INTERSECTION MATRICES OF WILD CYCLIC QUOTIENT SINGULARITIES

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Abstract. Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathbb{Z}/p\mathbb{Z}$ acts on $A := k[[u,v]]$ by k-linear automorphisms and let $A^{\mathbb{Z}/p\mathbb{Z}}$ denote the ring of invariants. Let $\pi : X \to \operatorname{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ be a minimal resolution of this quotient singularity with an exceptional divisor E consisting in n smooth irreducible components meeting with normal crossings. We study in this article the properties of the intersection matrix $N \in M_n(\mathbb{Z})$ associated with E. We show for instance that for any prime p, and for any $n \geq p+3$, there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity with intersection matrix of size n. We also show that for a large class of $\mathbb{Z}/p\mathbb{Z}$-quotient singularities, the matrix N is such that N^{-1} has an integer coefficient on its diagonal, and often even a full integer column.

1. Introduction

Let p be a prime. Let k be an algebraically closed field of characteristic p. Let $A := k[[u,v]]$ denote the ring of formal power series in two variables. Assume that $\mathbb{Z}/p\mathbb{Z}$ acts on A by k-linear automorphisms, and let $A^{\mathbb{Z}/p\mathbb{Z}}$ denote the ring of invariants. We will say that the closed point of $\operatorname{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ is a *wild cyclic quotient singularity*, where the term *wild* refers here to the fact that the group acting on A has order divisible by the characteristic p.

Let $\pi : X \to \operatorname{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ be a resolution of the singularity, so that in particular X is a regular scheme. Let C_i, $i = 1, \ldots, n$, denote the irreducible components of the exceptional divisor of π, and form the *intersection matrix*

$$N := ((C_i \cdot C_j)_X)_{1 \leq i,j \leq n} \in M_n(\mathbb{Z}),$$

where $(C_i \cdot C_j)_X$ denotes the intersection number of C_i and C_j computed on the regular surface X. Attached to the resolution π is its *dual graph* Γ_N, with vertices $v_1, \ldots, v_n$, where v_i and v_j are linked by $(C_i \cdot C_j)_X$ distinct edges when $i \neq j$. Let $\operatorname{Ad}(\Gamma_N)$ denote the adjacency matrix of the graph Γ_N. The matrix N has the form $\operatorname{Diag}(c_{11}, \ldots, c_{nn}) + \operatorname{Ad}(\Gamma_N)$, where $c_{ii} = (C_i \cdot C_i)_X$ is the self-intersection number of C_i. It is well-known that the matrix N is negative-definite. The following is also known about such matrices N:

(i) When the exceptional divisor of π has smooth components with normal crossings, the components C_i are smooth projective lines and the graph Γ_N is a tree ([19], Theorem 2.8).
(ii) The *discriminant group* $\Phi_N := \mathbb{Z}^n/\operatorname{Im}(N)$ is an elementary abelian p-group ([19], Theorem 2.6), so that in particular $|\Phi_N| = |\det(N)| = p^s$ for some integer $s \geq 0$.
(iii) The *fundamental cycle* $Z \in \mathbb{Z}^n_{>0}$ of N is the minimal positive vector such that NZ is a non-positive vector. The self-intersection $Z \cdot Z := ({}^tZ)NZ$ is such that $|Z \cdot Z| \leq p$ ([19], Theorem 2.4).

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Let p be any prime. Motivated by the above theorems, we call an intersection matrix $N \in M_n(\mathbb{Z})$ *p-suitable* if it satisfies the following linear algebraic properties:

(a) There exists a connected tree Γ on n vertices, and integers $c_1, \dots, c_n \geq 2$, such that $N = \mathrm{Diag}(-c_1, \dots, -c_n) + \mathrm{Ad}(\Gamma)$.
(b) The matrix N is negative definite and the group Φ_N is killed by p.
(c) The fundamental cycle Z of N is such that $|Z \cdot Z| \leq p$.

We will say that a p-suitable intersection matrix N *arises from a quotient singularity* if there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity $\mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ with a resolution of singularities $\pi : X \to \mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ such that all irreducible components C_i of the exceptional divisor E of π are *smooth*, and such that up to a choice of the ordering of the irreducible components C_i, the intersection matrix associated with E is equal to the given matrix N.

It is an open question to completely characterize the p-suitable intersection matrices which arise from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. Recent works on this question include [14], [24], [25], [26], and [28]. In this article, we establish some general properties of p-suitable matrices, and suggest some properties which might possibly be enjoyed by the matrices which arise from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity but not necessarily by all p-suitable matrices.

Recall that the *degree* (or *valency*) of a vertex v of a graph Γ is the number of edges of Γ attached to v. A vertex v with degree at least three is called a *node*, and a vertex v with degree one is called *terminal*. A graph is called a *chain* or a *path* if it is connected and does not contain any node. The graph is called *star-shaped* if it is a connected tree with a unique node.

We present in this article several constructions of p-suitable intersection matrices. Our first two results in this introduction indicate that p-suitable matrices are abundant. In particular, given any large prime p, there exist many p-suitable matrices N of every size $n \geq 9$.

Theorem (see 4.1 for a more general statement). *Given any connected tree Γ on $n \geq 9$ vertices which properly contains the graph of the Dynkin diagram E_8, and given any prime p, there exists a p-suitable intersection matrix N with associated graph Γ and $|\Phi_N| = p$.*

Theorem 5.5. *For any prime p and any integer $\delta \geq 2$, there exists a p-suitable intersection matrix N whose associated graph has δ nodes and $|\Phi_N| \geq p^\delta$.*

Given a prime p and any integer $\delta > 1$, it is natural to wonder whether there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity whose minimal resolution of singularities has a resolution graph which is a tree with δ distinct nodes. Our current record is $\delta = 5$ when $p = 2$, found in 6.3.

The families of $\mathbb{Z}/p\mathbb{Z}$-quotient singularities whose resolution graphs are currently known have resolutions whose number of irreducible components increases with p. For instance, the intersection matrix A_{p-1} on the path on $n = p-1$ vertices arises as a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity (see [25], 9.4). For trees which have at least one node, we can prove the following theorem.

Theorem 8.1. *Let p be any prime. Let $n \geq p+3$ be any integer. Then there exists a p-suitable intersection matrix of size n which arises from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.*

In view of Theorems 8.1 and 4.2, it is natural to ask whether there exist a lower bound $n(p)$, with $\liminf n(p) = \infty$, such that if N is a p-suitable matrix of size n arising as a quotient singularity and whose graph is a tree with at least one node, then $n \geq n(p)$.

An ample supply of p-suitable intersection matrices with star-shaped graphs is provided by the resolutions of weighted homogeneous singularities of the form $z^p - x^a y^b (x^c - y^d) = 0$ with $a, b, c, d \geq 1$ subject to certain mild conditions (see [25], Proposition 4.9). Some of these hypersurface singularities are known to be quotient singularities, such as the Brieskorn singularities $z^p + x^{pr+1} + y^{ps+1} = 0$ ([25], Theorem 5.3). We provide in this article two new classes of weighted homogeneous singularities which are $\mathbb{Z}/p\mathbb{Z}$-quotient singularities. In the classification of [29], page 61, the Brieskorn singularities are of Type I, and our next two singularities are of Type II and Type III, respectively.

Theorem 7.6. *Let k be an algebraically closed field of characteristic p. Let $r, s \in \mathbb{Z}_{>0}$. Let $f = z^p + x^{pr+1}y + y^{ps+1}$ or $f = z^p + x^{pr+1}y + y^{ps}x$. Let $B := k[[x, y]][z]/(f)$. Then there exists a k-linear action of $\mathbb{Z}/p\mathbb{Z}$ on $A := k[[u, v]]$ such that B is isomorphic to $A^{\mathbb{Z}/p\mathbb{Z}}$.*

The $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in Theorem 7.6 have resolutions which are star-shaped. They belong to a larger class of quotient singularities introduced in 7.2 which provides many examples whose resolutions have graphs with more than one node.

We prove in Theorem 9.2 that a certain class of resolutions of quotient singularities arising when constructing regular models of curves has associated intersection matrices N with the following in additional property: *The matrix N^{-1} has at least one integer coefficient on its diagonal.* This naturally leads us to ask the following question: Assume that N is a p-suitable matrix arising from a quotient singularity. Assume that the graph of N has at least one node. Is it possible for the matrix N^{-1} to have no integer coefficient?

We present a large class of intersection matrices N where N^{-1} does not have any integers on its diagonal in Proposition 10.6. However, often enough, a p-suitable intersection matrix N not only is such that N^{-1} has an integer coefficient, but N^{-1} also has an integer column, as in the following theorem.

Theorem 10.1. *Let p be prime. Let N be an intersection matrix such that Φ_N is killed by p. Assume that the graph Γ associated with N is a star-shaped tree. If $|\Phi_N| \neq p$, then N^{-1} has at least one integer column.*

In many examples of p-suitable matrices N arising as quotient singularities presented in this article and in [23], the fundamental cycle $Z \in \mathbb{Z}^n_{>0}$ of N is such that $-Z$ is an integer column of N^{-1}. This is the case for instance if Φ_N is trivial (see 10.8 (c)). When $p = 2$, we can show:

Theorem 10.9. *Let $p = 2$. Let N be a p-suitable intersection matrix with fundamental cycle Z. Then either $-Z$ or $-Z/p$ is a column of N^{-1}.*

When Z is a multiple of a column of N^{-1}, we obtain the following bound for $|\Phi_N|$.

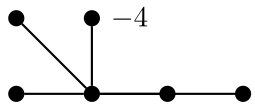
Theorem (see 11.1). *Let N be a p-suitable intersection matrix. Let $p(Z)$ denote the arithmetic genus of the fundamental cycle Z of N. Let K be the canonical cycle of N (see 2.2).*

(a) *Assume that $-Z$ or $-Z/p$ is a column of N^{-1}. Then $\log_p(|\Phi_N|)(p - 1) \leq Z \cdot K + p$.*
(b) *Let $p = 2$. Then $|\Phi_N|$ divides $p^{2p(Z)+2}$.*

2. Notation

Let $N \in M_n(\mathbb{Z})$ be a p-suitable intersection matrix whose associated graph is a connected tree Γ on n vertices $v_1, \ldots, v_n$. Thus by our definition, there exist integers $c_1, \ldots, c_n \geq 2$, such that $N = \mathrm{Diag}(-c_1, \ldots, -c_n) + \mathrm{Ad}(\Gamma)$. In this article, we will describe N using its tree Γ, and adorn each vertex v_i with the negative integer $-c_i$. We follow the established custom and omit to adorn v_i if the integer $-c_i$ is -2.

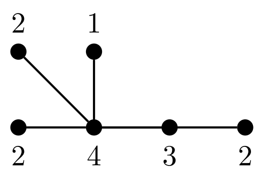
Example 2.1. We use the decorated tree Γ on the left in (a) below to represent the 6×6-matrix N on the right after having made a choice of ordering of the vertices of the tree Γ.

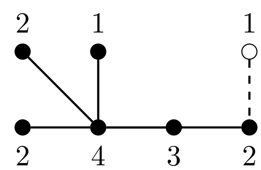
(a) 

$$N = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Let N be any intersection matrix. Let $Z \in \mathbb{Z}^n_{>0}$ denote the fundamental cycle of N. We represent the vector Z with ${}^tZ := (z_1, \ldots, z_n)$ by adorning the vertex v_i of Γ with the positive integer z_i. In the case of the above matrix N, we have ${}^tZ := (4, 2, 2, 1, 3, 2)$, which we record on the left below.

We found it efficient to record the vector NZ on the same drawing as we draw the vector Z. We use the following convention. Let ${}^t(NZ) = (s_1, \ldots, s_n)$, with $s_i \leq 0$ for all $i = 1, \ldots, n$. For each index i such that $s_i \neq 0$, add a white vertex to the graph of Γ, and link it with a dashed line to the vertex v_i. Adorn the new white vertex with the coefficient $|s_i|$. In the example of the matrix N above, we find that ${}^t(NZ) = (0, \ldots, 0, -1)$, which we record in (b) on the right below.

Z 

(b) 

Note that the information provided in the diagram (b) above, namely, the graph Γ, the vector Z, and the vector NZ, allows the recovery of the diagonal elements of the matrix N, and thus this data is sufficient to describe N itself. For the convenience of the reader, we will often include the information of the diagonal of N explicitly, and will provide a pair of diagrams as in (a) and (b) above to describe a matrix N, even if only one diagram would suffice.

The drawing of Z and NZ allows for a quick computation of the self-intersection $|Z^2| := |({}^tZ)NZ|$ by simply multiplying the integers linked by dashed lines, and adding the results of the multiplications together. In the example above, we find that $|Z^2| = 1 \cdot 2 = 2$.

Note that in the given example, NZ is equal, up to a sign, to a standard vector of $\mathbb{Z}^n$. When such is the case and Γ is any tree, the drawing of ${}^tZ = (z_1, \ldots, z_n)$ allows for a quick computation of $|\Phi_N|$. Indeed, let d_i denote the degree in Γ of the vertex v_i. If $NZ = -e_j$, then $|\Phi_N| = z_j \prod_{i=1}^n z_i^{d_i-2}$ (use [19], Theorem 3.14). For instance, in the example above, we obtain that $|\Phi_N| = 2\frac{4^2}{2\cdot 2\cdot 2} = 4$. When the order of Φ_N is not prime, the precise group structure of Φ_N needs to be determined using for instance the Smith Normal form of N.

2.2. When describing an intersection matrix N in later sections, we might also indicate whether N is numerically Gorenstein. Recall that this is a purely linear algebraic condition which can be expressed as follows. Write $N = \mathrm{Ad}(\Gamma_N) - \mathrm{Diag}(c_1, \ldots, c_n)$, with $c_i \geq 2$ for $i = 1, \ldots, n$. Let ${}^tH := (c_1 - 2, \ldots, c_n - 2)$. Since N is invertible, the equation $NK = H$ has a unique solution $K \in \mathbb{Q}^n$. The vector K is called the *canonical cycle* of N.

The $n \times n$ intersection matrix N is *numerically Gorenstein* if $K \in \mathbb{Z}^n$. If a p-suitable intersection matrix arises from a hypersurface quotient singularity, then the matrix N is numerically Gorenstein (see [25], Lemma 10.3). In the explicit example introduced above, the matrix N is numerically Gorenstein because every 2-suitable intersection matrix is numerically Gorenstein ([25], Proposition 10.5).

Given any vector $R \in \mathbb{Z}^n$ with ${}^tR = (r_1, \ldots, r_n)$, we have ${}^tRNK = \sum_{i=1}^n r_i(c_i - 2)$, and the integer ${}^tRNR + {}^tRNK$ is even. The integer $p(R) := \frac{1}{2}({}^tRNR + {}^tRNK) + 1$ is called the *arithmetical genus* of R.

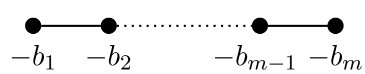
In later sections, we will title each paragraph describing a p-suitable intersection matrix N by either **Intersection Matrix** or **Quotient Singularity**. By convention, we use the title **Intersection Matrix** when we do not know whether the p-suitable intersection matrix N described in that paragraph actually arises as a quotient singularity. This is the case in particular for the matrix N described in 2.1. When $p = 2$, this matrix N is the smallest 2-suitable intersection matrix for which we do not know if it arises from a quotient singularity. When we know that a given p-suitable intersection matrix N arises as a quotient singularity, we use the title **Quotient Singularity** and we include a description of the quotient singularity.

2.3. For later use in describing intersection matrices, we record here the following standard construction. Given an ordered pair of positive integers r and s with $\gcd(r, s) = 1$ and $r > s$, we construct an associated intersection matrix $N = N(r, s)$ with vector $R = R(r, s)$ and such that $({}^tR)N = (-r, 0, \ldots, 0)$.

Indeed, we can find an integer $m \geq 1$ and integers $b_1, \ldots, b_m > 1$ and $s_1 := s > s_2 > \cdots > s_m = 1$ such that $r = b_1 s - s_2$, $s_1 = b_2 s_2 - s_3$, and so on, until we get $s_{m-1} = b_m s_m$. These equations are best written in matrix form:

$$\begin{pmatrix} -b_1 & 1 & \ldots & 0 \\ 1 & -b_2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & \ldots & 1 & -b_m \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ \vdots \\ s_m \end{pmatrix} = \begin{pmatrix} -r \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We let N denote the above square matrix, and let R be the first column matrix above. It is well-known that $\det(N) = \pm r$ (see, e.g, [18], 2.6). The matrix N is an intersection matrix whose associated graph is a path of length m:



Similarly, starting with a matrix N represented by the above path with $b_1, \ldots, b_m \geq 2$ and setting $s_m := 1$, it is possible to sequentially solve for integers $1 < s_{m-1} < \cdots < s_1$ such that the associated vector ${}^tR = (s_1, \ldots, s_{m-1}, 1)$ is such that $({}^tR)N = (-1)^{m-1}\det(N)(1, 0, \ldots, 0)$.

As usual, if $X, Y \in \mathbb{Z}^n$, we write $X > 0$ (resp., $X \geq 0$) if all coefficients of X are positive (resp., if all coefficients are non-negative). We write $X > Y$ if $X - Y > 0$, and we write $X \geq Y$ if $X - Y \geq 0$. In particular, the fundamental cycle Z of an intersection matrix N is such that $Z > 0$ and $NZ \leq 0$.

3. Constructing new p-suitable matrices from old ones

In this section, starting with a p-suitable matrix N such that N^{-1} has an integer column, we construct in several instances a new p-suitable matrix of larger size. A similar result is obtained in 9.14 assuming the existence of a column of N^{-1} which is not an integer column, but such that the diagonal element on that column is an integer.

3.1. Let N be any symmetric integer matrix with negative integers on the diagonal, and non-negative integers off the diagonal, and assume that its associated graph Γ is connected. In general, such a matrix need not be negative definite or semi-definite. However, as recalled in

[19] 3.3, *if there exists any integer vector* $Z > 0$ *such that* $NZ \leq 0$, *then either* $NZ = 0$ *and* N *is negative semi-definite, or* $NZ < 0$ *and* N *is non-singular and negative definite.*

Let $N \in M_n(\mathbb{Z})$ be an intersection matrix with associated graph Γ. We let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n$. When v is a vertex of Γ and no ordering of the vertices of Γ has been chosen, we let e_v denote the standard basis vector of $\mathbb{Z}^n$ associated with v. We let $\overline{v}$ denote the class of e_v in the quotient $\Phi_N := \mathbb{Z}^n/\mathrm{Im}(N)$.

