 2x^6 + 8x^5 - x^4 - 8x^3 - 2x^2 + 3x + 1$	5	3366	-110852311
17(2)	$x^9 - x^8 - x^7 + x^6 - 3x^4 + x^3 + 4x^2 - 1$	5	3294	-113501567
11(6), 13, 31	$x^9 - 2x^8 + x^6 - x^5 + 14x^4 - 28x^3 + 19x^2 - 2x - 1$	5	3246	-114479303
11(6), 17	$x^9 - 2x^7 - 3x^6 + x^5 + 3x^4 + x^3 - x^2 + 1$	5	3204	-116188367
11(6), 13(3), 19	$x^9 - 2x^8 - x^7 + 5x^6 - 2x^5 - 2x^4 + x^3 - x^2 + x + 1$	5	3222	-117283087
11(12), 17	$x^9 - 2x^8 + x^7 + x^6 - x^5 - x^4 + x^3 - 2x + 1$	5	3156	-119747759
11(4), 17	$x^9 - 2x^8 + 2x^6 - x^5 + 2x^4 - 2x^3 - 2x^2 + 2x + 1$	5	3102	-123595631
11(2), 17	$x^9 - x^8 - 2x^7 + 5x^6 - x^5 - 5x^4 + 6x^3 - 3x + 1$	5	3072	-123668767
11(4), 17	$x^9 - x^7 - 3x^6 - x^5 + 3x^4 + 4x^3 + x^2 - 2x - 1$	5	3072	-124007591
11(6), 17	$x^9 - 2x^7 - x^6 + 2x^5 - x^4 - 3x^3 + 3x^2 + x - 1$	5	3048	-125785223
11(6), 13, 19	$x^9 - x^8 - 3x^7 + 4x^6 + 5x^5 - 5x^4 - 5x^3 + 2x + 1$	5	2880	-133731799
11(6), 23	$x^9 - x^8 - x^7 + x^6 + 2x^5 - 4x^3 - x^2 + x + 1$	5	2808	-138182311
11(4), 19	$x^9 - x^8 + x^7 - 2x^6 - x^5 + 2x^2 + 1$	5	2724	-142603319
13, 17	$x^9 - x^6 + x^5 + x^2 - 2x - 1$	5	2712	-143792279
11(4), 19	$x^9 - 3x^7 - x^6 + 5x^5 - 3x^4 - 3x^3 + 4x^2 - 1$	5	2712	-144146159
13, 17	$x^9 - x^8 - 2x^7 + 4x^6 - 5x^4 + x^3 + 3x^2 - x - 1$	5	2676	-145894103
11(6), 17	$x^9 - 3x^8 + 3x^7 - 3x^6 + 4x^5 - 4x^4 + 5x^3 - 4x^2 + 3x - 1$	5	2688	-148366271
13, 19 *	$x^9 - 3x^8 + 4x^7 - 4x^6 + 3x^4 - 4x^3 + 4x^2 - x + 1$	5	2628	-150782119
11(2), 13, 17	$x^9 - 2x^7 - 3x^4 + 4x^3 + x^2 - 3x + 1$	5	2640	-151174007
11(2), 17	$x^9 - 3x^7 - 4x^6 + 2x^5 + 7x^4 + 4x^3 - 3x^2 - 4x - 1$	5	2430	-166035007
11(6), 13, 19	$x^9 - x^8 - x^7 + 3x^6 - 3x^5 + x^4 - 3x^3 + 4x^2 - 3x + 1$	5	2430	-168139799

The last field in the above table is the nonic field K/\mathbb{Q} of rank 5 with the largest discriminant (in absolute value) that our search found with an 19-special elliptic curve. Fields with larger discriminants were found with a point of order $N = 17$. We include below only the largest discriminants found, as well as some fields with $N = 11, 13$, where one of the curves found has potentially good reduction.