3.2. Let $N \in M_n(\mathbb{Z})$ be an intersection matrix. Let $(N^{-1})_i$ denote the i-th column of the matrix N^{-1}. Recall that each coefficient of the matrix N^{-1} is negative ([32], Corollaire p. 387). Let $p_i \geq 1$ denote the smallest positive integer such that the vector $R_i := -p_i(N^{-1})_i$ has non-negative integer coefficients. By minimality of p_i, the greatest common divisor of the coefficients of the integer vector R_i is 1. By construction, we have $NR_i = -p_i e_i$, showing that the order of the class of e_i in Φ_N is p_i. By definition of Z, we also have $Z \leq R_i$ for each $i = 1, \dots, n$.

Lemma 3.3. *Let* p *be prime. Let* $N \in M_n(\mathbb{Z})$ *be a* p*-suitable intersection matrix. Assume that for some* i, *the integer vector* R_i *(defined in* 3.2*) is such that* $({}^tR_i)NR_i = -1$. *Let* $N' \in M_n(\mathbb{Z})$ *denote the matrix which differs from* N *only at the* (i,i)*-entry, with* $N'_{ii} = N_{ii} - (p-1)$. *Then*
(a) N' *is* p*-suitable, and* $|\Phi_{N'}| = p|\Phi_N|$.
(b) *Assume that* N *is numerically Gorenstein, with an integer vector* ${}^tK := (k_1, \dots, k_n)$ *such that* ${}^tKN = -(N_{11}+2, \dots, N_{nn}+2)$. *Then* N' *is numerically Gorenstein if and only if* p *divides* $k_i + 1$.

Proof. (a) Let N^{ii} denote the $(n-1)\times(n-1)$-matrix obtained from N by removing its i-th row and i-th column. The hypothesis that $({}^tR_i)NR_i = -1$ implies that $NR_i = -e_i$ and that the ith coefficient of R_i is 1. Without loss of generality, we can assume that $i = 1$. We now show that the same row and column operations produce the Smith Normal Form of both N and N'. Write ${}^tR_1 = (1, r_2, \dots, r_n)$. Let N_i denote the i-th column of N. Add the linear combination $\sum_{j=2}^n r_j N_j$ to the column N_1. Similarly, add the linear combination $\sum_{j=2}^n r_j N'_j$ to the column N'_1. Proceed similarly with adding the same linear combination of the rows of N to the first row of N, and do the same for N'. At the end of these operations, we find that N is similar to the matrix on the left below, and N' is similar to the matrix on the right:

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & N^{11} & \\ 0 & & & \end{pmatrix}, \qquad \begin{pmatrix} -p & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & N^{11} & \\ 0 & & & \end{pmatrix}.$$

It is clear then that $\Phi_{N'} \cong \mathbb{Z}/p\mathbb{Z} \times \Phi_N$.

Let Z (resp. Z') denote the fundamental vector of N (resp. N'). It follows from 10.8(b) that $R_i = Z$. Since $N'R_i = NR_i - (p-1)e_i = -pe_i$, we find that $Z' \leq R_i$. In particular, $|({}^tZ')N'Z'| \leq |({}^tR_i)N'R_i| = p$, as desired.

(b) Define

$$K' := K + \frac{(k_i+1)(p-1)}{p} R_i.$$

It is easy to check that $({}^tK')N' = ({}^tK)N + (p-1)e_i$, so that K' is the canonical cycle of N'. Since the i-th coefficient of R_i is equal to 1, we find that the vector K' has integer coefficients if and only if p divides $k_i + 1$. □

Let N be a p-suitable intersection matrix of size n. Suppose that the matrix N^{-1} has an integer column. We use below this column to create a new p-suitable matrix $\overline{N}$ of size $n+1$. Without loss of generality, we can assume that the first column of N^{-1} is an integer vector. In other words, the integer vector $R_1 \in \mathbb{Z}^n_{>0}$ is such that $NR_1 = -e_1$. Let $r_1 \in \mathbb{Z}_{>0}$ denote the first coefficient of R_1. Set

$$\overline{N} := \begin{pmatrix} -(r_1+1) & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & N & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}.$$

Theorem 3.4. *Let N be a p-suitable intersection matrix of size n. Suppose that the first column of N^{-1} is an integer vector. Then*

(a) *The matrix $\overline{N}$ is p-suitable of size $n+1$, with $|\Phi_{\overline{N}}| = |\Phi_N|$. The vector $\overline{R} := {}^t(1, {}^tR_1)$ is the fundamental vector of $\overline{N}$ and ${}^t\overline{R}\,\overline{N}\,\overline{R} = -1$.*

(b) *The matrix $\overline{N}'$ constructed in* 3.3 *using $\overline{N}$ and $\overline{R}$ is p-suitable of size $n+1$, with $|\Phi_{\overline{N}'}| = p|\Phi_N|$.*

Proof. (a) Label the standard basis of $\mathbb{Z}^{n+1}$ as $\{e_0, e_1, \ldots, e_n\}$. It is immediate to check that $\overline{N} \cdot \overline{R} = -e_0$. Write ${}^tR_1 := (r_1, \ldots, r_n)$. To show that $\Phi_{\overline{N}}$ is isomorphic to Φ_N, we proceed with the following row and column operations. Add the sum of columns $\sum_{j=1}^n r_j \overline{N}_{j+1}$ to the first column of $\overline{N}$. Similarly, add the same linear combination of the last rows to the first row of $\overline{N}$. After these operations, we find that $\overline{N}$ is similar to

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & N & \\ 0 & & & \end{pmatrix}.$$

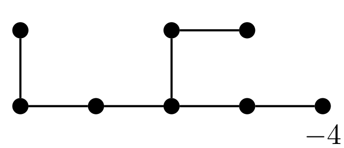
It is clear then that $\Phi_{\overline{N}} \cong \Phi_N$.

Since $\overline{NR} = -e_0$, we find that ${}^t\overline{RNR} = -1$. It follows from Proposition 10.8 (b) that $\overline{R}$ is the fundamental vector of $\overline{N}$. The statement of (b) follows immediately from Lemma 3.3 (a). □

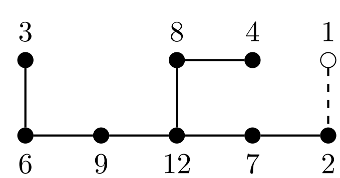
We illustrate below the constructions in Theorem 3.4 when $p = 3$. An example when $p = 2$ and the Dynkin diagram D_m is found in 6.2.

Quotient Singularity 3.5. ($n = 8$) The following matrix is p-suitable for any prime p:

Diagonal of N

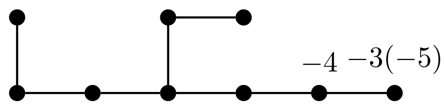


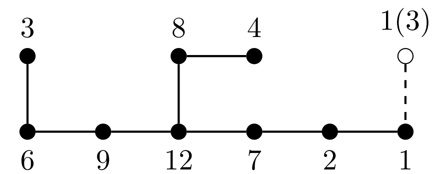
Z



The associated group Φ_N is trivial and $Z^2 = -2$. This matrix arises from the resolution of the hypersurface singularity given by $f = z^3 + x^4 + y^7 = 0$. It is shown to arise from a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity in [25], Theorem 7.1, or Theorem 5.3.

Quotient Singularity 3.6. ($n = 9$) Let $p = 3$. Using the matrix N in 3.5 and its fundamental vector, Theorem 3.4 constructs the p-suitable matrices $\overline{N}$ and $\overline{N}'$ below.

$\overline{N}(\overline{N}')$


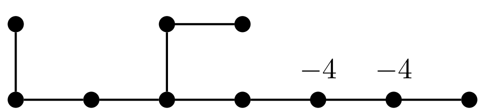
$\overline{Z}(\overline{Z}')$


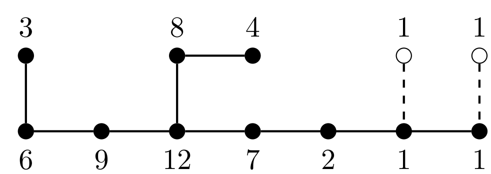
The associated group $\Phi_{\overline{N}}$ is trivial and $\overline{Z}^2 = -1$. The matrix $\overline{N}$ arises from the resolution of the hypersurface singularity given by $f = z^3 + x^4 + y^{19} = 0$. It is shown to arise from a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity in [25], Theorem 5.3.

The associated group $\Phi_{\overline{N}'}$ has order 3 and $(\overline{Z}')^2 = -3$. We do not know if the matrix $\overline{N}'$ arises from a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity. Note that the matrix $\overline{N}'$ is not numerically Gorenstein, even though the matrix $\overline{N}$ is.

Remark 3.7. The singularities in 3.5, 3.6, and 4.23 in [23] with equation $z^3 + y^7 + x^{25} = 0$, illustrate the following phenomenon. The ring $B := k[[x, y]][z]/(f)$, with $f = z^p + x^{pr+1} + y^{ps+1}$, is a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity ([25], Theorem 5.3 (i)). The intersection matrix N associated with the resolution of $\operatorname{Spec} B$ is such that N^{-1} always has two integer columns R and S such that the construction in Theorem 3.4 (a) with N and R (resp. S) produces the intersection matrix associated with the $\mathbb{Z}/p\mathbb{Z}$-quotient singularities given by $f_R = z^p + x^{pr+1} + y^{ps+1+p(pr+1)}$ (resp. $f_S = z^p + x^{pr+1+p(ps+1)} + y^{ps+1}$). This fact is at the core of the proof of Theorem 8.1.

Quotient Singularity 3.8. ($n = 10$) Given the matrix $\overline{N}$ in 3.6 and its fundamental cycle $\overline{Z}$, Theorem 3.9 below constructs the following matrix $\overline{N}''$.

$\overline{N}''$


$\overline{Z}''$


The associated group $\Phi_{\overline{N}''}$ has order 3 and $(\overline{Z}'')^2 = -2$. Note that in this example, the fundamental cycle $\overline{Z}''$ is not a multiple of a column of $(\overline{N}'')^{-1}$.

The matrix $\overline{N}$ in 3.6 is associated with the resolution of $f = z^3 + x^4 + y^{19} = 0$. Perform the blow-up of the origin of the hypersurface $f = 0$. In the chart with coordinates $z/y, x/y, y$, the strict transform is given by $(z/y)^3 + (x/y)^4 y + y^{16} = 0$. It turns out that the singularity given by $g = z^3 + x^4 y + y^{16} = 0$ has resolution matrix equal to $\overline{N}''$. Theorem 7.6 shows that the singularity $g = 0$ is a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity. This blow-up construction of a new quotient singularity from an old one motivated our next theorem, which is purely linear algebraic.

Theorem 3.9. *Let $p \geq 3$. Let $N \in M_n(\mathbb{Z})$ be a p-suitable intersection matrix. Assume that for some $i \in [1, n]$, the i-th column of N^{-1} is an integer column. Let $r := |(N^{-1})_{ii}|$ and assume in addition that $r \leq (p-1)/2$. Then there exists a new p-suitable intersection matrix $N'' \in M_{n+p-r-1}(\mathbb{Z})$ with the following properties:*

(a) $|\Phi_{N''}| = p|\Phi_N|$.

(b) *Let Z and Z'' denote the fundamental vectors of N and N''. Then $|Z^2| \leq r$, and $|Z''^2| \leq 2r$.*

(c) *When $p = 3$, then $r = 1$, $|Z''^2| = 2$, and Z'' is not a column of $(N'')^{-1}$.*

Proof. Let A_{p-r-1} denote a chain of $p-r-1$ consecutive vertices $w_1, w_2, \dots, w_{p-r-1}$, with w_1 being a vertex of degree 1 on the chain. Set all self-intersections of A_{p-r-1} to be -2. Let $\Gamma_{N''}$ denote the union of the graphs Γ_N and A_{p-r-1} with an additional edge linking $v_i \in \Gamma_N$ to $w_1 \in A_{p-r-1}$. The diagonal element of the matrix N'' at vertices of Γ_N are those of N, except at v_i, where we set $N''_{ii} := N_{ii} - 1$. The diagonal elements of N'' at vertices of A_{p-r-1} are all -2.

(a) Without loss of generality, we may assume that $i = n$ and that the vertex $v := v_n$ is the last vertex in the chosen ordering of the graph Γ_N and of the columns of N. We let N^v denote the matrix obtained from N by deleting the row and the column of N corresponding to v. To show that $|\Phi_{N''}| = p|\Phi_N|$, we compute $\det(N'')$ as a sum of two determinants, as follows. Write

$$N'' = \begin{pmatrix} N^v & \vdots & & & & \\ \cdots & N_{nn}-1 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}.$$

Then

$$\det(N'') = \det \begin{pmatrix} N^v & \vdots & & & & \\ \cdots & N_{nn} & 1 & & & \\ & 0 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} + \det \begin{pmatrix} N^v & 0 & & & & \\ \cdots & -1 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}.$$

Hence

$$\det(N'') = \det(N)(-1)^{p-r-1}(p-r) + \det(N^v)(-1)^{p-r}.$$

By construction, $(N^{-1})_{nn} = \det(N^v)/\det(N) = -r$. It follows that $\det(N^v) = -\det(N)r$. Therefore

$$\begin{aligned} \det(N'') &= (-1)^{p-r-1}p\det(N) - (-1)^{p-r-1}r\det(N) - (-1)^{p-r}\det(N)r \\ &= (-1)^{p-r-1}p\det(N), \end{aligned}$$

as desired.

(b) We continue to assume that $i = n$. Since the n-th column of N^{-1} is an integer column by hypothesis, the positive vector R_n introduced in 3.2 is such that $NR_n = -e_n$. Also by hypothesis, $({}^tR_n)NR_n = -r$. It follows that $|Z^2| \le |R_n^2| = r$ (see proof of Proposition 10.8 (b)). By hypothesis, $r \le p-1-r$. Set ${}^t\tilde{Z}'' := ({}^tR_n, r, \dots, r, r-1, r-2, \dots, 2, 1)$, to obtain

$$({}^t\tilde{Z}'')N'' = (0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0)$$

and $({}^t\tilde{Z}'')N''\tilde{Z}'' = -2r$. It follows that N'' is negative definite (3.1), and that $Z'' \le \tilde{Z}''$, so that $|({}^tZ'')N''Z''| \le |({}^t\tilde{Z}'')N''\tilde{Z}''| = 2r$.

To finish the proof that N'' is p-suitable, it remains to show that $\Phi_{N''}$ is killed by p. For this, we will show that the class of every vertex of $\Gamma_{N''}$ is killed by p. Let us start with the class of w_{p-r-1}. Consider the vector ${}^tR_{w_{p-r-1}} := ({}^tR_n, r+1, r+2, \dots, p-1)$. It is easy to check that

$$N''R_{w_{p-r-1}} = -pe_{w_{p-r-1}}.$$

Since r is a coefficient of R_n and $\gcd(r, r+1) = 1$, this equality shows that the class of w_{p-r-1} in $\Phi_{N''}$ has order p. Using this fact and Lemma 10.2, we conclude that the classes of v_n, $w_1, \ldots, w_{p-r-2}$ also have order p. Consider now a vertex v_j of Γ_N with $j < n$, with the relation $NR_j = -p_j e_j$ and $p_j \in \{1, p\}$. Let r_j denote the coefficient of R_j at the vertex $v = v_n$. Let

$$^tS_j = (^tR_j, r_j, \ldots, r_j).$$

We have the relation

$$(^tS_j)N'' = (0, \ldots, -p_j, \ldots, 0, 0, \ldots, 0, -r_j). \tag{3.1}$$

Since the matrix N^{-1} is symmetric and we assume that the n-th column has integer coefficients, we find that either (1) $p_j = p$, in which case r_j is divisible by p, or (2) $p_j = 1$.

In case (1), the relation (3.1) shows that the order of e_j in $\Phi_{N''}$ is equal to $p_j = p$. In case (2), we have two possibilities. Either (2)(i): r_j is divisible by p, in which case again (3.1) shows that the order of e_j in $\Phi_{N''}$ is equal to $p_j = 1$, or (2)(ii): r_j is not divisible by p, in which case (3.1) shows that the order of e_j in $\Phi_{N''}$ is equal to the order of e_{p-r-1}, which we showed above to be p.

(c) Let $p = 3$. Then $^tR_n N R_n = -1$ by hypothesis. It follows from Proposition 10.8 (b) that $Z = R_n$. Set $^t\tilde{Z}'' := (^tZ, 1)$. Then $^t(N''\tilde{Z}'') = (0, \ldots, 0, -1, -1)$ and $^t\tilde{Z}''N''\tilde{Z}'' = -2$. We claim that $Z'' = \tilde{Z}''$. Indeed, if $Z'' < \tilde{Z}''$, then it follows from the proof of Proposition 10.8 (b) that $|Z''^2| < |\tilde{Z}''^2|$. This is not possible because the coefficients of $\tilde{Z}''$ at v_n and w are equal to 1, and this implies that the coefficients of Z'' at v_n and w also have to equal 1. Then $|Z''^2| \geq 2$, which is a contradiction. □

Remark 3.10. It may happen that the initial matrix N in Theorem 3.9 is numerically Gorenstein, but the larger matrix N'' is not. Such an example occurs in [23], 6.15, where $p = 5$ and N is the intersection matrix of the resolution of $z^5 + x^2 + y^8 = 0$.

4. Existence of p-suitable matrices of small sizes

Fix a finite connected tree Γ on n vertices. For a given prime p, one may wonder whether there exists a p-suitable matrix N with associated graph Γ. We show in this section that such matrix might not exist when p is small (see Proposition 4.7). On the other hand, it is likely that for most graphs Γ, and for all primes p large enough (depending on Γ), such a p-suitable matrix does exist. We will not attempt in this article to exhibit evidence for this expectation beyond Theorem 4.1 (see, e.g., [22] Remark 1.4). We show then in Proposition 4.6(a) that for any given p, the number of p-suitable matrices N with graph Γ is always finite.