N	field K (degree 9 rank 5, continued)	rk	$ex(K)$	$discr(K)$
17(2)	$x^9 - 3x^8 + 5x^7 - 10x^6 + 13x^5 - 12x^4 + 12x^3 - 7x^2 + 3 - 1$	5	1440	-321576947
11(2)*	$x^9 - x^8 + 2x^7 - 3x^5 + 7x^4 - 9x^3 + 8x^2 - 5 + 1$	5	1326	-356722399
11(2), 17	$x^9 - x^7 - 4x^6 - x^5 + 3x^4 + 4x^3 + x^2 - 3 - 1$	5	1362	-364057919
17	$x^9 - 3x^8 + 4x^7 - 6x^5 + 7x^4 - 2x^3 - 2x^2 + 3 - 1$	5	1206	-398752247
11(2)*	$x^9 - 2x^8 + 3x^6 - 2x^4 - 3x^3 + 3x^2 + 2 - 1$	5	1242	-403349507
17*	$x^9 - 3x^7 - x^6 + x^5 + x^4 + 2x^3 - x^2 - 2 + 1$	5	1236	-409265327
11(2), 13*, 17	$x^9 - x^8 - x^7 + 3x^6 - 2x^5 - 2x^4 + 3x^3 + x^2 - 2 - 1$	5	1188	-430320007
11(2)*	$x^9 - 2x^7 - 3x^6 + 5x^4 + 5x^3 - 2x^2 - 4 - 1$	5	978	-530050543
13*	$x^9 - 2x^8 + x^6 + x^5 - 3x^4 - 4x^3 + 9x^2 - 5 + 1$	5	894	-576500327
13, 17	$x^9 - 2x^8 + 5x^7 - 5x^6 + 4x^5 - 5x^3 + 5x^2 - 5 + 1$	5	984	-592732439

N	field K (degree 9 and rank 6)	rk	$ex(K)$	$discr(K)$
11(4), 13, 17	$x^9 - 2x^7 - x^6 + x^5 + 4x^4 - 4x^3 - 3x^2 + 4x - 1$	6	7956	460131857
11(2), 13, 29	$x^9 - x^8 - 5x^7 + 5x^6 + 7x^5 - 8x^4 - 2x^3 + 4x^2 - x - 1$	6	7416	507490937
11(4), 17	$x^9 - x^8 - 3x^7 + 5x^6 - x^4 - 3x^3 - 2x^2 + 4x - 1$	6	7074	537751889
11(2), 19*	$x^9 - 2x^8 + x^7 + 3x^6 - 6x^5 + 3x^3 - x^2 + x + 1$	6	7026	538684561
11(6), 17	$x^9 - 2x^8 - 2x^7 + 9x^6 - 10x^5 + 2x^4 + 5x^3 - 4x^2 + x + 1$	6	5838	677984681
11(6)*	$x^9 - x^8 - 3x^7 + 3x^6 + x^5 - 4x^4 + 5x^3 + 2x^2 - 4x + 1$	6	5706	709699969
11(2), 17	$x^9 - 3x^7 - 2x^6 + 3x^5 + 3x^4 - x^3 + x^2 - 1$	6	5628	710045729
17	$x^9 - x^8 - 5x^7 + 4x^6 + 11x^5 - 4x^4 - 12x^3 + 4x + 1$	6	5352	755663113
11(4), 17	$x^9 - 2x^7 - 3x^6 + 4x^4 + 3x^3 + x^2 - 2x - 1$	6	5244	798186349
11(2), 17	$x^9 - 3x^8 + 4x^6 + x^4 - 4x^2 - x + 1$	6	5070	798333049
13*	$x^9 - 3x^8 + x^7 + 7x^6 - 6x^5 - 9x^4 + 7x^3 + 6x^2 - 2x - 1$	6	5250	803996713

We ran searches through nonic fields of rank 7 and 8 with high discriminants. In both cases, our limited searches did not find instances of fields with N -special elliptic curves E/K with $N \geq 17$.

The analytic rank of $X_1(11)$ is 0 over the totally real field $\mathbb{Q}(\zeta_{19})^+$ of degree 9, with torsion subgroup reduced to $\mathbb{Z}/5\mathbb{Z}$. (The same holds for the totally real field $\mathbb{Q}(\zeta_{23})^+$ of degree 10.) The field $\mathbb{Q}(\zeta_{19})^+$ has 28398 exceptional units, a number which is slightly too large for our naive search algorithm to complete in a reasonable amount of time. We did not find any points where both coordinates are exceptional units in $\mathbb{Q}(\zeta_{19})^+$ on any plane curve $F_N(r, s) = 0$ except for the cusps when $N = 19$.

In view of this data, it is natural to wonder whether there are only finitely many nonic fields with an N -special elliptic curve E/K with $N \geq 19$.

We noted the following cluster of totally real nonic fields for $N = 13$. Every field below contains a totally real cubic field, of discriminant either 148 or 169. The 13-special elliptic curves E/K that we found all have j -invariant in the cubic subfield, and either have everywhere good reduction, or have a small set of primes of bad reduction, of surprisingly small norms. It would be interesting to determine if there is some theoretical reason for the existence of such cluster.

In the table below, the curve $X_1(13)$ has a closed point over the given totally real nonic field. The column ‘prime’ is the norm of the prime(s) where the unique elliptic curve found over that field has semi-stable reduction. The number 1 indicates that the associated curve has everywhere good reduction. For comparison in the size of the discriminants, we include at the end the totally real nonic field of fourth smallest discriminant, and second smallest among the nonic fields with a cubic subfield ([1], [5]). Coincidentally, over this field also, there exists a 13-special elliptic curve E/K with j -invariant in the cubic subfield, of discriminant 229.

Totally real nonic fields K and $N = 13$	$ex(K)$	$discr(K)$	prime
$x^9 - 4x^8 - 13x^7 + 51x^6 + 61x^5 - 190x^4 - 162x^3 + 211x^2 + 181x + 1$	660	47082882548032	23^3
$x^9 - 4x^8 - 13x^7 + 25x^6 + 71x^5 + 2x^4 - 80x^3 - 55x^2 - 13x - 1$	300	36882285775168	1
$x^9 - x^8 - 17x^7 + 12x^6 + 100x^5 - 39x^4 - 243x^3 + 29x^2 + 206x + 19$	384	28614782539072	17^3
$x^9 - 2x^8 - 14x^7 + 8x^6 + 64x^5 + 40x^4 - 34x^3 - 38x^2 - 11x - 1$	396	21919316785249	79, 79, 79
$x^9 - x^8 - 12x^7 + 16x^6 + 30x^5 - 56x^4 + 12x^3 + 16x^2 - 8x + 1$	1350	6962194128409	1
$x^9 - 13x^7 - 9x^6 + 41x^5 + 52x^4 - 2x^3 - 23x^2 - 9x - 1$	696	5760304545088	107^3
$x^9 - x^8 - 13x^7 + 20x^6 + 26x^5 - 57x^4 + 15x^3 + 15x^2 - 8x + 1$	1080	5253417947968	29^3
$x^9 - 4x^8 - 5x^7 + 31x^6 - 17x^5 - 34x^4 + 32x^3 - x^2 - 5x + 1$	804	3222370424128	1
$x^9 - 2x^8 - 9x^7 + 11x^6 + 28x^5 - 18x^4 - 34x^3 + 8x^2 + 13x + 1$	26718	16440305941	2^6

A.3 Number fields of degree 10.

N	field K (degree 10 and rank 4)	rk	$ex(K)$	$discr(K)$
11(20), 31 ($j = 0$)	$x^{10} + 2x^8 - 3x^7 + 3x^6 - 7x^5 + 8x^4 - 7x^3 + 7x^2 - 4x + 1$	4	1502	-224415603
11(6), 17	$x^{10} - x^8 - 2x^7 - x^6 + 2x^5 + 2x^4 - x^2 + 1$	4	1488	-226173952
11(2), 13, 17	$x^{10} - 2x^9 + 2x^8 - 2x^6 + 4x^5 - 4x^4 + x^3 + 3x^2 - 3x + 1$	4	1404	-240232739
11(2), 13, 19	$x^{10} - x^9 + x^7 - x^6 - x^4 + x^3 + x^2 - x + 1$	4	1398	-244175707
17	$x^{10} - x^9 + x^6 - x^5 - x^4 + 3x^3 - x^2 - x + 1$	4	1382	-246944619
11(4), 13, 19	$x^{10} - x^8 - 2x^7 + x^6 + 3x^5 - 2x^4 - 4x^3 + x^2 + 3x + 1$	4	1328	-260270739
11(4), 17	$x^{10} + 3x^8 - x^7 + 4x^6 - x^5 + 4x^4 - x^3 + 2x^2 - x + 1$	4	1236	-281268947
11(4), 13, 19	$x^{10} - x^9 + x^8 - x^7 - x^5 + 3x^4 - 4x^3 + 4x^2 - 2x + 1$	4	1230	-285267739
11(4), 19(2)	$x^{10} - 3x^9 + 6x^8 - 10x^7 + 13x^6 - 13x^5 + 11x^4 - 8x^3 + 5x^2 - 2x + 1$	4	1122	-325246087
17	$x^{10} + x^8 - x^7 + x^6 - 2x^5 + x^4 - x^3 + x^2 - x + 1$	4	1068	-335570327
13, 19	$x^{10} - 2x^9 + 2x^8 - 5x^7 + 5x^6 - 3x^5 + 5x^4 - 2x^3 + 2x^2 - 3x + 1$	4	968	-381570507
17	$x^{10} - 2x^9 + 4x^8 - 6x^7 + 8x^6 - 8x^5 + 8x^4 - 7x^3 + 5x^2 - 3x + 1$	4	912	-417166579
13, 19	$x^{10} - 3x^9 + 3x^8 - 2x^7 + 2x^6 - x^5 + x^4 - 2x^3 - x^2 + 2x + 1$	4	794	-488663523
17	$x^{10} - 4x^9 + 8x^8 - 9x^7 + 6x^6 - x^5 - x^4 - x^3 + 4x^2 - 3x + 1$	4	782	-511382079
19	$x^{10} - 2x^9 + 2x^8 - 2x^6 + 3x^5 - 5x^3 + 7x^2 - 4x + 1$	4	636	-678743987
13, 17	$x^{10} - x^9 + x^8 - 3x^7 + 4x^6 - 3x^5 + 5x^4 - 6x^3 + 4x^2 - 2x + 1$	4	608	-750450339