Theorem 4.1. *Let Γ_0 be a finite connected tree such that for some prime ℓ, there exists an ℓ-suitable matrix N_0 with associated graph Γ_0 such that $|\Phi_{N_0}| = 1$. Let Γ be any finite connected tree which strictly contains Γ_0 as an induced subgraph. Let p be any prime. Then there exists a p-suitable matrix N with associated graph Γ such that $|\Phi_N| = p$.*

Proof. Since both Γ_0 and Γ are connected trees, our hypothesis implies that there exists at least one terminal vertex of Γ which is not contained in Γ_0. In fact, if Γ has s more vertices than Γ_0, we can consider a sequence of connected trees

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{s-1} \subset \Gamma_s = \Gamma$$

such that for each $j = 1, \ldots, s$, Γ_j is obtained from Γ_{j-1} by adding a single vertex to Γ_{j-1} and linking it by a single edge to an already existing vertex of Γ_{j-1}.

For each $j = 1, \ldots, s$, use Theorem 3.4 (a) to produce an intersection matrix N_j with graph Γ_j such that $|\Phi_{N_j}| = 1$. Then use Theorem 3.4 (b) to modify the matrix N_s to obtain a new p-suitable matrix with graph $\Gamma_s = \Gamma$ such that $|\Phi_N| = p$. □

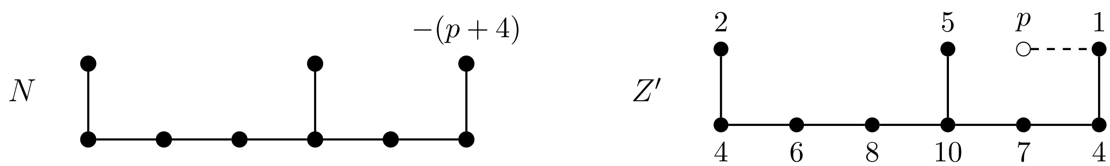
Corollary 4.2. *Given any connected tree Γ which properly contains the graph of the Dynkin diagram E_8, and given any prime p, there exists a p-suitable intersection matrix N with associated graph Γ and $|\Phi_N| = p$.*

Proof. Corollary 4.2 follows immediately from the more precise Theorem 4.1, since it is known that the Dynkin diagram E_8 has $\Phi_{E_8} = (0)$. □

Remark 4.3. Using [33], Corollary 3.11, we find that a graph as in Corollary 4.2 cannot be associated with the resolution of a rational singularity.

Remark 4.4. For further information on the intersection matrices N such that $|\Phi_N| = 1$, we refer the reader to [5], [6], [7], [8], and [9]. There are eight known such intersection matrices of minimal size $n = 8$, and they are listed in [23], Section 7. One such example is exhibited in 3.5.

Intersection Matrix 4.5. The graph Γ displayed below on $n = 9$ vertices contains the graph of the Dynkin diagram E_8. The proof of Theorem 4.1 leads to the following explicit intersection matrix:



The associated group Φ_N has order p and $|Z^2| \leq p$ since $Z'^2 = -p$. The case $p = 1$ gives the intersection matrix of the resolution of $z^2 + x^{13} + y^5 = 0$. The case $p = 2$ gives the matrix of the resolution of the blow-up $z^2 + x^9 + y^5 x = 0$. Both of these matrices arise from $\mathbb{Z}/2\mathbb{Z}$-quotient singularities (see [25] Theorem 5.3 (i), and Theorem 7.6). When $p \geq 3$, the matrix N is not numerically Gorenstein. When $p \geq 11$, the matrix N^{-1} has no integer column.

Proposition 4.6. *Let Γ be a connected graph on n vertices.*

(a) *Fix a prime p. Then there exist only finitely many intersection matrices of the form $N = \mathrm{Diag}(-c_1, \ldots, -c_n) + \mathrm{Ad}(\Gamma)$ with $c_1, \ldots, c_n \in \mathbb{Z}_{\geq 1}$ and such that Φ_N is killed by p.*

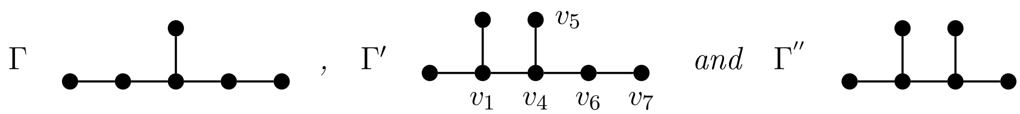
(b) *Assume that Γ is a tree. Let t denote the length of the longest path in Γ. Let $N = \mathrm{Diag}(-c_1, \ldots, -c_n) + \mathrm{Ad}(\Gamma)$, with $c_1, \ldots, c_n \in \mathbb{Z}$. Then the group Φ_N can be generated by $n - t + 1$ elements.*

Proof. (a) It is proved in Theorem 1 of [16] that for a given integer d, there exist at most finitely many matrices $-N = \mathrm{Diag}(c_1, \ldots, c_n) - \mathrm{Ad}(\Gamma)$ which are positive definite and have $\det(-N) = d$.

In our case, the matrix N has size n, so that the group Φ_N can be generated by n elements. Hence, when Φ_N is killed by p, $|\Phi_N|$ divides p^n. It follows that for any given prime p, there are only finitely many possibilities for the values taken by $\det(N)$.

(b) Suppose that the vertices $v_1, \ldots, v_t$ are the consecutive vertices of Γ on a path of longest length in Γ. The top left $t \times t$ submatrix M of N is a tridiagonal matrix. Let M' denote the submatrix of M obtained by removing its first row and last column. Every coefficient of the diagonal of M' is equal to 1. Since Γ is a tree, every coefficient of M' below the diagonal of M' is 0. Hence, M has a $(t-1 \times t-1)$-submatrix with determinant equal to 1. This shows that the Smith Normal Form $D := \mathrm{Diag}(d_1, \ldots, d_n)$ of N (with $d_1 \mid \ldots \mid d_n$) must have $d_1 = \cdots = d_{t-1} = 1$. Thus Φ_N, which is isomorphic to Φ_D, can be generated by $n - (t-1)$ elements. $\square$

Proposition 4.7. *Consider the graphs*



(a) *There exist no* 2*-suitable intersection matrices with graph* Γ.
(b) *There exist no* 2*-suitable or* 3*-suitable intersection matrices with graph* Γ' *or* Γ''.

Proof. (a) Consider the matrix $N := \mathrm{Diag}(-x_1, \ldots, -x_6) + \mathrm{Ad}(\Gamma)$, where $x_1, \ldots, x_6$ are variables and $\mathrm{Ad}(\Gamma)$ is the adjacency matrix of Γ. Then $\det(N)$ is a polynomial $f(x_1, \ldots, x_6)$. The set of integer values taken by this polynomial when $x_1, \ldots, x_6 \geq 2$ is discussed in [22], 5.3 (c). The smallest value is $|f(-2, \ldots, -2)| = 3$. When exactly one of the variables is increased to 3 and the others are left at 2, we obtain the values $|f(x_1, \ldots, x_6)| = 7, 9, 13$, Thus this polynomial does not take any value in $\{1, 2, 4\}$ when $x_1, \ldots, x_6 \geq 2$. This suffices to prove Part (a), since Γ has a path of length 5, so that when Φ_N is killed by 2, we have $|\Phi_N| \in \{1, 2, 4\}$ by 4.6 (b).

(b) Consider the matrix $N(-x_1, \ldots, -x_7) := \mathrm{Diag}(-x_1, \ldots, -x_7) + \mathrm{Ad}(\Gamma)$, where $x_1, \ldots, x_7$ are variables. Then $\det(N)$ is a polynomial $f(x_1, \ldots, x_7)$. Since Γ has a path of length 5, we must have $|\Phi_N| \in \{1, 2, 3, 4, 8, 9, 27\}$ by 4.6 (b).

For any $x_7 \geq 2$, the matrix $N(-2, \ldots, -2, -x_7)$ is not positive definite since its determinant is constant, equal to -4. The tuple $(x_1, \ldots, x_7) = (2, 2, 2, 2, 3, 2, 2)$ produces a matrix N of determinant 0 which is positive semi-definite of rank 6. The tuple $(2, 3, 2, 2, 2, 2, 3)$ produces a positive definite matrix N with $\Phi_N = \mathbb{Z}/3\mathbb{Z}$. The tuple $(2, 4, 2, 2, 2, 2, 2)$ produces a positive definite matrix N with $\Phi_N = \mathbb{Z}/2\mathbb{Z}$. This information suffices to produce an explicit effective bound B such that, if $N(-x_1, \ldots, -x_7)$ is positive definite with determinant at most 27, then $2 \leq x_1, \ldots, x_7 \leq B$. We leave the details to the reader, using [22] 2.1(c). We also need in addition that Φ_N is killed by $p = 2$ or 3. There are three examples of such N, with $\Phi_N = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ given above, and with $(x_1, \ldots, x_7) = (2, 2, 2, 2, 3, 2, 3)$ producing a matrix N with $\Phi_N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In each case, we leave it to the reader to check that we have $|Z^2| > p$, where Z denotes the fundamental cycle of N. Thus these matrices are not p-suitable.

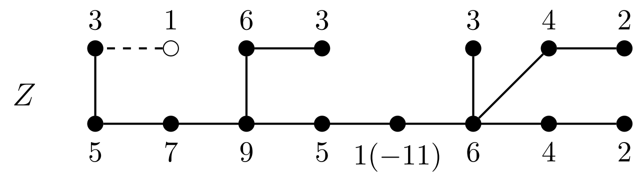
Consider now the matrix $N := \mathrm{Diag}(-x_1, \ldots, -x_6) + \mathrm{Ad}(\Gamma'')$, where $x_1, \ldots, x_6$ are variables. Then $\det(N)$ is a polynomial $f(x_1, \ldots, x_6)$. We leave it to the reader to show that this polynomial does not take any value in $\{1, 2, 3, 9, 27\}$ when $x_1, \ldots, x_6 \geq 2$. Since Γ has a path of length 4, we have $|\Phi_N| \in \{1, p, p^2, p^3\}$ by 4.6 (b). The values $|\Phi_N| = 4$ or 8 both occur, but the reader will check that in all occurrences, the group Φ_N has exponent 4. Hence, these matrices are not p-suitable. $\square$

5. Gluing two graphs to obtain new p-suitable matrices

We show in this section how to start with two p-suitable intersection matrices and build a third one. This construction will let us build in Theorem 5.5 p-suitable matrices whose graphs have any number of nodes. Let us start with the following example.

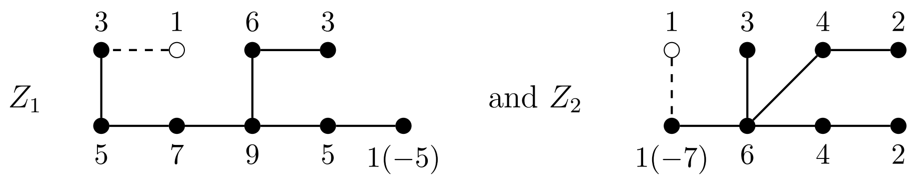
Quotient Singularity 5.1. ($n = 14$) We describe below the smallest known $\mathbb{Z}/3\mathbb{Z}$-quotient singularity to date having a graph with at least two nodes and a 3-suitable resolution matrix.

(Because the matrix N has a unique coefficient on the diagonal which is smaller than -2, we only give below the vectors Z and NZ.)

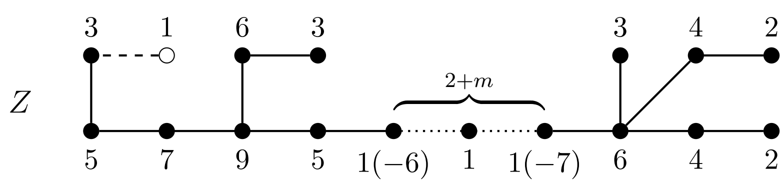


The associated group Φ_N has order 3^2 and $Z^2 = -3$. This intersection matrix is the resolution matrix of the singularity $f := z^p - (abxy)^{p-1}z - a^p xy - b^p y = 0$ with $a := x^3 + xy$ and $b := y^2 + x^3 y$. It follows from Theorem 7.5 that this is a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity.

The graph above with two nodes is obtained by gluing together the graphs of the 3-suitable intersection matrices N_1 and N_2 below. Note that the left matrix is known to arise from a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity ([23], 4.17(a2), $z^3 + x^4 y + y^4 = 0$, blow up of 3.5).



Intersection Matrix 5.2. **(**$n = 15 + m, m \geq 0$**)** Starting with the two intersection matrices N_1 and N_2, one can construct an infinite family of 3-suitable matrices. Theorem 5.4 generalizes this construction.



The associated group Φ_N has order 3^2 and $Z^2 = -3$.

5.3. To generalize the construction in 5.2, we need to introduce the following notation. Let N_1 be a p-suitable matrix of size n_1 with fundamental cycle Z_1. Assume that

(i) There exists a vertex v of Γ_{N_1} such that the coefficient of Z_1 corresponding to v is 1.
(ii) The coefficient of the vector $N_1 Z_1$ corresponding to the vertex v is 0.

Let N_2 be a p-suitable matrix of size n_2 with fundamental cycle Z_2. Assume that

(iii) $({}^t Z_2) N_2 Z_2 = -1$, so that in particular there exists a vertex w on the graph Γ_{N_2} such that the coefficient of Z_2 corresponding to this vertex is 1, and such that $-N_2 Z_2$ is the standard basis vector of $\mathbb{Z}^{n_2}$ corresponding to w (see proof of 10.8 (a)).

Fix a positive integer m. We now describe a new intersection matrix N of size $n_1 + m + n_2$. If $m = 0$, then the graph Γ_N is simply the union of the graphs Γ_{N_1} and Γ_{N_2} joined by a single edge linking v and w. If $m > 0$, let $u_1, \ldots, u_m$ denote the consecutive vertices on the graph of a chain A_m of length m. All the self-intersections of the matrix A_m are equal to -2. Since the vertices are consecutive, we will assume that u_1 and u_m have degree 1. Then the graph Γ_N is the union of the graphs Γ_{N_1}, A_m and Γ_{N_2} with one added edge linking v to u_1 and a second added edge linking u_m to v.

If $-c = (N_1)_{vv}$ denotes the diagonal element of N_1 corresponding to the vertex v, then we set to $-c-1$ the diagonal element of N corresponding to v in Γ_N. All other diagonal elements of N are those found already in N_1, A_m, or N_2.

Theorem 5.4. *Let p be prime. Let N_1 and N_2 be two p-suitable matrices satisfying the conditions* 5.3 *above. Then the matrix N introduced in* 5.3 *is p-suitable with $\Phi_N = \Phi_{N_1} \times \Phi_{N_2}$. If Z denotes the fundamental vector of N, then $|({}^tZ)NZ| \leq |({}^tZ_1)N_1Z_1|$.*

Proof. Let Z' denote the vector in $\mathbb{Z}_{>0}^{n_1+m+n_2}$ where Z' restricted to N_1 is Z_1, where Z' restricted to N_2 is Z_2, and where Z' restricted to A_m is ${}^t(1,\dots,1)$. The vector Z' has strictly positive coefficients. By our construction, the vector NZ' has non-zero coefficients exactly where the vector N_1Z_1 has non-zero coefficients. In fact, the non-zero coefficients of NZ' equal the non-zero coefficients of N_1Z_1, so that $({}^tZ')NZ' = ({}^tZ_1)N_1Z_1$. It follows that N is negative definite (3.1), and that the fundamental vector Z of N is such that $Z \leq Z'$. Since $|Z_1^2| \leq p$, we find that $|Z^2| \leq p$.

To show that N is p-suitable, it remains to show that Φ_N is killed by p. Since both Φ_{N_1} and Φ_{N_2} are killed by p, it suffices to show that $\Phi_N = \Phi_{N_1} \times \Phi_{N_2}$. For this we proceed with a row and column reduction of the matrix N.

Recall that the coefficient of Z_2 is 1 at w by hypothesis. Moreover, $-NZ_2$ is the standard basis vector corresponding to w. We use this fact and add the following linear combination of columns of N to its column corresponding to w: multiply each column of N corresponding to a vertex in Γ_{N_2} by the corresponding coefficient of Z_2, and add everything to the column corresponding to w. This operation almost clears out that column, leaving a -1 at the w-row, and a 1 at the u_m-row. A similar linear combination of the rows will almost clear out the w-row, leaving on the w-row a coefficient -1 in the w-column, and a coefficient 1 in the u_m-column. After this operation, we find that the group Φ_N is the product of two groups. It is easy to check one of them is Φ_{N_2}, and the second one can be determined to be Φ_{N_1}. □

Theorem 5.5. *Let p be prime. Let $\delta \in \mathbb{Z}_{\geq 2}$. Then there exists a p-suitable intersection matrix N whose associated graph is a tree with δ nodes and with $|\Phi_N| \geq p^\delta$.*

Proof. There are many ways of obtaining a p-suitable matrix whose graph is a tree with δ nodes. We exhibit below one such convenient way. Let N_1 and N_2 be two p-suitable matrices with star-shaped graphs as in Lemma 5.6. Let $m = \delta - 2$ and apply the construction of Theorem 5.4 to the matrices N_1 and N_2 using this m. We obtain in this way a new graph Γ_N with two nodes and a chain of m vertices $u_1, \dots, u_m$ linking the graphs of N_1 and N_2. It is easy to check that the matrix N satisfies Conditions (i) and (ii) at the vertex u_1. We can thus apply Theorem 5.4 to the pair (N, u_1) and the matrix N_2 to construct a new matrix $N^{(1)}$ whose graph has three nodes and is obtained as the union of the graphs of N and N_2 linked by one edge. We can continue this process with the vertex u_2 associated with the matrix $N^{(1)}$ to obtain a new matrix $N^{(2)}$ whose graph has four nodes. Repeating this process $\delta - 4$ times, we obtain a matrix $N^{(\delta-2)}$ whose graph has δ nodes. In each step in our process, Theorem 5.4 describe the associated finite group, and we find that since we chose $|\Phi_{N_1}|, |\Phi_{N_2}| \geq p$, the group $\Phi_{N^{(\delta-2)}}$ has order at least p^δ. □

Lemma 5.6. *Let p be prime. Then there exist a p-suitable matrix N_1 satisfying Conditions (i) and (ii) in 5.3, and a p-suitable matrix N_2 satisfying Condition (iii). Moreover, N_1 and N_2 can be chosen so that their graphs are star-shaped and $|\Phi_{N_1}|, |\Phi_{N_2}| \geq p$.*

Proof. When $p = 2$, the matrix N_r ($r \geq 1$) in 8.4 satisfies all three conditions. When $p \geq 3$, the matrix N in Remark 8.6 satisfies Conditions (i) and (ii). The construction (a) in Theorem 3.4, applied to the pair (N, Z) in 8.6 produces a new matrix $\overline{N}$ which satisfies Condition (iii). $\square$

Remark 5.7. Using [33], Theorem 5.1, we find that a graph as in Theorem 5.5 cannot be associated with the resolution of a rational singularity as soon as $\delta > |Z^2| - 2$ when $|Z^2| \geq 3$.