The fields above are among the first 700 decic fields of rank 4 found in the database [3]. We searched through an additional 90,000 fields of degree 10 and unit rank 4 and did not find other occurrences of N -special elliptic curves E/K with $N \geq 17$ and such that K does not contain any degree 5 subfield appearing in the quintic fields listed in [4], 6.6, and 6.7.

N	field K (degree 10 and rank 5)	rk	$ex(K)$	$discr(K)$
17, 19(2)	$x^{10} - 3x^9 + 3x^8 - x^7 + x^5 - 3x^4 + 4x^3 - x^2 - x + 1$	5	3900	801214577
11(24), 17	$x^{10} - 2x^8 - x^7 - x^6 + x^5 + 3x^4 - 2x^3 - 2x^2 + x + 1$	5	3894	801589013
11(6), 13, 17, 37	$x^{10} - 3x^9 + 7x^8 - 11x^7 + 12x^6 - 8x^5 + 2x^4 + 4x^3 - 6x^2 + 4x - 1$	5	3876	803282693
11(6), 13, 19	$x^{10} + x^8 - 2x^7 + x^4 + 2x^3 - x^2 - 1$	5	3828	817298432
11(2), 13, 17	$x^{10} - 2x^9 + 2x^7 - x^6 + x^2 - x - 1$	5	3810	819649013
13, 17(2), 29	$x^{10} - 2x^9 + 2x^8 - 5x^7 + 7x^6 - 4x^5 + 4x^4 - 8x^3 + 5x^2 - 1$	5	3780	820675277
11(2), 17	$x^{10} - x^9 + 4x^8 - 4x^7 + 7x^6 - 8x^5 + 7x^4 - 7x^3 + 4x^2 - 3x + 1$	5	3792	827119069
17(2)	$x^{10} - x^9 + x^8 + 2x^7 - 4x^6 + 3x^4 - 2x^3 - 2x^2 + 2x + 1$	5	3768	828364813
11(6), 17, 23	$x^{10} - x^9 - 2x^8 + x^7 + x^6 + 2x^5 + x^4 - 2x^3 + x - 1$	5	3786	830597309