6. Trees with more than one node in characteristic 2

In view of Theorem 5.5, it is natural to wonder, given a prime p and any integer $\delta > 1$, whether there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity whose minimal resolution of singularities has a resolution graph which is a tree with δ distinct nodes. Our record below is family of 2-suitable intersection matrices whose graphs are trees with 5 nodes, and which are likely to arise from a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. A family with 3 nodes is discussed in 9.10. The equations given for these singularities can be checked to arise from a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity using 7.1 and Theorem 7.5.

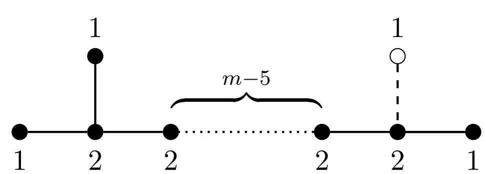
Remark 6.1. We used Magma [4] to compute explicitly the resolutions in this section. We include a generic code below.

```
p := 2; k := FiniteField(p^60); A < x, y, z >:= AffineSpace(k, 3);
a := x^2; b := y^3; f := z^p - (abxy)^(p-1) z - a^p xy + yb^p;
S := Surface(A, f); P := Scheme(A, [x, y, z]);
R := ResolveSingByBlowUp(S, P);
D := IntersectionMatrix(R); a; b; f; D; ElementaryDivisors(D);
nn := NumberOfBlowUpDivisors(R); nn; for i := 1 to nn do
B := BlowUpDivisor(S, R, i); i, IsSingular(B); Genus(B); end for;
```

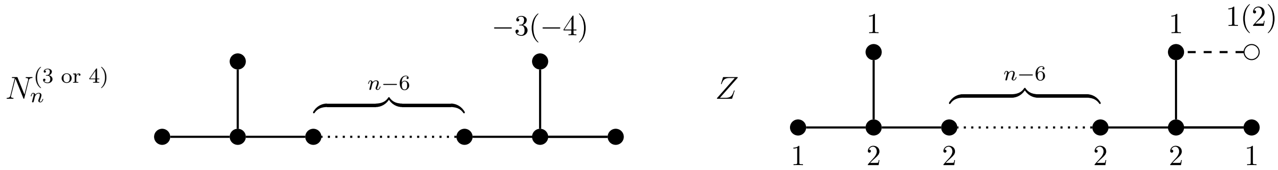
When $n = 6$, there exists only one tree with two nodes, and it is not associated with any 2-suitable intersection matrix (see Proposition 4.7(c)). When $n = 7$, there exist three trees with two nodes. One such tree is not associated with any 2-suitable intersection matrix (see Proposition 4.7(b)). The other two occur with 2-suitable intersection matrices in [23] 3.10 and in [23] 3.24. The matrix [23] 3.10 is known to arise from a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. When $n = 8$, there are already ten different connected trees with two nodes.

Quotient Singularity 6.2. ($n = 4\ell + 1 \geq 9$ **and two nodes**) It is well-known that the Dynkin diagram D_m on m vertices is a 2-suitable intersection matrix only when m is even, in which case we have $\Phi_{D_m} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. When m is odd, $\Phi_{D_m} = \mathbb{Z}/4\mathbb{Z}$. We represent below D_m with its fundamental vector.

D_m



Let $m = 2r$. It is known that D_{2r} arises as a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity (see 9.7, [2], [30]). We represent below the two extensions of D_m obtained from Theorem 3.4 using its fundamental cycle. We let $n := 2r + 1$ denote the number of vertices of the two extensions. We denote these matrices by $N_n^{(3)}$ and $N_n^{(4)}$.



The group $\Phi_{N_n^{(3)}}$ has order 2^2 and $Z^2 = -1$. The group $\Phi_{N_n^{(4)}}$ has order 2^3 and $Z^2 = -2$.

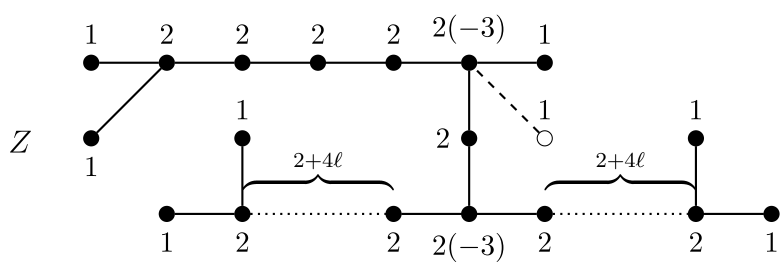
Computations suggest that we always have the following quotient singularities when $n = 4\ell + 1$ and $\ell \geq 2$:

- The matrix $N_n^{(3)}$ occurs as the resolution matrix of the $\mathbb{Z}/2\mathbb{Z}$-singularity given by the equation $f = 0$, where $f = z^p - (ab)^{p-1}z - a^p x - b^p y$ with $a = x$ and $b = y^{\ell+3} + xy$.
- The matrix $N_n^{(4)}$ occurs as the resolution matrix of the $\mathbb{Z}/2\mathbb{Z}$-singularity given by the equation $g = 0$, where $g = z^p - (abxy)^{p-1}z - a^p xy - b^p y$ with $a = x$ and $b = y^{\ell+2} + xy$.

In the case $\ell = 1$ and $n = 5$, the analogues of the matrices $N_n^{(3)}$ and $N_n^{(4)}$ have graphs with one node only, and occur in 8.4 (case $r = 1$) and in 7.7.

Surprisingly, *we have not been able to provide evidence that the 2-suitable matrix $N_n^{(3)}$ arises from a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity when $n = 2r + 1$ and $r \geq 3$ is odd.* On the other hand, the matrix $N_7^{(4)}$ is the intersection matrix associated with the resolution of the $\mathbb{Z}/2\mathbb{Z}$-quotient singularity $f := z^p - (abxy)^{p-1}z - a^p xy - b^p y = 0$ with $a := x^3 + xy$ and $b := y^3 + x^2 y$. Similarly, when $n = 11$ (resp. $n = 15$), the matrix $N_n^{(4)}$ is the intersection matrix associated with the resolution of f with $a := x^3 + xy$ and $b := y^3 + xy^2$ (resp. $b := y^3$).

Quotient Singularity 6.3. ($n = 18 + 8\ell$ **and five nodes**)



Computations indicate that the matrix occurs as the intersection matrix in the resolution of the hypersurface singularity $f = 0$, where $f := z^p - (ab)^{p-1}z - a^p y - b^p x$ with $a := x^{5+\ell} + y(x^3 + xy)$ and $b := y(x^3 + xy + y^3)$ when $\ell = 0, 1, 2$. The associated group Φ_N has order 2^6 and $Z^2 = -2$.

7. Explicit quotient singularities

We first recall in this section a family of $\mathbb{Z}/p\mathbb{Z}$-quotient hypersurface singularities introduced in [24], section 7. We then discuss a variation that allows for new parametrized families, as in [25], section 8.

7.1. Let k be an algebraically closed field of characteristic $p > 0$. Fix a system of parameters a, b in $k[[x, y]]$. Let $\mu \in k[[x, y]]$, and consider the equation

$$z^p - (\mu ab)^{p-1}z - a^p y + b^p x = 0, \tag{7.1}$$

and the associated ring

$$B_\mu = B := k[[x, y, z]]/(z^p - (\mu ab)^{p-1}z - a^p y + b^p x).$$

(a) *Assume that μ is a unit in $k[[x, y]]$.* It is shown in [24], 7.1, that B is isomorphic to the ring of invariants $A^{\mathbb{Z}/p\mathbb{Z}}$ of an explicit wild action of $\mathbb{Z}/p\mathbb{Z}$ on $A := k[[u, v]]$ ramified precisely at the origin. More precisely, after identifying A with the ring

$$k[[x, y]][u, v]/(u^p - (\mu a)^{p-1}u - x, v^p - (\mu b)^{p-1}v - y),$$

the action is determined by the automorphism σ with $\sigma(u) = u + \mu a$ and $\sigma(v) = v + \mu b$. The morphism $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathbb{Z}/p\mathbb{Z}}$ is ramified only at the maximal ideal $\mathfrak{m}$. Such actions are called *moderately ramified* in [24], and we refer the reader to [24] for further information on these actions.

(b) *Assume that μ is not a unit in $k[[x, y]]$, that $\mu \neq 0$, and that it is coprime to both a and b.* Then B is again isomorphic to the ring of invariants $A^{\mathbb{Z}/p\mathbb{Z}}$ for the action on $A := k[[u, v]]$ described above. However, in this case the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathbb{Z}/p\mathbb{Z}}$ is ramified in codimension 1

7.2. Consider now the following variation. Assume that $a, b, \mu \in k[[x, y]] \setminus \{0\}$ and that xy divides μ. Set

$$A_0 := k[[x, y]][U, V]/(U^p - (\mu a)^{p-1}U - x, V^p - (\mu b)^{p-1}V - xy).$$

Define $\tau_U : A_0 \to A_0$ with $\tau_U(U) := U + \mu a$ and $\tau_U(V) := V$. Similarly, define $\tau_V : A_0 \to A_0$ with $\tau_V(U) := U$ and $\tau_V(V) := V + \mu b$.

Proposition 7.3. *Assume that $a, b, \mu \in k[[x, y]] \setminus \{0\}$ and that xy divides μ. Then the ring A_0 is a domain. The maps τ_U and τ_V are $k[[x, y]]$-automorphisms of A_0 generating a group H isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. We have $k[[x, y]] = A_0^H$.*

Proof. The polynomial $U^p - (\mu a)^{p-1}U - x$ is irreducible in $k[[x, y]][U]$ because of our assumption that x divides μ and the Eisenstein-Schöneman Theorem applied to the prime ideal (x). The ring $R := k[[x, y]][U]/(U^p - (\mu a)^{p-1}U - x)$ is then a domain, with a unique maximal ideal generated by y and U. Since R is finite of rank p over $k[[x, y]]$, we find that its dimension is 2. Since the maximal ideal of R is generated by two elements, we find that the noetherian local ring R is regular.

Consider now $V^p - (\mu b)^{p-1}V - xy \in R[V]$. This polynomial is irreducible in $R[V]$ because of our assumption that y divides μ and the Eisenstein-Schöneman Theorem applied to the prime ideal (y). Hence $A_0 = R[V]/(V^p - (\mu b)^{p-1}V - xy)$ is a domain with maximal ideal (y, U, V).

It is clear that when $ab\mu \neq 0$, the maps τ_U and τ_V are automorphisms of order p of A_0 which generate a subgroup H of automorphisms of A_0 of order p^2. Let L denote the field of fractions of A_0 and let K be the field of fractions of $k[[x, y]]$. Then the extension L/K is Galois with group H. Since A_0 is integral over $k[[x, y]]$, any element of A_0 fixed by H is in K and is integral over $k[[x, y]]$. Since $k[[x, y]$ is integrally closed because it is regular, we find that $k[[x, y]] = A_0^H$. □

Let L denote the field of fractions of A_0. Let A' denote the subring $A_0[\frac{V}{U}]$ of L.

Proposition 7.4. *Assume that $a, b, \mu \in k[[x, y]] \setminus \{0\}$ and that xy divides μ. The ring homomorphism $A' \to A := k[[u, v]]$, which sends U to u and V/U to v, is a k-isomorphism.*

Proof. The equation $U^p - (\mu a)^{p-1}U - x = 0$ first shows that x/U is in the maximal ideal of A_0, and then that x/U^p is in A_0 and is a unit. The ring A_0 is not integrally closed, since it is clear from the equation $V^p - (\mu b)^{p-1}V - xy = 0$ that

$$\left(\frac{V}{U}\right)^p - \left(\frac{\mu b}{U}\right)^{p-1}\left(\frac{V}{U}\right) - \frac{x}{U^p}y = 0$$

is an integral relation for $\frac{V}{U}$ over A_0 since x divides μ and x/U is in A_0. The ring $A' := A_0[\frac{V}{U}]$, viewed as a subring of L, is a local ring of dimension 2 with maximal ideal generated by $(y, U, V, V/U)$. Since y and V can be expressed in terms of U and V/U, we find that the maximal ideal can be generated by two elements and, hence, A' is regular, and is thus isomorphic to the power series ring $k[[u, v]]$, with $u := U$ and $v := V/U$. □

Consider the automorphism $\tau_U \circ \tau_V = \sigma : A_0 \to A_0$ of order p with

$$\sigma(U) := U + \mu a, \quad \text{and} \quad \sigma(V) := V + \mu b.$$

The group $\langle\sigma\rangle$ acts on A', since

$$\sigma(V/U) = (V/U + \mu b/U)(1 + \mu a/U)^{-1}$$

and $1 + \mu a/U$ is a unit in A_0.

Let $z := aV - bU$. Then $\sigma(z) = z$, and we find that

$$z^p - (\mu ab)^{p-1}z - a^p xy + b^p x = 0. \tag{7.2}$$

Consider the ring

$$B' := k[[x, y]][Z]/(Z^p - (\mu ab)^{p-1}Z - a^p xy + b^p x),$$

and let B denote the subring $k[[x, y]][z]$ of A_0, image of the natural map $\varphi : B' \to B \subseteq A^{\langle\sigma\rangle}$ which sends Z to z.

Theorem 7.5. *Assume that $a, b, \mu \in k[[x, y]] \setminus \{0\}$ and that xy divides μ. Assume also that (x, y) is the radical of the ideal (a, b) in $k[[x, y]]$. Then the ring B' is a domain and the map φ is injective. This map induces an isomorphism between the field of fractions of B' and the field of fractions of $A^{\langle\sigma\rangle}$. The homomorphism $\varphi : B' \to A^{\langle\sigma\rangle}$ is an isomorphism if B' is regular in codimension 1. This latter condition is satisfied for instance if either $a = x^r$ and $b = y^s$, or if $a = y^r$ and $b = x^s$, for some integers $r, s \geq 1$.*

Proof. The ring B' is a domain because the polynomial $f := Z^p - (\mu ab)^{p-1}Z - a^p xy + b^p x$ is irreducible in $k[[x, y]][Z]$. Indeed, we assume that x divides μ, and it is easy to check that x cannot divide $a^p y + b^p$ under our hypotheses. We can then apply the Eisenstein-Schöneman criterion. One checks then that (f) is the kernel of the map $k[[x, y]]Z] \to A'$, so that the homomorphism φ is injective. By degree considerations, we find that the field of fractions of B' is isomorphic, under the natural extension of φ, to the field of fractions of $A^{\langle\sigma\rangle}$. The ring B' is Cohen–Macaulay since it is free as a module over the regular ring $k[[x, y]]$. Thus B' is normal as soon as it is regular in codimension 1.

Because of the special forms of a and b in the Theorem, we can show that B' is regular in codimension 1 by using the Jacobian criterion of Nagata ([12], IV.22.7.3). We claim that if a prime ideal $\mathfrak{p}$ of B' contains the classes of f and of the partial derivatives f_x, f_y, f_Z, then $\mathfrak{p}$

contains (x,y,Z). Let us assume first that $p>2$. Then

$$\begin{aligned}\frac{\partial f}{\partial Z} &= -(\mu ab)^{p-1}.\\ \frac{\partial f}{\partial x} &= Z(\mu ab)^{p-2}\frac{\partial \mu ab}{\partial x} - a^p y + b^p.\\ \frac{\partial f}{\partial y} &= Z(\mu ab)^{p-2}\frac{\partial \mu ab}{\partial y} - a^p x.\end{aligned}$$

If $a,b\in(x,y)$, then we conclude that $\mathfrak{p}$ contains a factor of μab, a factor of $a^p x$ and a factor of $-a^p y+b^p$.

If $a=x^r$, then $\mathfrak{p}$ contains x. If then $b=y^s$, then $\mathfrak{p}$ either contains y or a factor of $-x^{rp}+y^{ps-1}$. But if it contains $-x^{rp}+y^{ps-1}$ and x, it always also must contain y, as desired.

If $a=y^r$, then $\mathfrak{p}$ contains x or y since it contains $a^p x$. If then $b=x^s$ and $\mathfrak{p}$ contains x, then since it contains $-a^p y+b^p$, it must contain y also. If $b=x^s$ and $\mathfrak{p}$ contains y, then since it contains $-a^p y+b^p$ it must contain x also. Once the ideal $\mathfrak{p}$ contains x and y, the relation $f=0$ shows that it must contain Z.

We now consider the case where $p=2$. We have in this case

$$\begin{aligned}\frac{\partial f}{\partial x} &= Z(ab\frac{\partial \mu}{\partial x} + a\mu\frac{\partial b}{\partial x} + b\mu\frac{\partial a}{\partial x}) - a^p y + b^p.\\ \frac{\partial f}{\partial y} &= Z(ab\frac{\partial \mu}{\partial y} + a\mu\frac{\partial b}{\partial y} + b\mu\frac{\partial a}{\partial y}) - a^p x.\end{aligned}$$

Since $\mathfrak{p}$ contains at least one of a,b,μ, and since μ is divisible by xy by hypothesis, we find that we need only consider two cases, when $x\in\mathfrak{p}$ and when $y\in\mathfrak{p}$. In both cases, we find that $\mathfrak{p}$ contains a factor of $-a^p y+b^p$ and a factor of $-a^p x$. Suppose first that $x\in\mathfrak{p}$. Then $\mathfrak{p}$ contains a factor of $-a^p y+b^p$. Then using the expression $-a^p y+b^p$, we find that either $a=y^r$ and $y\in\mathfrak{p}$, or $a=x^r$, and again $y\in\mathfrak{p}$.

Suppose now that $y\in\mathfrak{p}$. Using the expression $-a^p y+b^p$, we find that $\mathfrak{p}$ contains a factor of b, and thus contains x when $b=x^s$. If $a=x^r$, then the expression $-a^p x$ shows that $x\in\mathfrak{p}$, as desired. □

We provide now two new classes of weighted homogeneous singularities which are $\mathbb{Z}/p\mathbb{Z}$-quotient singularities. The method of proof of Theorem 7.6 below follows the same argument as in the proof of Theorem 5.3 in [25].

Theorem 7.6. *Let k be an algebraically closed field of characteristic p. Let $r,s\in\mathbb{Z}_{>0}$. Let $g=z^p+x^{pr+1}y+y^{ps}x$ or $g=z^p+y^{pr+1}x+x^{ps+1}$. Let $B:=k[[x,y]][z]/(g)$. Then there exists a k-linear action of $\mathbb{Z}/p\mathbb{Z}$ on $A:=k[[u,v]]$ such that B is isomorphic to $A^{\mathbb{Z}/p\mathbb{Z}}$.*

Proof. Fix a,b in $k[[x,y]]$ such that either $a=x^r$ and $b=y^s$, or $a=y^r$ and $b=x^s$, for some integers $r,s\geq 1$. Consider the family of hypersurface singularities $\operatorname{Spec} B_\mu$, $\mu\in(xy)k[[x,y]]$, with

$$B_\mu := k[[x,y,z]]/(z^p-(\mu ab)^{p-1}z-a^p xy+b^p x).$$

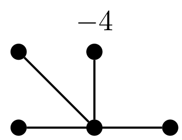
Theorem 7.5 shows that when $\mu\neq 0$, the ring B_μ is isomorphic to the ring of invariants $A^{\mathbb{Z}/p\mathbb{Z}}$ of an explicit action of $\mathbb{Z}/p\mathbb{Z}$ on $A=k[[u,v]]$.

Set $\mu=0$ in $z^p-(\mu ab)^{p-1}z-a^p xy+b^p x$ with $a=x^r$ and $b=y^s$, to obtain $f=z^p-x^{pr+1}y+y^{ps}x$. Similarly, setting $\mu=0$ with $a=y^r$ and $b=x^s$ produces $f=z^p-y^{pr+1}x+x^{ps+1}$. We now claim that it is possible to find a polynomial μ of large enough degree such that $B:=k[[x,y,z]]/(f)$ is isomorphic over k to B_μ. To prove the existence of a k-isomorphism from $B:=k[[x,y,z]]/(f)$ to B_μ, we use the Lemma in [11], 2.6, page 345. For the details of the proof of this Lemma, the authors of [11] refer the reader to the paper [3]. Recall that the Tjurina ideal of f is $j(f):=(f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$, and that there exists an integer $n>0$ such that $(x,y,z)^n\subseteq j(f)$ if and only if the Tjurina number $\tau := \dim_k(k[[x,y,z]]/j(f))$ is finite. This is indeed the case for our polynomials f. Then the Lemma in [11], 2.6, implies that if $\deg(\mu h) > 2\tau$ (with $h \in k[[x,y,z]]$), then $B := k[[x,y,z]]/(f)$ is isomorphic over k to $k[[x,y,z]]/(f+\mu h)$.

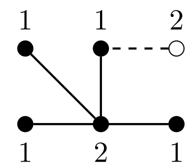
To conclude the proof, we note that since k is algebraically closed, we can make the change of variables $(x,y,z) = (\zeta X, Y, \sqrt[p]{\zeta} Z)$ to transform $z^p - x^m y + y^n x = 0$ into $Z^p + X^m Y + Y^n X = 0$, with $\zeta^{m-1} = -1$. Similarly, the change of variables $(x,y,z) = (\zeta X, Y, \sqrt[p]{-\zeta} Z)$ transforms $z^p - y^m x + x^n = 0$ into $Z^p + Y^m X + X^n = 0$ when $\zeta^{n-1} = -1$. □

Quotient Singularity 7.7. When $p = 2$, the smallest p-suitable resolutions of the singularities $g = 0$ in Theorem 7.6 have $n = 1$ in 8.7, and $n = 5$ below:

Diagonal of N



Coefficients of Z



The associated group Φ_N has order 2^3, and $Z^2 = -2$. The matrix N arises as the resolution matrix of the singularity $f := z^p - (abxy)^{p-1} z + a^p xy - b^p y$ with $a := x$, and $b := y^3$. It follows from Theorem 7.5 that this is a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. When $p = 3$, the smallest p-suitable resolutions that we found have $n = 8$ ([23] 4.13 and 4.17).

Remark 7.8. *The triple groupings.* Let $p = 2$. The singularity $z^2 + x^c + y^d = 0$ is a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity when both c and d are odd ([25], Theorem 5.3(i)).

Assume that $1 < c < d$. The blow-up of the maximal ideal produces in the chart $(z/y, x/y, y)$ the singularity $(z/y)^2 + (x/y)^c y^{c-2} + y^{d-2} = 0$, which we normalize (with abuse of notation) to $z^2 + x^c y + y^{d-c+1} = 0$. This is again a quotient singularity by Theorem 7.6 (use the case $f = z^p + y^{pr+1} x + x^{ps+1}$ and change the role of x and y).

Assume now that $1 < c < d < 2c$. Then we can perform the blowup of $z^2 + x^c y + y^{d-c+1} = 0$ at the origin to get $z^2 + x^{c-1} y + y^{d-c+1} x^{d-c-1} = 0$, which we normalize to $z^2 + x^{2c-d+1} y + y^{d-c+1} x = 0$. This is again a quotient singularity by Theorem 7.6 (use the case $f = z^p + x^{pr+1} y + y^{ps} x$ and change the role of x and y).

Starting this process with the E_8 singularity $z^2 + x^3 + y^5 = 0$ produces two new $\mathbb{Z}/2\mathbb{Z}$-quotient singularities whose associated graphs are the graphs of the Dynkin diagrams E_7 and D_6, respectively. Note that the size of the intersection matrix does not always decrease after a blow-up. Quite frequently when $p = 2$, if the Brieskorn singularity has matrix N of size n, then the resolution of the blow-up has the matrix N' of size n constructed from N in Lemma 3.3. For instance, the matrix N_1 in 8.4 is associated with $z^2 + x^3 + y^9 = 0$, and the resolution of the blow-up $z^2 + x^3 y + y^7 = 0$ is the matrix 7.7. For an example of a blow-up when $p = 3$, see 3.8.

8. Existence of quotient singularities with resolutions of small size

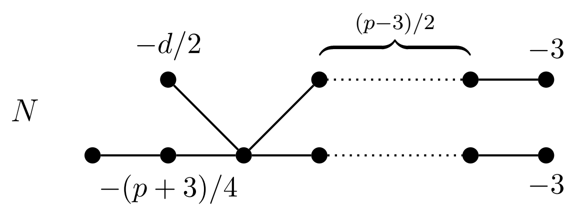
It is known that the intersection matrix A_{p-1} on the path on $n = p - 1$ vertices arises as a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity (see [25], 9.4, for $p > 2$, and 8.7 below for $p = 2$). We exhibit in this section examples of families of p-suitable intersection matrices of size n where it is known that they arise from a quotient singularity and where the graph is star-shaped. As the next theorem shows, we have not been able to produce examples where n is small compared to p, suggesting the possibility that such examples might not exist.

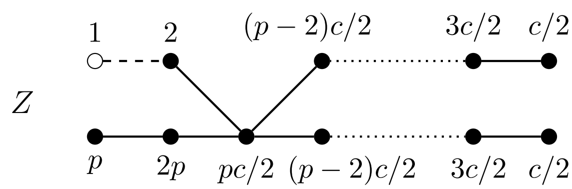
Theorem 8.1. *Let p be any prime. Let $n \geq p+3$ be any integer. Then there exists a p-suitable intersection matrix of size n which arises from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.*

Proof. We divide the proof into three cases, when (i) $p \geq 5$, (ii) $p = 3$, and (iii) $p = 2$.

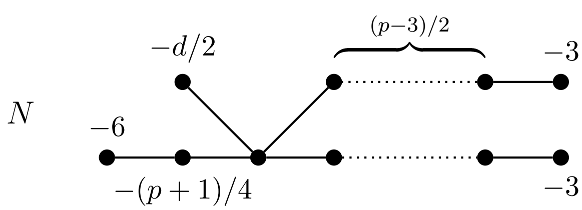
(i) Assume that $p \geq 5$. We first establish the case $n = p + 3$ of the theorem with the following two claims. Recall that any Brieskorn singularity given by an equation of the form $z^p + x^{pr+1} + y^{ps+1} = 0$ for some $r, s \geq 1$ is a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity ([25], Theorem 5.3).

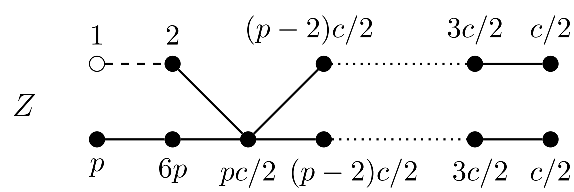
8.2. (a) *Assume that $p = 4k + 1$. Let $c := p + 1$ and $d := p(p + 1)/2 + 1$. Then the Brieskorn singularity $z^p + x^c + y^d = 0$ has a resolution whose associated intersection matrix N is p-suitable of size $n = p + 3$. The intersection matrix is represented below, with $|\Phi_N| = p$ and $Z^2 = -2$.*





(b) *Assume that $p = 4k + 3$. Let $c := 3p + 1$ and $d := p(3p + 1)/2 + 1$. Then the Brieskorn singularity $z^p + x^c + y^d = 0$ has a resolution whose associated intersection matrix N is p-suitable of size $n = p + 3$. The intersection matrix is represented below, with $|\Phi_N| = p$ and $Z^2 = -2$.*



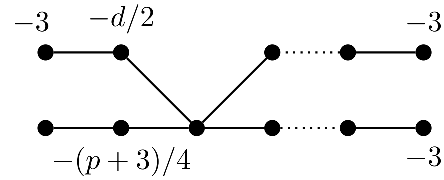


In both case (a) and case (b), we exhibit a vector Z such that $-Z$ is a column of N^{-1}. Our notation suggests that this vector is the fundamental vector of N, but we will not need, or prove, this fact here. Recall that given N and Z, Theorem 3.4(a) exhibits a new p-suitable matrix $\overline{N}$ with a vector $\overline{Z}$ such that $-\overline{Z}$ is a column of $\overline{N}^{-1}$ and $|\overline{Z}^2| = 1$. Since $-\overline{Z}$ is a column of $\overline{N}^{-1}$, we can apply Theorem 3.4(a) again to $\overline{N}$ and $\overline{Z}$ to obtain a p-suitable matrix $\overline{N}^{(2)}$ and vector $\overline{Z}^{(2)}$ such that $\overline{Z}^{(2)}$ is a column of $(\overline{N}^{(2)})^{-1}$, and so on, leading for each $i \geq 2$ to a p-suitable matrix $\overline{N}^{(i)}$ and vector $\overline{Z}^{(i)}$ such that $\overline{Z}^{(i)}$ is a column of $(\overline{N}^{(i)})^{-1}$.

The key to finish the proof of Theorem 8.1 when $p \geq 5$ is the following claim: *if N is associated with the resolution of $z^p+x^c+y^d$, then $\overline{N}$ is associated with the resolution of $z^p+x^c+y^{d+pc} = 0$, and for all $i \geq 2$, $\overline{N}^{(i)}$ is associated with the resolution of $z^p + x^c + y^{d+ipc} = 0$.* Since $d + ipc$ is of the form $pm + 1$, Theorem 5.3 of [25] can be applied to show that $\overline{N}^{(i)}$ arises from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity, of size $n = p + 3 + i$.

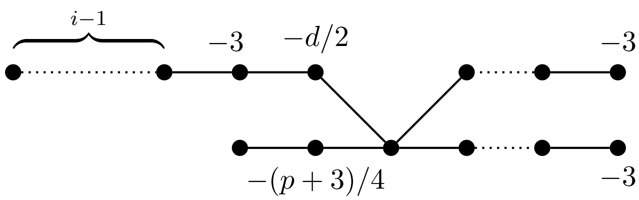
We discuss case (a) below, and leave the details of the proof in case (b) to the reader. We will need to show that the resolution of $z^p + x^c + y^{d+pc} = 0$ has the intersection matrix

(8.1) $\overline{N}$



and the resolution of $z^p + x^c + y^{d+ipc} = 0$ has intersection matrix

(8.2) $\overline{N}^{(i)}$



We use the notation introduced in [25], Theorem 5.1, to describe the intersection matrix of the resolution of the singularity of $z^p + x^c + y^d = 0$. Let $g := \gcd(c, d)$, and

$$a_1 := c/g, \quad a_2 := d/g, \text{ and } a_0 := p.$$

Set $\ell_1 := dp/g$, $\ell_2 := cp/g$ and $\ell_0 := cd/g$, and define b_j by $b_j\ell_j + 1 \equiv 0 \mod a_j$ and $0 \leq b_j < a_j$. The resolution is star-shaped, and each terminal chain is determined by a fraction a_j/b_j using the construction 2.3 with the pair (a_j, b_j). The unique node of the graph has self-intersection $-s_0$, where

$$s_0 := g^2/cdp + b_1/a_1 + b_2/a_2 + gb_0/p.$$

Since $p = 4k + 1$ in case (a), we find that $d = p(p + 1)/2 + 1$ is even. Thus $g = 2$. It is easy to check that $b_1/a_1 = 2/(c/g)$, $b_2/a_2 = 1/(d/g)$ and $b_0/a_0 = (p - 2)/p$. One checks that the associated chains are of lengths $2, 1$, and $(p-1)/2$, respectively. Since $g = 2$, there are two chains of type $(p-2)/p$. Thus the total number of components in the resolution is $p+3$. Each self-intersection on each of the chains is at most -2. It is easy to check that $s_0 = 2$.

Let us now describe the intersection matrix of the resolution of the singularity of $z^p + x^c + y^{d+icp} = 0$. Note that we have $g = \gcd(c, d + icp)$. Let

$$a'_1 := c/g, \quad a'_2 := (d + icp)/g, \text{ and } a'_0 := p.$$

Set $\ell'_1 := (d + icp)p/g$, $\ell'_2 := cp/g$ and $\ell'_0 := c(d + icp)/g$, and define b'_j by $b'_j\ell'_j + 1 \equiv 0 \mod a'_j$ and $0 \leq b'_j < a'_j$. The resolution is star-shaped, and again each terminal chain is determined by a fraction a'_j/b'_j using the construction 2.3. The unique node of the graph has self-intersection $-s'_0$, where

$$s'_0 := g^2/c(d + icp)p + b'_1/a'_1 + b'_2/a'_2 + gb'_0/p.$$

It follows immediately from the definitions and from $a_1 = a'_1$ that $b_1 = b'_1$. Similarly, it follows from $a_0 = a'_0$ that $b_0 = b'_0$. Consider now the equality

$$b_2(cp/g) + 1 = \alpha(d/g)$$

where $0 \leq b_2 < a_2$. It follows that $\alpha \leq cp/g$. Then we can write

$$(b_2 + \alpha i)(cp/g) + 1 = \alpha(d/g + icp/g)$$

and we obtain $b'_2 := b_2 + \alpha i < d/g + \alpha i \leq d/g + icp/g$.

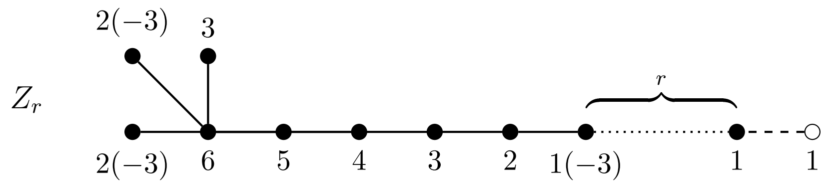
We claim that $s_0' = s_0$. Indeed

$$\begin{aligned} s_0' - s_0 &= \tfrac{g^2}{c(d+icp)p} - \tfrac{g^2}{cdp} + \tfrac{g(b_2+\alpha i)}{d+icp} - \tfrac{gb_2}{d} \\ &= \tfrac{g^2}{cp}(\tfrac{1}{d+icp} - \tfrac{1}{d}) + gb_2(\tfrac{1}{d+icp} - \tfrac{1}{d}) + \tfrac{g\alpha i}{d+icp} \\ &= \tfrac{gi}{d(d+icp)}(-g - b_2cp + \alpha d) = 0. \end{aligned}$$

To complete the proof of the claim, it suffices to check that the terminal chain obtained from the fraction b_2'/a_2' using the construction 2.3 with the pair (a_2', b_2') is the one depicted in (8.1) and (8.2). This is not hard using the values $c = p+1$ and $d = pc/2+1$, and we leave the details to the reader.

(ii) We now address the case $p = 3$ of Theorem 8.1. Examples of $\mathbb{Z}/3\mathbb{Z}$-quotient singularities with 3-suitable intersection matrices are found in 8.6 with size $n = 6$, and in [23] 4.4 with size $n = 7$. For the cases where $n \geq 8$, we proceed with the following family.

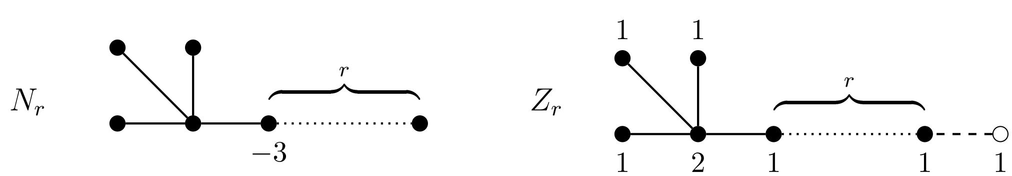
Quotient Singularity 8.3. ($n = 8+r$, $r \geq 0$) The matrix N_r associated with the graph below has three diagonal coefficients smaller than -2. We give these coefficients below along with the coefficients of Z_r and N_rZ_r.