N	field K (degree 10, rank 5, continued)	rk	$ex(K)$	$discr(K)$
11(2), 17	$x^{10} - 2x^9 + 4x^8 - 7x^7 + 8x^6 - 8x^5 + 6x^4 - 3x^3 + x - 1$	5	3798	834771989
13, 17(2)	$x^{10} - x^9 - 2x^8 + 3x^7 + x^6 - x^5 - x^4 - x^3 + 2x^2 + x - 1$	5	3774	838803593
11(12), 13, 17(2)	$x^{10} - x^7 - 3x^6 + 3x^5 + 3x^4 - x^3 - 1$	5	3690	848429441
11(6), 13, 19	$x^{10} - x^8 - 2x^7 + x^6 + 2x^5 - x^4 - 2x^3 + 2x + 1$	5	3708	856749056
11(2), 13, 19	$x^{10} - 2x^9 + 4x^8 - 5x^7 + 6x^6 - 7x^5 + 7x^4 - 8x^3 + 5x^2 - 3x + 1$	5	3690	859848569
11(6), 13(3), 17, 19	$x^{10} - 2x^8 - 3x^7 + 5x^5 + 5x^4 - 3x^2 - 3x - 1$	5	3660	867798793
11(2), 13(3), 17	$x^{10} - x^9 - x^8 + 3x^7 - 3x^6 - x^5 + 5x^4 - x^3 - 3x^2 + x + 1$	5	3654	875003125
11(10), 13(3), 17	$x^{10} - 2x^9 + 4x^7 - 4x^6 + 2x^4 - x^3 + x^2 - x + 1$	5	3612	888484253
11(6), 17(2)	$x^{10} - x^9 + 3x^8 - 5x^7 + 5x^6 - 7x^5 + 5x^4 - 5x^3 + 3x^2 - x + 1$	5	3612	889458133
11(12), 17	$x^{10} - x^8 - x^7 + x^6 - 2x^4 + x^2 + x - 1$	5	3570	894158693
11(2), 19	$x^{10} - 2x^9 + 2x^8 - 2x^7 + x^6 + 2x^5 - 9x^4 + 12x^3 - 9x^2 + 4x - 1$	5	3576	894453953
11(12), 17	$x^{10} + x^8 - 3x^7 - x^6 - x^5 + x^4 + 3x^3 + x - 1$	5	3576	896659541
11(2), 13, 23	$x^{10} - x^9 - x^8 + 3x^7 - x^6 - 3x^5 + 3x^4 + 2x^3 - x - 1$	5	3504	914778125
13, 29	$x^{10} - 2x^7 + x^5 + 2x^4 + 3x^3 - 3x^2 - 2x + 1$	5	3510	917035997
11(6), 23	$x^{10} - 2x^9 + 2x^7 + x^6 - 5x^4 + 2x^3 + 4x^2 - x - 1$	5	3450	936901549
11(4), 17(2)	$x^{10} - 2x^8 - x^7 - x^6 - x^5 + 4x^4 + 4x^3 - x^2 - 3x - 1$	5	3468	938671133
11(2), 13, 17	$x^{10} - x^9 - x^8 + 3x^7 - x^6 - 2x^5 + 3x^4 - x^3 - x^2 - 1$	5	3468	941778433
11(2), 17	$x^{10} - 3x^9 + 6x^8 - 8x^7 + 9x^6 - 8x^5 + 5x^4 - 2x^3 - 1$	5	3426	942600293
11(2), 17(2)*	$x^{10} - x^9 - 2x^8 + 5x^7 - 3x^6 + x^5 - 3x^4 + 5x^3 - 2x^2 - x + 1$	5	3408	946838273
11(6), 23	$x^{10} - x^9 - x^8 + 3x^7 - 2x^5 + 2x^4 - 1$	5	3408	951955141
13, 17	$x^{10} - x^8 - x^7 + 2x^6 + 2x^5 - x^3 - x^2 + x - 1$	5	3372	959628125
11(2), 13, 17, 19	$x^{10} - x^9 - 2x^8 + 2x^6 + 2x^5 + x^4 - 3x^3 - 2x^2 + 2x + 1$	5	3390	962143313
11(6), 17	$x^{10} - 2x^9 + 2x^8 - 3x^6 + 2x^5 - 2x^4 + 4x^3 - 6x^2 + 4x - 1$	5	3348	963051893
11(6), 17	$x^{10} - 2x^8 - x^7 + 3x^6 + 3x^5 - 5x^4 + 3x^2 - 2x + 1$	5	3348	967461881
11(2), 13, 19	$x^{10} - 2x^9 + 3x^8 - x^7 - 2x^6 + 5x^5 - 4x^4 - 2x^3 + 4x^2 - 11$	5	3342	971937941
11(2), 13(6), 17	$x^{10} + 3x^8 - 5x^7 - 11x^5 - 5x^3 + 3x^2 + 1$	5	3324	976008137
11(2), 13, 17	$x^{10} - x^9 - 2x^8 + 3x^7 + 2x^6 - 7x^5 + 4x^4 + 2x^3 - 5x^2 + 3x - 1$	5	3324	982657741
11(6), 19	$x^{10} - 2x^9 + 3x^7 - x^6 - 5x^5 + 5x^4 - 4x^2 + 3x - 1$	5	3318	988598813
11(2), 13(2), 17	$x^{10} - x^9 + 2x^7 - x^6 - 3x^5 + 3x^4 - 3x^2 + 2x - 1$	5	3294	994689209
11(2), 13, 19	$x^{10} - 3x^9 + 5x^8 - 6x^7 + 6x^6 - 4x^5 + 3x^4 - 2x^3 + x^2 - x - 1$	5	3270	995628125
11(2), 13, 17	$x^{10} - x^9 + 2x^7 - 3x^6 + 3x^5 + x^4 - 4x^3 + 3x^2 - 1$	5	3282	997436953
11(2), 17	$x^{10} + x^8 - x^7 - 3x^6 - 2x^4 + 2x^3 + 2x^2 + 1$	5	2946	1111067821
11(2), 17	$x^{10} - x^9 - x^7 + 2x^5 - 2x^4 + 2x^2 - x - 1$	5	3006	1117913089

N	field K (degree 10, rank 5, continued)	rk	$ex(K)$	$discr(K)$
11(4), 13, 17	$x^{10} - 2x^9 + 3x^8 - 3x^7 + 3x^6 - 2x^5 + x^4 - x^2 - 1$	5	2958	1120006561
13(3), 19	$x^{10} - 3x^9 + 7x^8 - 10x^7 + 11x^6 - 9x^5 + 5x^4 - x^3 - x^2 + 2 - 1$	5	2988	1129862893

A.4 Number fields of degree 11.