The associated group Φ_{N_r} has order 3 and $Z_r^2 = -2$ if $r = 0$, and $Z_r^2 = -1$ if $r > 0$. The matrix N_r, $r \geq 0$, arises as the resolution of a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity in characteristic 3: It is the resolution of the hypersurface singularity $f = z^3 + x^4 + y^{10+12r} = 0$ ([25], 8.3). The matrix N_{r+1} is obtained from N_r and Z_r using Theorem 3.4 (a).

(iii) We now address the case $p = 2$ of Theorem 8.1.

Quotient Singularity 8.4. ($n = 4 + r$, $r \geq 0$)



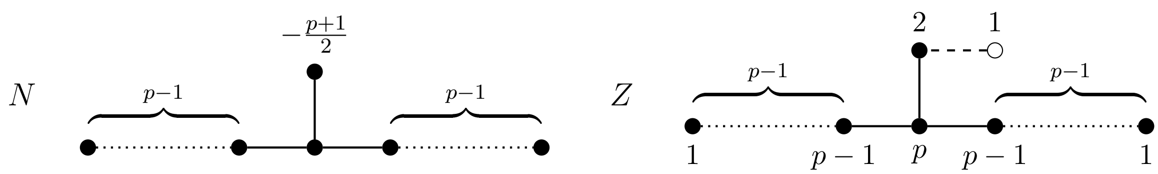
We have $Z_r^2 = -2$ when $r = 0$, and $Z_r^2 = -1$ when $r > 0$. The associated group Φ_{N_r} has order 2^2. The matrix N_r, $r \geq 0$, arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2: It is the resolution of the hypersurface singularity $f = z^2 + x^3 + y^{3+6r} = 0$ ([25], 8.3). The matrix N_{r+1} is obtained from N_r and Z_r using Theorem 3.4 (a). The matrix N_r with $r = 0$ also appears in [15], Theorem C (iv). □

Corollary 8.5. *For each prime p, there exist infinitely many p-suitable matrices with $Z^2 = -1$ and arising from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.*

Proof. The statement is immediate from the list of matrices exhibited in the proof of Theorem 8.1. □

It would be interesting to prove that for each integer $1 < s \leq p$, there exist at least one (or better, infinitely many) p-suitable matrices with $Z^2 = -s$ and arising from a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. When $s \neq 1, 2, (p+1)/2$, and p, examples of $\mathbb{Z}/p\mathbb{Z}$-quotient singularities with $Z^2 = -s$ are not known in general. An example with $p = 7$ and $s = 6$ is given in [23] 6.9. We do not know of an example with $p = 7$ and $s = 5$.

Remark 8.6. Fix a prime $p \geq 3$. We remark here that in general, the graph Γ alone does not determine a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. Indeed, consider the following two $\mathbb{Z}/p\mathbb{Z}$-quotient singularities having the same graph on $2p$ vertices. Let



with $|\Phi_N| = p$ and $Z^2 = -2$. The matrix N is numerically Gorenstein and is shown to arise as a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity in [25], Theorem 6.3. The matrix N exhibited in 9.16 has the same graph as above, is not numerically Gorenstein, and is shown to arise as a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity in [20], Theorem 6.8, or [21] Theorem 1.1, or [28], Corollary 7.13.

Quotient Singularity 8.7. The only known case so far where the matrix $N = (-p)$ arises as a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity is when $p = 2$. This is obtained with $a := x^i$ and $b := y^{i+1}$ in the equation $f := z^p - (abxy)^{p-1}z - a^p xy - b^p y = 0$ (see Theorem 7.5). Note that the matrix $N = (-p)$ is not numerically Gorenstein when $p > 2$.

9. Quotient singularities on models of curves

We review in this section how one can naturally generate interesting quotient singularities when constructing regular models of curves. As we will see in Theorem 9.2, the intersection matrices N associated with these singularities must be such that N^{-1} has at least one integer coefficient. Motivated by the setup of models of curves, we show in Theorem 9.5 how to start with the discrete data of the reduction of a curve and obtain infinitely many new p-suitable matrices which might arise as $\mathbb{Z}/p\mathbb{Z}$-quotient singularities.

9.1. Let K be a complete discrete valuation field with valuation v, ring of integers $\mathcal{O}_K$, uniformizer π_K, and residue field k of characteristic $p > 0$, assumed to be algebraically closed. Let X/K be a smooth proper geometrically connected curve of genus $g > 0$. When $g = 1$, assume in addition that $X(K) \neq \emptyset$. Assume that X/K does not have semi-stable reduction over $\mathcal{O}_K$, and that it achieves good reduction after a cyclic extension L/K of degree p.

Let H denote the Galois group of L/K. Let $\mathcal{Y}/\mathcal{O}_L$ be the smooth model of X_L/L. Let σ denote a generator of H. By minimality of the model $\mathcal{Y}$, σ defines an automorphism of $\mathcal{Y}$ also denoted by σ (but note that $\sigma : \mathcal{Y} \to \mathcal{Y}$ is not a morphism of $\mathcal{O}_L$-schemes). We also denote by σ the automorphism of the special fiber $\mathcal{Y}_k$ induced by the action of σ on $\mathcal{Y}$. Let $\mathcal{Z}/\mathcal{O}_K$ denote the quotient $\mathcal{Y}/H$, and let $\alpha : \mathcal{Y} \to \mathcal{Z}$ denote the quotient map. The scheme $\mathcal{Z}$ is normal. The map α induces a natural map $\mathcal{Y}_k \to \mathcal{Z}_k^{red}$ which factors as follows:

$$\mathcal{Y}_k \xrightarrow{\ \rho\ } \mathcal{Y}_k/\langle\sigma\rangle \longrightarrow \mathcal{Z}_k^{red}.$$

The map ρ is Galois of order $|H|$, and the second map is the normalization map of $\mathcal{Z}_k^{red}$ (see [20], 5.1).

Let $P_1, \ldots, P_d$, be the ramification points of the map $\mathcal{Y}_k \to \mathcal{Y}_k / \langle\sigma\rangle$. Let $Q_1, \ldots, Q_d$ be their images in $\mathcal{Z}$. The normal scheme $\mathcal{Z}$ is singular exactly at $Q_1, \ldots, Q_d$ (see [20], 5.2). Consider the regular model $\mathcal{X} \to \mathcal{Z}$ obtained from $\mathcal{Z}$ by a minimal desingularization. After finitely many blow-ups $\mathcal{X}' \to \mathcal{X}$, we can assume that the model $\mathcal{X}'$ is such that $\mathcal{X}'_k$ has smooth components and normal crossings, and is minimal with this property. Let f denote the composition $\mathcal{X}' \to \mathcal{Z}$. Let C_0/k denote the strict transform in $\mathcal{X}'$ of the irreducible closed subscheme $\mathcal{Z}_k^{red}$ of $\mathcal{Z}$. The curve C_0 has multiplicity $|H|$ in $\mathcal{X}'$. Let $D_1, \ldots, D_d$ denote the irreducible components of $\mathcal{X}'_k$ that meet C_0. Let r_i denote the multiplicity of D_i, $i = 1, \ldots, d$. We assume $d \geq 1$.

Theorem 9.2. *Let X/K be a smooth proper geometrically connected curve of genus $g > 0$ be as 9.1. Keep the above notation. In particular, let $f : \mathcal{X}' \to \mathcal{Z}$ denotes a resolution of the quotient singularities of $\mathcal{Z}$. Let $\Gamma_{_i}$ denote the graph attached to the exceptional divisor $f^{-1}(Q_i)$ associated with the resolution of Q_i. Let Γ denote the graph associated with the special fiber $\mathcal{X}'_k$. Then, for all $i = 1, \ldots, d$,*

(a) *The graph Γ_{Q_i} contains a node of Γ, and p divides r_i.*
(b) *Choose an ordering of the vertices of Γ_{Q_i} such that D_i is the first vertex in that ordering. Let N_i denote the intersection matrix associated with this ordering. Then the top left coefficient $(N_i^{-1})_{11}$ of N_i^{-1} is an integer.*

Proof. Part (a) is proved in Theorem 5.3 of [20]. We show now that Part (b) is an immediate consequence of Part (a). Let n_i denote the size of the matrix N_i, and let e_1 denote the first standard vector in $\mathbb{Z}^{n_i}$. Removing the component C_0 of multiplicity p disconnects the special fiber $\mathcal{X}'_k$ into the d connected curves $f^{-1}(Q_i)$, $i = 1, \ldots, d$. Each component of $f^{-1}(Q_i)$ has a multiplicity in $\mathcal{X}'_k$, and we thus have a vector $R_i \in \mathbb{Z}_{>0}^{n_i}$ such that $N_i R_i = -pe_1$, and such that ${}^tR_i = (r_i, \ldots)$ because of our choice of ordering of the components of $f^{-1}(Q_i)$. It follows from the equality $N_i R_i = -pe_1$ that $-R_i/p$ is the first column of the matrix N_i^{-1}. Since we known that p divides r_i, we find that the top left coefficient of N_i^{-1} is an integer. □

9.3. Recall that to any regular model $\mathcal{X}/\mathcal{O}_K$, one associates a linear algebraic object called an *arithmetical graph* which describes the combinatorics of the special fiber $\mathcal{X}_k$. We recall below the definition of an arithmetical graph for the convenience of the reader.

Let Γ be a finite connected graph on s vertices. An *arithmetical structure* (Γ, M, R) on Γ is a matrix $M \in M_s(\mathbb{Z})$ of the form $M = \mathrm{Diag}(-c_1, \ldots, -c_s) + \mathrm{Ad}(\Gamma)$ with $c_i \in \mathbb{Z}_{>0}$ for $i = 1, \ldots, s$, and a vector $R \in \mathbb{Z}_{>0}^s$ such that M is positive semidefinite of rank $s - 1$ and $MR = 0$. Writing ${}^tR = (r_1, \ldots, r_s)$, we always assume that $\gcd(r_1, \ldots, r_s) = 1$. Such triple (Γ, M, R) will also be called an *arithmetical graph*.

Theorem 9.5 below constructs infinitely many p-suitable matrices starting with an arithmetical graph with some additional properties (specified below in 9.4). The quotient construction of models of curves used in Theorem 9.2 suggests that the p-suitable matrices constructed in Theorem 9.5 might arise in some cases from quotient singularities in models of curves. We explain in more detail this motivation in the case of elliptic curves of reduction type I_m^* in 9.12.

9.4. Let v be a vertex of the arithmetical graph (Γ, M, R). Consider the submatrix M^v obtained from M by removing the row and the column of M corresponding to the vertex v. Let Γ_v denote the induced subgraph of Γ obtained by removing from Γ the vertex v and all the edges of Γ attached to v. If Γ_v is a connected graph, then M^v is an intersection matrix associated with Γ_v. The discussion below does not assume that Γ_v is connected.

Let $m \in \mathbb{Z}_{\geq 1}$. Consider the following intersection matrix N on $n := s + mp - 1$ vertices with graph Γ_N. The graph Γ_N is obtained from the graph Γ by attaching to the vertex v a chain of $mp - 1$ new vertices. More precisely, consider the path A_{mp-1} with vertices $w_1, \dots, w_{mp-1}$, labeled in such a way that w_1 and w_{mp-1} are the terminal vertices of the path. The graph Γ_N is obtained by linking with one edge the vertex v of Γ with the vertex w_1 of A_{mp-1}. The diagonal elements of N are those of M for every vertex of Γ except for the vertex v. Denoting by $-c_v$ the diagonal element of M corresponding to v, we set the diagonal element of N for the vertex v to be $-c_v - 1$. The diagonal element of the new vertex w_i is set to be -2, for $i = 1, \dots, pm - 1$.

Theorem 9.5. *Let p be prime. Let (Γ, M, R) be an arithmetical structure on a finite connected graph Γ on s vertices. Suppose that v is a vertex of Γ such that the coefficient of R corresponding to v is equal to p. Assume that the group $\mathbb{Z}^{s-1}/\mathrm{Im}(M^v)$ is killed by p. Assume also that the coefficients on the diagonal of M are at most equal to -2, except possibly for the coefficient $-c_v$ corresponding to v, which could equal -1. Let $m \geq 1$. Then*

(a) *The matrix $N \in M_{s+mp-1}(\mathbb{Z})$ described in 9.4 is a p-suitable intersection matrix associated with Γ_N, and $\Phi_N = \mathbb{Z}^{s-1}/\mathrm{Im}(M^v)$.*
(b) *The column of N^{-1} corresponding to v (resp. $w_{p(m-1)-1}$) is an integer column when $m = 1$ (resp. $m > 1$).*

Proof. Let us prove first the case $m = 1$. We choose an ordering of the vertices of Γ so that v is the last vertex in that ordering. The matrix N can be represented as follows:

$$N = \begin{pmatrix} M^v & & \vdots & & & & \\ \dots & & -c_v - 1 & 1 & 0 & \dots & 0 \\ & & 1 & -2 & 1 & & \vdots \\ & & 0 & \ddots & \ddots & \ddots & 0 \\ & & \vdots & & 1 & -2 & 1 \\ & & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

Let ${}^tZ' := (({}^tR), p-1, p-2, \dots, 2, 1)$. By construction, since $MR = 0$, we find that

$$({}^tZ')N = (0, \dots, 0, -1, 0, \dots, 0), \tag{9.1}$$

where the only non-zero entry is in the s-th column, the column corresponding to v. This fact follows in an essential way from the fact that we have added exactly $p - 1$ vertices to Γ. The equation (9.1) shows that the s-th column of N^{-1} is an integer vector. It follows from the minimality of the fundamental cycle Z of N that $Z \leq Z'$, and $|Z^2| \leq |Z'^2| = p$.

To compute the group Φ_N, we explicitly describe a row and column reduction of the matrix N. First, add to the last column of N the sum of the other columns, weighted by the coefficient of the column in Z'. We obtain the matrix N' below:

$$N' = \begin{pmatrix} M^v & & \vdots & & & & 0 \\ \dots & & -c_v - 1 & 1 & 0 & \dots & -1 \\ & & 1 & -2 & 1 & & 0 \\ & & 0 & \ddots & \ddots & \ddots & \vdots \\ & & \vdots & & 1 & -2 & 0 \\ & & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

It is clear from the shape of N' that $\mathbb{Z}^{s+p-1}/\mathrm{Im}(N')$ is isomorphic to $\mathbb{Z}^{s-1}/\mathrm{Im}(M^v)$. We leave it to the reader to describe the row and column operations needed to establish this isomorphism. Since $\mathbb{Z}^{s-1}/\mathrm{Im}(M^v)$ is killed by p by hypothesis, we find that N is p-suitable, and the case $m = 1$ is proved.

Given the matrix N obtained above in the case of $m = 1$, consider the following new arithmetical graph (Γ_1, M_1, R_1). Recall that the vertices of the graph Γ_N are the vertices of Γ and new vertices $w_1, \ldots, w_{p-1}$, with w_{p-1} the terminal vertex on the new chain on Γ_N. Let Γ_1 be the graph Γ_N along with a new vertex w_p attached by one edge to w_{p-1}. Let $R_1 \in \mathbb{Z}^{s+p}$ be such that ${}^tR_1 := ({}^tR, p, \ldots, p)$. Let $M_1 \in M_{s+p}(\mathbb{Z})$ be the matrix with associated graph Γ_1 whose coefficient on the diagonal corresponding to w_p is -1, and whose other diagonal coefficients are as in N. Then $M_1R_1 = 0$ and so (Γ_1, M_1, R_1) is an arithmetical graph with a vertex w_p of multiplicity p.

Since $M_1^{w_p} = N$, we find that $\mathbb{Z}^{s+p-1}/\mathrm{Im}(M_1^{w_p}) = \mathbb{Z}^{s-1}/\mathrm{Im}(M^v)$. We can thus prove the case $m = 2$ of the Theorem by applying the case $m = 1$ to the arithmetical graph (Γ_1, M_1, R_1) with the vertex w_p. It is clear that this process can be continued and that the general case can be obtained by a sequence of m applications of the case $m = 1$. □

Example 9.6. Let p be prime. We describe below a class of star-shaped arithmetical trees $\Gamma(p, r_1, \ldots, r_t)$ with a unique vertex v_0 of multiplicity p to which the construction in Theorem 9.5 can be applied.

Let $t \geq 2$. Consider integers r_i, $i = 1, \ldots, t$, such that $1 \leq r_i < p$. Assume that $\sum_{i=1}^t r_i = cp$ for some integer c. Each pair (p, r_i) determines an intersection matrix $N_i = N(p, r_i)$ as in 2.3 whose graph Γ_{N_i} is a path, along with a vector $R_i = R(p, r_i)$ such that $({}^tR_i)N_i = (-p, 0, \ldots, 0)$. Note that this construction uses a chosen order of the vertices of Γ_{N_i}, and we denote by w_i the first vertex of Γ_{N_i} in this ordering. In the construction, w_i is then a vertex of degree 1 of Γ_{N_i}.

Let $\Gamma := \Gamma(p, r_1, \ldots, r_t)$ denote the graph with unique node v_0 to which we attach each path Γ_{N_i} with one edge linking v_0 to w_i. Let s denote the total number of vertices of Γ. Let $M \in M_s(\mathbb{Z})$ denote the matrix of the form $M = \mathrm{Diag}(-c_1, \ldots, -c_s) + \mathrm{Ad}(\Gamma)$ such that M restricted to the vertices on the path Γ_{N_i} is the matrix N_i, and such that the self-intersection of the central vertex v_0 is $-c$. Let $R \in \mathbb{Z}^s_{>0}$ denote the vector such that R restricted to the vertices on the path Γ_{N_i} is the vector R_i and such that the coefficient corresponding to v_0 is p. Then we have $MR = 0$ by construction, and (Γ, M, R) is an arithmetical structure on Γ.

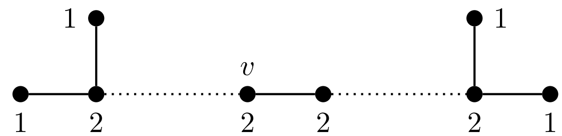
Removing from Γ the vertex v_0 and the edges adjacent to v_0 in Γ leaves us with the disjoint union of the graphs Γ_{N_i}. Since $\Phi_{N_i} = \mathbb{Z}/p\mathbb{Z}$ for each $i = 1, \ldots t$, it follows that $\mathbb{Z}^{s-1}/\mathrm{Im}(M^{v_0})$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^t$. We can thus apply Theorem 9.5 to the arithmetical graph (Γ, M, R) at the vertex v_0. Choosing $m \geq 1$, we obtain an intersection matrix N of size $n = s + pm - 1$ whose graph is star-shaped with $t + 1$ terminal chains, and whose group Φ_N is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^t$.

9.7. The p-suitable matrix N in 9.16 is obtained from $\Gamma(p, 1, p-1)$ using Theorem 9.5. This matrix is not numerically Gorenstein. It does arise from a quotient singularity.

Let now $p = 2$. The Dynkin diagrams D_{2d} (see 6.2) are 2-suitable intersection matrices obtained from the arithmetical tree $\Gamma(2, 1, 1)$ using Theorem 9.5. The Dynkin diagrams D_{2d} are known to arise as $\mathbb{Z}/2\mathbb{Z}$-quotient singularities (see, e.g., [1], Examples on page 64, [2], [30] (2.6), or [19], Theorem 4.1, or [27], III.3.1.5.1. The earliest appearance of D_4 and D_8 as quotient singularities might be in [31], § 6). It is interesting to note that the Dynkin diagrams D_{2d} arise in two different ways. Let $\mathbb{Z}/2\mathbb{Z}$ act on $A := k[[u, v]]$ such that $\mathrm{Spec}\, A^{\mathbb{Z}/2\mathbb{Z}}$ has a resolution of type D_{2d}. Consider the associated morphism $\varphi : \mathrm{Spec}\, A \to \mathrm{Spec}\, A^{\mathbb{Z}/2\mathbb{Z}}$. When d is even, the morphism φ is only ramified at the maximal ideal. When d is odd, φ is ramified in codimension 1. In particular, only the case D_{2d} with d even can arise in the context of regular models of elliptic curves. We present below an intersection matrix $N(r,s,t)$, obtained from the Kodaira type I_m^* with $m = 2(r+s)$, which seems to arise in the context of elliptic curves only when $r+s$ is even (see 9.12).

Intersection Matrix 9.8. The Kodaira type I_m^* is the arithmetical graph with $m+5$ vertices described below. We fix a vertex v to apply the construction in Theorem 9.5.

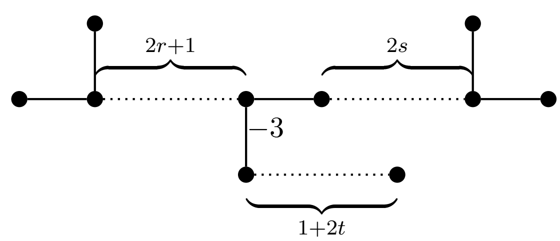
I_m^*



To the left of the vertex v is a Dynkin diagram D_a (on a vertices, see 6.2) and to the right of v is a Dynkin diagram D_b. Since the vertex v can a priori be any of the vertices of multiplicity 2 on the graph, we slightly generalize the definition of the Dynkin diagram D_a to include the cases $a = 2$ and $a = 3$. We set D_2 to be the disjoint union of two vertices of self-intersection -2, so that $\Phi_{D_2} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We set D_3 to be the path on three vertices, each of self-intersection -2, so that $\Phi_{D_3} = \mathbb{Z}/4\mathbb{Z}$. In general, it is well-known that $\Phi_{D_a} = \mathbb{Z}/4\mathbb{Z}$ if a is odd, and $\Phi_{D_a} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if a is even.

Let $r, t \geq 0$ and $s \geq 1$ be integers. Theorem 9.5 shows that the following intersection matrix $N = N(r,s,t)$, constructed from the arithmetical graph I_{2r+2s}^*, is always 2-suitable:

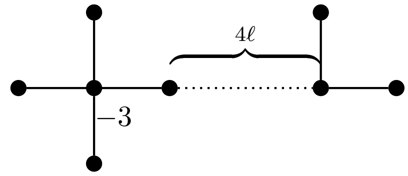
$N = N(r,s,t)$



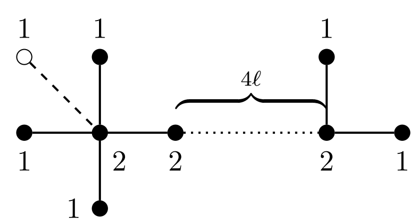
The group Φ_N has order 2^4 since it is isomorphic to $\Phi_{D_{2r+2}} \times \Phi_{D_{2s+2}}$.

Quotient Singularity 9.9. (**$n = 6 + 4\ell$ and two nodes.**) Set $r = t = 0$ in the above intersection matrix $N(r,s,t)$ and let $s = 2\ell$, with $\ell \geq 1$.

$N = N(0, 2\ell, 0)$



Z

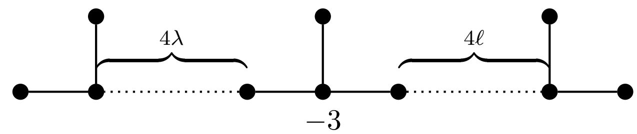


The group Φ_N has order 2^4 and $Z^2 = -2$. Computations indicate that this intersection matrix arises in the resolution of the singularity $f := z^p - (ab)^{p-1}z - a^p y - b^p x = 0$ when $a := x^2 + xy$ and $b := y^{2+\ell} + xy$ with $\ell \geq 1$. The quotient singularity 3.6 in [23] can be interpreted as the case $\ell = 0$ in this construction.

Our computations thus make it likely that, with the parameters $r = t = 0$ and $s = 2\ell$, the 2-suitable matrix N in 9.8 does arise from a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. *We do not know if it* *arises when s is odd.* A similar phenomenon related to the congruence class modulo 4 of the length of a chain connecting two nodes in the graph is also noted in 6.2.

Quotient Singularity 9.10. **($n = 6 + 4\lambda + 4\ell$ and three nodes.)** Set again $t = 0$ in the intersection matrix N in 9.8, and let $r = 2\lambda$ and $s = 2\ell$, with $\lambda, \ell \geq 1$.

$N = N(2\lambda, 2\ell, 0)$



The group Φ_N has order 2^4 and $Z^2 = -2$. Computations indicate that this intersection matrix might arise in the resolution of the singularity $f := z^p - (ab)^{p-1}z - a^p y - b^p x = 0$ when $a := x^{2+\lambda} + xy$ and $b := y^{2+\ell} + xy$ with $\lambda, \ell \geq 1$.

Remark 9.11. In the 2-suitable intersection matrices above in 9.8, 9.9, and 9.10, the diagonal elements are all equal to -2, except for one single coefficient equal to -3. Intersection matrices with this property have been completely classified in [35].

Remark 9.12. Let us return to the set-up of Theorem 9.2. In particular, let $\mathcal{O}_K$ be a discrete valuation ring with algebraically closed residue field k of characteristic $p = 2$. Let X/K be an elliptic curve. Assume that there exists a quadratic extension L/K such that X_L/L has a smooth model $\mathcal{Y}/\mathcal{O}_L$. Assume that the special fiber $\mathcal{Y}_k$ is a supersingular curve. The normal quotient $\mathcal{Z} := \mathcal{Y}/\mathrm{Gal}(L/K)$ has then a unique singular point. Let $\mathcal{X}' \to \mathcal{Z}$ denote the minimal desingularization of $\mathcal{Z}$, and let $\mathcal{X}_0$ denote the minimal regular model of X/K, with contraction morphism $\mathcal{X}' \to \mathcal{X}_0$.

Assume that the Kodaira type of the special fiber of $\mathcal{X}_0$ is I_m^* for some $m \geq 1$. Since $[L : K] = 2$, we can apply Theorem 1 in [10] to find that the component group of the Néron model of X/K must be killed by 2, so that m has to be even. We can thus write $m = 2r + 2s$ for some $r \geq 0$ and $s \geq 1$ such that the intersection matrix of the desingularization $\mathcal{X}'$ is of the form $N(r, s, t)$ in 9.8. Since we assume that the elliptic curve has potentially good supersingular reduction, we can further use [34], Theorem 1.4, to show that in fact m is divisible by 4.

Hence, using elliptic curves, we can only produce examples of $\mathbb{Z}/2\mathbb{Z}$-quotient singularities with intersection matrices $N(r, s, t)$ as in 9.8 with the additional constraint that $r + s$ is even. *We do not know if the matrix $N(r, s, t)$ also arises as a quotient singularity when $r + s$ is odd.*

9.13. Let p be prime. Let N be a p-suitable intersection matrix of size n. Suppose that v is a vertex of Γ_N such that the corresponding column $(N^{-1})_v$ of the matrix N^{-1} is not an integer column, but the diagonal element $(N^{-1})_{vv}$ is an integer. Let $m \geq 1$. Consider the following matrix N_m of size $n + pm$. Let $w_1, \ldots, w_{pm}$ be the ordered vertices of a chain of length pm. Consider the graph Γ_{N_m} obtained by attaching the vertex v of Γ_N with the initial vertex w_1 of the chain using a single edge. Set the matrix N_m associated with Γ_{N_m} to have the following diagonal elements: if w is a vertex of Γ_N, use the diagonal element of N. Set the diagonal element corresponding to w_1 to be $(N^{-1})_{vv} - 1$. Set all other diagonal elements corresponding to $w_2, \ldots, w_{pm}$ to be -2. The matrix N is a submatrix of N_m in the top left corner.

Corollary 9.14. *Let p be prime. Let N be a p-suitable intersection matrix of size n. Suppose that v is a vertex of Γ_N such that the corresponding column $(N^{-1})_v$ of the matrix N^{-1} is not an integer column, but the diagonal element $(N^{-1})_{vv}$ is an integer. Let $m \geq 1$. Then the matrix N_m of size $n + pm$ described in 9.13 is p-suitable. Its group Φ_{N_m} is isomorphic to Φ_N.*

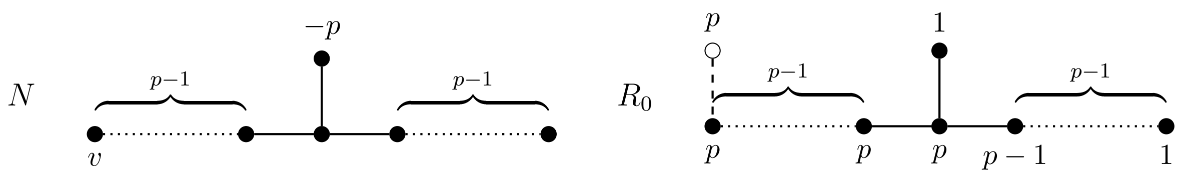
Proof. The corollary follows directly from Theorem 9.5 applied to the following arithmetical graph (G, M, R). Let G denote the graph obtained by linking the vertex v of Γ_N to an additional vertex w_1 by a single edge. The matrix M with graph G is set to have the matrix N in its top left corner. The bottom right diagonal element is set to be $(N^{-1})_{vv}$. Since all coefficients of the matrix N^{-1} are negative, the vector ${}^tR := (-p\,{}^t(N^{-1})_v, p)$ has positive coefficients. It is easy to check that $MR = 0$. Since ${}^t(N^{-1})_v$ is not an integer column, the coefficients of R are coprime. The triple (G, M, R) is an arithmetical graph and the vertex w_1 has multiplicity p. The top left minor of M has group killed by p since it is isomorphic to Φ_N. Thus we can apply the construction of Theorem 9.5 to (G, M, R) and w_1. □

Let us say that a matrix N *can be extended* if there exists a matrix N' that contains N and such that the diagonal coefficients of N are also on the diagonal of N'.

Corollary 9.15. *Let p be prime. Let N be a p-suitable intersection matrix of size n. Suppose that v is a vertex of Γ_N such that the diagonal element $(N^{-1})_{vv}$ is an integer. Then the matrix N can be extended to a larger p-suitable matrix N'.*

Proof. If the column $(N^{-1})_v$ is not an integer column, then the matrix $N' := N_m$ in Corollary 9.14 is the desired extension of N. If the column $(N^{-1})_v$ is an integer column, then the matrix $N' := \overline{N}$ in Theorem 3.4 (a) is an extension of N. □

Quotient Singularity 9.16. The following p-suitable matrix N and its vertex v on the left produce, using Corollary 9.14, new matrices N_m for every $m \geq 1$. The matrix N and all the new matrices N_m all arise as $\mathbb{Z}/p\mathbb{Z}$-quotient singularities. We indicate on the right the coefficients of the vector $R_0 = -p(N^{-1})_v$ such that ${}^tR_0N = (-p, 0, \ldots, 0)$. We have $|\Phi_N| = p^2$.



The matrix N is not numerically Gorenstein, and is shown to arise as a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity in [20], Theorem 6.8, or [21] Theorem 1.1, or [28], Corollary 7.13. For N_m, use [21] Theorem 1.3 and 3.12.

10. Existence of integer coefficients in N^{-1}

We investigate in this section when a p-suitable matrix N has the property that N^{-1} has an integer coefficient, or an integer column. The geometric motivation for studying this question comes from Theorem 9.2(b).

Let $N \in M_n(\mathbb{Z})$ be an intersection matrix with associated graph Γ. We let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{Z}^n$. When v is a vertex of Γ and no ordering of the vertices of Γ has been chosen, we let e_v denote the standard basis vector of $\mathbb{Z}^n$ associated with v. We let $\overline{v}$ denote the class of e_v in the quotient $\Phi_N := \mathbb{Z}^n/\mathrm{Im}(N)$.

Let $(N^{-1})_v$ denote the column of N^{-1} corresponding to v, so that we have $N(N^{-1})_v = e_v$. It follows that $\overline{v}$ is trivial in Φ_N if and only if the column $(N^{-1})_v$ is an integer vector.

Theorem 10.1. *Let p be a prime. Let $N \in M_n(\mathbb{Z})$ be an intersection matrix such that Φ_N is killed by p. Assume that the graph Γ associated with N is a star-shaped tree. If $|\Phi_N| \neq p$, then the matrix N^{-1} has at least one integer column.*

Proof. Let v_0 denote the unique node of Γ. Removing v_0 from Γ as well as all the edges of Γ adjacent to v_0 produces the disjoint union of m terminal chains $\Gamma_1, \dots, \Gamma_m$. Order the vertices of each terminal chain from the vertex connected to v_0 to the terminal vertex of the chain. Let N_{Γ_i} denote the intersection matrix of the chain with that ordering. Let $a_i > b_i$ denote the coprime integers associated with N_{Γ_i} in 2.3. In particular, there exists a vector ${}^tR_i = (b_i, \dots, 1)$ such that $({}^tR_i)N_{\Gamma_i} = (-a_i, 0, \dots, 0)$. Let $-c$ denote the diagonal element of N associated with the vertex v_0. Then, as seen in [25], Proposition 1.3, we have

$$|\det(N)| = \left(\frac{\prod_{i=1}^m a_i}{\operatorname{lcm}(a_1, \dots, a_m)}\right)\left(\operatorname{lcm}(a_1, \dots, a_m)(c - \sum_{i=1}^m b_i/a_i)\right).$$

Assume that Φ_N is killed by p and that for all vertices v of Γ, $\overline{v} \neq 0$. Then by Lemma 10.2 (b), we have a_i coprime to p for $i = 1, \dots, m$. It follows that $\frac{\prod_{i=1}^m a_i}{\operatorname{lcm}(a_1,\dots,a_m)}$ can only be a power of p if $\frac{\prod_{i=1}^m a_i}{\operatorname{lcm}(a_1,\dots,a_m)} = 1$. Hence,

$$|\det(N)| = \operatorname{lcm}(a_1, \dots, a_m)(c - \sum_{i=1}^m b_i/a_i).$$

It follows from [25], Proposition 1.3, that $\operatorname{lcm}(a_1, \dots, a_m)(c - \sum_{i=1}^n b_i/a_i)$ is the order of $\overline{v_0}$ in Φ_N. By hypothesis, this order is p. Hence, we have shown that $|\det(N)| = p$, as desired. □

Lemma 10.2. *Let p be a prime. Let $N \in M_n(\mathbb{Z})$ be an intersection matrix with associated graph Γ. Let $v_1, \dots, v_t$ denote the consecutive vertices of a terminal chain T of Γ. More precisely, assume that v_t is a terminal vertex (and so has degree 1), and that when $t > 1$, v_i is linked to v_{i+1} by exactly one edge for $i = 1, \dots, t-1$, and v_1 has degree 2. Let N_T denote the matrix of this chain, and let $s_0 > s_1 > \dots > s_t = 1$ denote the integers associated with this terminal chain as in 2.3. Letting ${}^tS := (s_1, \dots, s_t)$, we have ${}^tSN_T = (-s_0, 0, \dots, 0)$. Let v_0 be the vertex of Γ linked to v_1 that is not on the given terminal chain. Then*

(a) *We have $\overline{v}_0 = s_0\overline{v}_t$. We also have $s_1\overline{v}_0 = s_0\overline{v}_1$ and, more generally, $s_{i+1}\overline{v}_i = s_i\overline{v}_{i+1}$.*
(b) *Assume that the group Φ_N is killed by p, and that $\overline{v}_t \neq 0$ in Φ_N. Then the elements $\overline{v}_0, \dots, \overline{v}_{t-1}$ are not trivial in Φ_N if and only if p does not divide $s_0, s_1, \dots, s_{t-1}$.*

Let $-c_i$ denote the coefficient corresponding to the vertex v_i on the diagonal of N. When $p = 2$, the condition p does not divide $s_0, s_1, \dots, s_{t-1}$ is equivalent to the condition $c_1, \dots, c_{t-1}$ are even and c_t is odd.