N	field K (degree 11 and rank 5)	rk	$ex(K)$	$discr(K)$
17	$x^{11} - 2x^{10} + x^9 + 3x^8 - 5x^7 + 2x^6 + 4x^5 - 6x^4 + 2x^3 + 3x^2 - 3x + 1$	5	4446	-5901091967
13, 17	$x^{11} + x^9 - 2x^8 - 2x^7 - x^6 + 3x^4 + x^3 + x^2 - 1$	5	4422	-5939843699
11(2), 13, 17	$x^{11} - 3x^{10} + 6x^9 - 10x^8 + 13x^7 - 13x^6 + 11x^5 - 6x^4 + 2x^3 + x^2 - 2x + 1$	5	4422	-5953134259
11(10), 13, 17	$x^{11} - 2x^{10} + x^9 + 3x^8 - 5x^7 + x^6 + 5x^5 - 5x^4 + x^3 + 2x^2 - 2x + 1$	5	4386	-5999947987
11(2), 13, 17(2)	$x^{11} - 4x^{10} + 8x^9 - 13x^8 + 17x^7 - 17x^6 + 15x^5 - 11x^4 + 7x^3 - 5x^2 + 2x - 1$	5	4398	-6052527179
11(2), 13, 17(2)	$x^{11} - 2x^{10} + 3x^9 - 2x^8 + 2x^7 - 3x^6 + 6x^5 - 9x^4 + 9x^3 - 7x^2 + 4x - 1$	5	4344	-6079177579
11(12), 17	$x^{11} - 2x^9 - 2x^8 + x^7 + 4x^6 + 2x^5 - x^4 - x^3 - x^2 - x - 1$	5	4242	-6229245439
11(2), 17	$x^{11} - 3x^{10} + 6x^9 - 8x^8 + 8x^7 - 6x^6 + 3x^5 - 2x^3 + 2x^2 - x + 1$	5	4272	-6260560763
11(10), 19	$x^{11} + 3x^9 - 2x^8 + 3x^7 - 5x^6 + 2x^5 - 4x^4 + 2x^3 - x^2 + x - 1$	5	4266	-6313976939
19	$x^{11} - 2x^{10} + 2x^9 - 4x^7 + 7x^6 - 3x^5 - 4x^4 + 6x^3 - 2x^2 - x + 1$	5	4200	-6337515647
11(2), 13, 17	$x^{11} - 3x^{10} + 5x^9 - 3x^8 - 3x^7 + 8x^6 - 6x^5 - x^4 + 5x^3 - 2x^2 - x + 1$	5	4200	-6385658759
29	$x^{11} - 3x^{10} + 2x^9 + 3x^8 - 6x^7 + 2x^6 + 4x^5 - 4x^4 + 2x^2 - x - 1$	5	4146	-6472354879
11(2), 13, 17	$x^{11} - 3x^{10} + 2x^9 + 4x^8 - 8x^7 + 3x^6 + 4x^5 - 4x^4 + x - 1$	5	4104	-6526790267
11(6), 23	$x^{11} - 2x^9 - x^8 + x^7 + 3x^6 - x^5 - 3x^4 + x^2 + x - 1$	5	4110	-6558115807
11(2), 13(10)	$x^{11} - 3x^{10} + 6x^9 - 7x^8 + 6x^7 - 3x^6 - x^5 + 3x^4 - 3x^3 + x^2 - 1$	5	4086	-6612476603
11(2), 13, 17, 19	$x^{11} - x^{10} - 2x^9 + 3x^8 + x^7 - 5x^6 + 3x^5 + 4x^4 - 4x^3 - 2x^2 + 2x + 1$	5	4086	-6642253999
11(2), 17	$x^{11} - 3x^{10} + 6x^9 - 8x^8 + 6x^7 - 2x^6 - 2x^5 + 3x^4 - x^3 - x^2 + x - 1$	5	4032	-6683439479
11(4), 19	$x^{11} - x^{10} - x^9 + x^7 + x^4 - x^2 + 1$	5	2718	-11009745739
11(2), 19	$x^{11} - x^{10} + 2x^9 - 2x^6 + 6x^5 - 11x^4 + 11x^3 - 8x^2 + 4x - 1$	5	2736	-11077432423
17	$x^{11} - 3x^{10} + 6x^9 - 10x^8 + 15x^7 - 18x^6 + 20x^5 - 19x^4 + 15x^3 - 9x^2 + 4x - 1$	5	2628	-11079834679
11(2), 17	$x^{11} - x^{10} - x^9 + 3x^8 - x^6 - 2x^5 + x^4 + 4x^3 - 3x^2 - x + 1$	5	2808	-11091632459
17	$x^{11} - x^{10} + 2x^9 - 4x^8 + 2x^7 - 4x^6 + 4x^5 - x^4 + 3x^3 - x^2 + x - 1$	5	2592	-11309793311
13(3), 19	$x^{11} - 3x^{10} + 4x^9 - 2x^8 - x^7 + 2x^6 - x^5 - 2x^4 + 4x^3 - 4x^2 + 2x - 1$	5	2694	-11374313387
11(2), 17	$x^{11} - 4x^{10} + 8x^9 - 10x^8 + 6x^7 + 4x^6 - 11x^5 + 9x^4 - x^3 - 3x^2 + 3x - 1$	5	2634	-11377562459
11(2), 17	$x^{11} - 2x^{10} + 5x^9 - 7x^8 + 9x^7 - 9x^6 + 7x^5 - 4x^4 + x^3 + x^2 - 2x + 1$	5	2604	-11879585159

N	field K (degree 11 and rank 5, continued)	rk	$ex(K)$	$discr(K)$
11(2), 17	$x^{11} - 2x^{10} + x^8 + 2x^6 - x^5 - 2x^4 - x^3 + 2x + 1$	5	2502	-11901264959
11(2), 19	$x^{11} - 2x^{10} + x^9 + 2x^8 - 4x^7 + 4x^6 - 4x^4 + 4x^3 - 2x + 1$	5	2070	-16166694263

The three fields below with $N = 31$ are taken from points on $X_1(31)$ found in [6].

N	field K (degree 11 and rank 6)	rk	$ex(K)$	$discr(K)$
11(4), 13, 19	$x^{11} - x^{10} - 3x^9 + 7x^7 + 2x^6 - 8x^5 - x^4 + 4x^3 + x^2 - 2x - 1$	6	10836	23882289781
19	$x^{11} - 2x^{10} - x^8 + 3x^7 + 3x^6 - 5x^5 + x^4 + 2x - 1$	6	10710	24313061389
11(2), 13, 17	$x^{11} - 3x^{10} + 6x^9 - 8x^8 + 6x^7 - 6x^5 + 10x^4 - 7x^3 + 3x^2 - 1$	6	10626	24350040329
13, 17	$x^{11} - 2x^8 + 4x^4 - x^3 - 3x^2 + x + 1$	6	10512	24963663301
31	$x^{11} - x^{10} + 2x^8 + x^6 - 7x^5 + x^4 + 4x^3 - x^2 + 2x - 1$	6	8688	32252418553
11(2), 13, 19	$x^{11} - 3x^{10} + 3x^9 + 3x^8 - 10x^7 + 9x^6 + x^5 - 9x^4 + 8x^3 - x^2 - 2 + 1$	6	8622	32338529681
11(4), 13, 19	$x^{11} - x^{10} - 2x^7 - x^6 - 3x^5 - 4x^4 - x^3 - 2x^2 - 3 - 1$	6	8496	32363454073

N	field K (degree 11 and rank 7)	rk	$ex(K)$	$discr(K)$
11(6), 31 **	$x^{11} - 8x^{10} + 23x^9 - 25x^8 + 8x^6 + 14x^5 - 10x^4 - 8x^3 + 10x^2 - 5x + 1$	7	20070	-145065457639
11(2), 31	$x^{11} - 2x^{10} - 6x^9 + 8x^8 + 9x^7 - x^6 - 6x^5 - 13x^4 - x^3 + 7x^2 + 4 + 1$	7	18060	-167720394923

A.5 Number fields of degree 12.

N	field K (degree 12 and rank 5)	$ex(K)$	$discr(K)$
11(2), 13(5), 17	$x^{12} - 2x^{11} + 2x^{10} - x^9 + 2x^8 - 5x^7 + 8x^6 - 7x^5 + 4x^4 - 3x^3 + 4x^2 - 3x + 1$	5334	41223887921
11(2), 17(2), 19(2)	$x^{12} + 2x^{10} - x^9 + x^8 - 5x^7 + 6x^6 - 10x^5 + 13x^4 - 10x^3 + 7x^2 - 4x + 1$	5270	42058512657
11(14), 13*, 43	$x^{12} - x^9 + 5x^8 - 2x^7 - x^6 - 7x^5 + 8x^4 - 4x^3 + 5x^2 - 4x + 1$	5204	42553255797
11(12), 13(2), 17	$x^{12} - x^{11} - 2x^{10} + 3x^9 + x^8 - x^7 - x^6 - 2x^5 + 3x^4 + 2x^3 - 2x^2 - x + 1$	5124	42925852301
17	$x^{12} - 2x^{11} + 4x^{10} - 6x^9 + 7x^8 - 10x^7 + 9x^6 - 9x^5 + 8x^4 - 5x^3 + 4x^2 - x + 1$	5082	43643638373
11(12), 17	$x^{12} - 3x^{11} + 7x^{10} - 13x^9 + 21x^8 - 28x^7 + 34x^6 - 35x^5 + 30x^4 - 21x^3 + 12x^2 - 5x + 1$	5084	44376802641
11(6), 13, 37	$x^{12} - 2x^{11} + 7x^{10} - 7x^9 + 14x^8 - 6x^7 + 13x^6 - 2x^5 + 10x^4 - 2x^3 + 5x^2 - x + 1$	4986	44533203125
11(6), 17	$x^{12} - 4x^{11} + 7x^{10} - 6x^9 + 2x^8 + x^6 - x^5 + x^2 - x + 1$	4926	45441432661
11(6), 23	$x^{12} - 2x^{11} + x^{10} + 3x^9 - 5x^8 + 2x^7 + 3x^6 - 4x^5 + x^4 + 2x^3 - x^2 - x + 1$	4824	46334726117
11(6), 19	$x^{12} - x^{11} - 2x^{10} + 4x^9 - 5x^7 + 4x^6 + x^5 - 3x^4 + 2x^3 - x + 1$	4776	47103726121
11(6), 17	$x^{12} - x^{11} + x^{10} - 2x^9 + 4x^8 - 4x^7 + 2x^6 - 3x^5 + 4x^4 - 2x^3 + 1$	4704	48231112297
11(6), 13(2), 17	$x^{12} - 2x^{11} + 2x^{10} - x^9 + 2x^6 - 4x^5 + 6x^4 - 6x^3 + 4x^2 - 2x + 1$	4656	48612725813
13(2), 37($j = 0$)	$x^{12} - 4x^{11} + 11x^{10} - 21x^9 + 32x^8 - 40x^7 + 45x^6 - 46x^5 + 40x^4 - 26x^3 + 12x^2 - 4x + 1$	4634	50551744653
11(2), 13, 17(2)	$x^{12} - 3x^{10} - 3x^9 + 3x^8 + 8x^7 + 4x^6 - 6x^5 - 9x^4 - 3x^3 + 4x^2 + 4x + 1$	4520	51152452317
11(12), 17	$x^{12} - x^{10} - 2x^9 + x^8 + 3x^7 - x^5 - x^4 + x^3 - x + 1$	4440	52090836797
17(2)	$x^{12} - 2x^{11} + x^{10} + x^9 - x^8 + x^7 - 3x^6 + 4x^5 - x^4 - 2x^3 + 3x^2 - 2 + 1$	4416	52765663201
11(4), 19	$x^{12} - x^{11} - x^{10} + x^9 - x^8 + 2x^7 + x^6 - x^5 - x^4 - x^3 + x^2 + 1$	4338	53996413117
11(2), 17	$x^{12} - x^{11} + x^{10} - x^8 + 3x^7 - 3x^6 + 2x^5 - 2x^3 + 2x^2 - 2 + 1$	4212	55151314501
11(6), 13(2), 19	$x^{12} - 3x^{11} + 6x^{10} - 7x^9 + 5x^8 - x^7 - x^6 - x^5 + 6x^4 - 8x^3 + 7x^2 - 4x + 1$	4164	55591687817
11(4), 17	$x^{12} - 2x^{11} + x^{10} + 2x^9 - x^8 - 3x^7 + 4x^6 + 2x^5 - 7x^4 + 4x^3 + 2x^2 - 3x + 1$	4092	56633340437
11(2), 19	$x^{12} - 3x^{11} + 6x^{10} - 8x^9 + 8x^8 - 5x^7 + 3x^6 - 3x^5 + 5x^4 - 7x^3 + 7x^2 - 4 + 1$	4104	57354370889
13(3), 17	$x^{12} - x^{11} + 2x^9 - x^7 + x^6 + 3x^5 + 2x^4 + 2x^2 + 3x + 1$	3978	59661667817
11(10), 17	$x^{12} - 4x^{11} + 6x^{10} - 3x^9 - x^8 + 5x^6 - 9x^5 + 10x^4 - 8x^3 + 5x^2 - 2x + 1$	3978	60109653997
11(2), 13(2), 17	$x^{12} - x^{11} + 3x^{10} - 5x^9 + 7x^8 - 8x^7 + 10x^6 - 9x^5 + 6x^4 - 6x^3 + 3x^2 - x + 1$	3938	60707892717
13, 19	$x^{12} - 2x^{10} - x^9 + 2x^8 + 3x^7 - 3x^5 + x^3 - x^2 + 1$	3882	60974918537

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