Proof. (a) Let us start by showing that $\overline{v}_0 - s_0\overline{v}_t = 0$. For this, it suffices to show that the vector $e_{v_0} - s_0e_{v_t}$ is in the image of N. Recall that $|\det(N_T)| = s_0$. Consider the sequence $1 = r_1 < r_2 < \dots < r_t < s_0$ such that, letting ${}^tR := (1, r_2, \dots, r_t)$, we have $({}^tR)N_T = (0, \dots, 0, -s_0)$. Let N_v denote the column of N corresponding to the vertex v. We obtain

$$e_{v_0} - s_0e_{v_t} = N_{v_1} + r_2N_{v_2} + \dots + r_tN_{v_t},$$

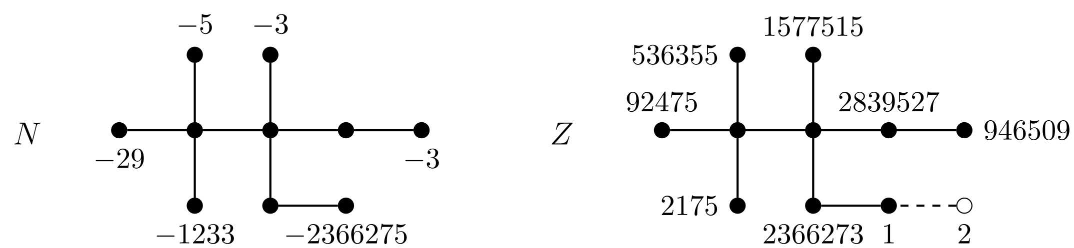
as desired. Shortening the chain $v_1, \dots, v_i, \dots, v_t$ to $v_i, \dots, v_t$ and applying the above result to the new chain $v_i, \dots, v_t$ shows that $\overline{v}_{i+1} - s_{i+1}\overline{v}_t = 0$. Using the relations $\overline{v}_i - s_i\overline{v}_t = 0$ and $\overline{v}_{i+1} - s_{i+1}\overline{v}_t = 0$ gives us $s_{i+1}\overline{v}_i = s_i\overline{v}_{i+1}$.

(b) Assume that $\overline{v}_t$ has order p and that the group Φ_N is killed by p. Suppose that there exists an index i in $[0, t-1]$ such that $\overline{v}_i$ is trivial. The relation $s_i\overline{v}_{i+1} = s_{i+1}\overline{v}_i$ shows then that either p divides s_i, as desired, or that $\overline{v}_{i+1} = 0$. We can repeat the argument with $\overline{v}_{i+1}$ if

p does not divide s_i. Since we assume that $\overline{v}_t$ is not trivial, we must have s_j divisible by p for some $j \in [i, t-1]$.

Assume now that p divides s_i for some $i \in [0, t-1]$. Then the relation $s_i\overline{v}_{i+1} = s_{i+1}\overline{v}_i$ shows that either $\overline{v}_i$ is trivial as desired, or that p divides s_{i+1}. Repeating the process if $\overline{v}_i$ is not trivial, we find that either one of the $\overline{v}_j$ is trivial, as desired, or we have p dividing $s_t = 1$, which is a contradiction. □

Intersection Matrix 10.3. We exhibit below a 2-suitable matrix N of size $n = 10$ with $|\Phi_N| = 2$ and such that N^{-1} does not have any integer coefficient. Suppose that such N exists. Then the condition $|Z^2| \leq 2$ implies that N contains a principal square submatrix N' of size $n-1$ such that $|\det(N')| = 1$. The terminal chains of the graph of N also have to satisfy the conditions of Lemma 10.2. It turns out that Graph (11) in Table I of [5] exhibits an 9×9 intersection matrix N' satisfying the above conditions, and from which one can obtain the desired example N of size $n = 10$:



The group Φ_N has order 2 and $Z^2 = -2$. The matrix N^{-1} has no integer coefficient. As the reader will have noted, one coefficient of N is very negative compared to the size of $|\Phi_N|$. The graph (5) in Table I of [5] is star-shaped on 9 vertices and leads to a similar example.

Proposition 10.4. *Let p be prime. Let $N \in M_n(\mathbb{Z})$ be a p-suitable intersection matrix such that $|\Phi_N| = p$. If N^{-1} has an integer coefficient, then it has an integer column.*

Proof. Let us assume that N^{-1} has no integer columns. Let N_j^{-1} denote the j-th column of N^{-1}. Since Φ_N is killed by p, we have $pN_j^{-1} \in \mathbb{Z}^n$ and $N(pN_j^{-1}) = pe_j$, showing that the class of e_j has order dividing p. Since the vector N_j^{-1} has at least one non-integer coefficient by hypothesis, we find that the class of e_j has order exactly p in $\Phi_N := \mathbb{Z}^n/\mathrm{Im}(N)$.

Assume now that $|\Phi_N| = p$. If $N = (-p)$, the proposition is true. Assume that $n \geq 2$. Since the matrix N is symmetric, for any $k \leq n$, we can find $j \leq n$ such that the j-th coefficient of the k-th line is not an integer. Let $i \neq j$. Then there exists an integer a_i coprime to p such that the class of $a_ie_i - e_j$ is trivial in Φ_N, since both the classes of e_i and of e_j have exact order p. Since we have $N(a_iN_i^{-1} - N_j^{-1}) = a_ie_i - e_j$, we find that $(a_iN_i^{-1} - N_j^{-1})$ must be an integer vector. Thus, if the k-th coefficient of N_j^{-1} is not an integer, then the k-th coefficient of N_i^{-1} cannot be an integer. It follows that the matrix N^{-1} has a k-th row, none of whose coefficients are integers. □

Proposition 10.5. *Let N be a p-suitable intersection matrix such that its associated graph Γ is a chain. Then the matrix N^{-1} has no integer coefficient.*

Proof. First, since Γ is a chain, the group Φ_N is always a non-trivial cyclic group ([19], Lemma 3.13). Since N is p-suitable, we find that $|\Phi_N| = p$. It follows from 10.4 that to show that N^{-1} has no integer coefficient, it suffices to show that it has no integer column. For this, we use 10.2 (a). Keeping the notation introduced in 10.2, let $v_0, v_1, \dots, v_t$ denote the consecutive vertices of Γ. We have $p = \det(-N) > s_0 > s_1 > \cdots > s_t$. In particular, every integer s_i is coprime to p for $i = 0, \dots, t-1$. Since $s_{i+1}\overline{v}_i = s_i\overline{v}_{i+1}$, we find that none of the classes $\overline{v}_i$ can be trivial, because if one were trivial, all would be, and this would contradict the fact that $\det(-N) > 1$. □

Proposition 10.6. *Let Γ be a connected graph on n vertices $v_1, \dots, v_n$. Let d_i denote the degree of the vertex v_i. Let $N := \mathrm{Diag}(-c_1, \dots, -c_n) + \mathrm{Ad}(\Gamma)$. Assume that $c_i \in \mathbb{Z}_{>d_i}$ for all $i = 1, \dots, n$. Then the matrix N is negative definite, has fundamental cycle ${}^tZ = (1, \dots, 1)$, and its inverse N^{-1} does not have any integer on its diagonal. In particular, N^{-1} does not have any integer column.*

Proof. Our hypothesis on the diagonal of N shows that $P := -N$ is strictly row diagonally dominant (see [13], p. 124). It follows then from Theorem 2.5.12 in [13] that P^{-1} is strictly diagonally dominant. This means that if we write $P^{-1} = ((q_{ij}))$, then for each i, and for all $j \neq i$, $q_{ii} > q_{ij}$.

Note that since $(1, \dots, 1)N > 0$, we find that N is negative definite (3.1). Without loss of generality, it suffices to prove that the diagonal element q_{11} on the first line Q_1 of P^{-1} is not an integer. Let N_1 denote the first column of N. Then

$$\begin{array}{rcl} Q_1(-N_1) & = & q_{11}c_1 - \sum_{j=2}^{n} q_{1j}\mathrm{Ad}(\Gamma)_{1j} = 1 \\ (1, \dots, 1)(-N_1) & = & c_1 - \sum_{j=2}^{n} \mathrm{Ad}(\Gamma)_{1j} = c_i - d_i \geq 1. \end{array}$$

It follows that $\sum_{j=2}^{n}(q_{1j} - q_{11})\mathrm{Ad}(\Gamma)_{1j} \geq q_{11} - 1$. Because the graph is connected, we find that the left hand side of the inequality is always strictly negative. On the other hand, if q_{11} is a (positive) integer, then $q_{11} - 1 \geq 0$, which is not possible. □

Remark 10.7. In the above proposition, assume only the weaker condition that $c_i \in \mathbb{Z}_{\geq d_i}$ for all i. One may wonder whether it would remain true under this weaker hypothesis that if $c_i > d_i$, then the diagonal coefficient $(N^{-1})_{ii}$ cannot be an integer.

Proposition 10.8. *Let $N \in M_n(\mathbb{Z})$ be an intersection matrix with fundamental vector Z.*
(a) *If $({}^tZ)NZ = -1$, then $-Z$ is an integer column of the matrix N^{-1}.*
(b) *Let $R_i \in \mathbb{Z}^n$ be as in 3.2. If $({}^tR_i)NR_i = ({}^tZ)NZ$, then $R_i = Z$. In particular, if for some i, $({}^tR_i)NR_i = -1$, then $R_i = Z$.*
(c) *Assume that $|\Phi_N| = 1$. Then the vector $-Z$ is a column of N^{-1}. More precisely, let z denote the minimum of the coefficients of the matrix $-N^{-1}$. Then there exists an index $j \in [1, n]$ such that $z = (-N^{-1})_{jj}$ and the vector $-Z$ is equal to the j-th column of N^{-1}.*

Proof. (a) Write ${}^tZ = (z_1, \dots, z_n)$. Recall that Z is a positive vector, and that NZ is a non-positive vector. It follows that when $({}^tZ)NZ = -1$, there exists an integer $i \in [1, n]$ such that $NZ = -e_i$ and $z_i = 1$. Hence, $-Z$ is equal to the i-th column of N^{-1}.

(b) Since by construction, R_i has positive coefficients and NR_i is a non-positive vector, the minimality property of Z implies that $Z \leq R_i$. Let us then write $R_i = Z + X$ with $X \geq 0$. It follows that

$$R_i^2 = Z^2 + 2({}^tZ)NX + X^2.$$

Since N is definite negative, $X^2 \leq 0$. Since $NZ \leq 0$ and $X \geq 0$, we have $({}^tZ)NX \leq 0$. Hence, $R_i^2 \leq Z^2$, and when $R_i^2 = Z^2$, we must have $X^2 = 0$ or, equivalently, $R_i = Z$.

Assume now that $({}^tR_i)NR_i = -1$. Since $R_i^2 \leq Z^2 \leq -1$, we find that in this case $R_i^2 = Z^2$, so that $R_i = Z$.

(c) Write $R_i = Z + X_i$ with $X_i \geq 0$. It follows that

$$Z \cdot R_i = -p_i z_i = Z^2 + Z \cdot X_i.$$

Since $|\Phi_N| = 1$, we find that $p_i = 1$. Let $j \in [1, n]$ be such that $z_j = \min_i(z_i)$. Since Z^2 is a linear combination (with negative coefficients) of $z_1, \ldots, z_n$, we find that the only possibility to have $-z_j = Z^2 + Z \cdot X_j$ with $Z \cdot X_j \leq 0$ is to have $NZ = -e_j$. Since $NR_j = -e_j$, we must have $Z = R_j$, as desired. Now that we know that Z is a column of $-N^{-1}$, and that $R_i \geq Z$ for all i, we find that z must be a coefficient of R_j. Since the matrix N^{-1} is symmetric, z must be the diagonal element of the j-th column of $-N^{-1}$. □

Theorem 10.9. *Let N be a 2-suitable intersection matrix with fundamental vector Z. Then there exists a column R of N^{-1} such that either $Z = -R$ or $Z = -2R$.*

Proof. Proposition 10.8 (a) shows that if $({}^tZ)NZ = -1$, then $-Z$ is an integer column of the matrix N^{-1}. Let ${}^tZ := (z_1, \ldots, z_n)$. Suppose now that $({}^tZ)NZ = -2$. Then either

(a) There exists $i \in [1, n]$ such that $NZ = -e_i$ and $z_i = 2$, or
(b) There exists $i \in [1, n]$ such that $NZ = -2e_i$ and $z_i = 1$, or
(c) There exist $i, j \in [1, n]$, $i \neq j$, such that $NZ = -e_i - e_j$ and $z_i = z_j = 1$.

In the first case, $-Z$ is a column of N^{-1}. The second case can only happen when N is 2-suitable, and in this case $-Z/2$ is a column of N^{-1}.

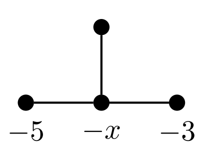
Assume now that we are in the third case and that Φ_N is killed by a prime p. Recall that $NR_i = -p_i e_i$ and $NR_j = -p_j e_j$ for some $p_i, p_j \in \{1, p\}$. It follows that $Z = R_i/p_i + R_j/p_j$. Since Z is an integer vector, such equality can only happen if $p_i = p_j = 1$ or $p_i = p_j = p$. Since Z is the fundamental cycle, we must have $Z \leq R_i$ and $Z \leq R_j$, and therefore only the case $p_i = p_j = p$ can happen, and we have $pZ = R_i + R_j$.

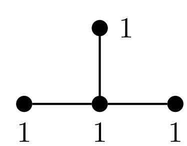
The hypothesis that $NZ = -e_i - e_j$ shows that $Z < R_i$ and $Z < R_j$. We obtain then that $2Z < R_i + R_j = pZ$. When $p = 2$, this inequality is impossible, and so (c) cannot happen. □

Remark 10.10. When $p > 2$, it may happen that the fundamental cycle is not related only to a single column of N^{-1}. The construction in Theorem 3.9 for instance is likely to produce p-suitable intersection matrices N with fundamental vector Z such that NZ has two non-zero coefficients (see 3.8 when $p = 3$ and $Z^2 = -2$ as in (c) above, or [23], 5.8 when $p = 5$).

We do not know how to construct families of p-suitable intersection matrices N such that NZ has at least three non-zero coefficients and $n \geq p$. Two sporadic such examples are found in [23], 5.1 with $p = 5$ and $n = 7$, and in [23], 6.17, with $p = 7$ and $n = 10$. It would be interesting to determine if such example exists when $p = 3$.

When p is large compared to n, such examples are easy to produce. Let $x \geq 4$ and consider the intersection matrix N below where NZ has $n = 4$ non-zero entries:

Diagonal of N


Coefficients of Z


The associated group Φ_N has order $30x - 31$. We have $|Z^2| = x + 4$. Dirichlet's Theorem implies the existence of infinitely many integers $x \geq 4$ such that $30x - 31$ is prime. For such $p := 30x - 31$, the matrix N is p-suitable since $x + 4 < 30x - 31$.

11. Bound for the size of the discriminant group

Let N be a p-suitable intersection matrix with associated graph Γ. When Z is a multiple of a column of N^{-1}, we produce below a tight upperbound for $|\Phi_N|$.

Let d_i denote the degree of the vertex $v_i \in \Gamma$. Write $N = -\mathrm{Diag}(c_1, \dots, c_n) + \mathrm{Ad}(\Gamma)$ with $c_1, \dots, c_n \geq 2$. Recall from 2.2 that given any vector $R \in \mathbb{Z}^n$ with ${}^tR = (r_1, \dots, r_n)$, we have ${}^tRNK = \sum_{i=1}^n r_i(c_i - 2)$. Let now $R = R_j \in \mathbb{Z}^n$ be as in 3.2, with $NR_j = -p_j e_j$ and $p_j \in \{1, p\}$. Then

$$(1, \dots, 1)NR = -p_j = \sum_{i=1}^n (-c_i + 2)r_i + \sum_{i=1}^n (d_i - 2)r_i,$$

and we find that

$$ {}^tRNK = p_j + \sum_{i=1}^n (d_i - 2)r_i. \tag{11.1}$$

Assume in addition that $R = R_j = Z$. Then $({}^tR)NR = -p_j r_j = Z^2 \geq -p$. Hence, $p_j r_j \leq p$.

Theorem 11.1. *Let N be a p-suitable intersection matrix. Let $p(Z)$ denote the arithmetic genus of the fundamental cycle Z of N. Let K be the canonical cycle of N.*

(a) *Assume that $-Z$ is a column of N^{-1}. Then $\log_p(|\Phi_N|)(p-1) \leq Z \cdot K + p$.*
(b) *Assume that $-Z/p$ is a column of N^{-1}. Then $\log_p(|\Phi_N|)(p-1) \leq Z \cdot K + 1$.*
(c) *Let $p = 2$. Then $|\Phi_N|$ divides $p^{2p(Z)+2}$.*

Proof. Recall the expression obtained in [19], Theorem 3.14, for the size of Φ_N using the vector $R := R_j$:

$$|\det(N)| = p_j r_j \cdot \prod_{i=1}^n r_i^{(d_i - 2)}.$$

This expression is strikingly similar to the expression for ${}^tRNK$ found in (11.1). We are going to relate these two expressions using an arithmetical graph obtained from N, to which we will apply the following result on arithmetical graphs.

Let $(G, M, \overline{R})$ be an arithmetical tree on s vertices, as in 9.3. Let ${}^t\overline{R} := (\overline{r}_1, \dots, \overline{r}_s)$, with $\gcd(\overline{r}_1, \dots, \overline{r}_s) = 1$. Let δ_i denote the degree of the vertex $v_i \in G$. The main integer invariant associated with $(G, M, \overline{R})$ is given by the formula

$$2g_0 - 2 := \sum_{i=1}^s \overline{r}_i(\delta_i - 2).$$

Let Φ_M denote the torsion subgroup of $\mathbb{Z}^s/\mathrm{Im}(M)$. Then [17], Theorem 4.7, shows that

$$\sum_{q \text{ prime}} \mathrm{ord}_q(|\Phi_M|)(q-1) \leq 2g_0.$$

(a) Let us assume now that $-Z = R_j$ is a column of N^{-1}, so that $p_j = 1$ and $r_j \leq p$. Using N, Γ, and R_j, we construct the following arithmetical graph $(G, M, \overline{R})$. The tree G is obtained from Γ by attaching with a single edge one new vertex w to the vertex v_j of Γ. The matrix M has then size $s = n + 1$. Assuming that the order of the vertices of G are $v_1, \dots, v_n, w$, we set the diagonal of M to be $(-c_1, \dots, -c_n, -r_j)$. Set ${}^t\overline{R} := (r_1, \dots, r_n, 1)$, and note that $M\overline{R} = 0$. It is easy to check that $|\Phi_M| = |\Phi_N|$ and that

$$\begin{aligned} 2g_0 - 2 &= \textstyle\sum_{i=1}^s \overline{r}_i(\delta_i - 2) = \sum_{i=1}^n r_i(d_i - 2) + r_j - 1 \\ &= ({}^tR)NK - p_j + r_j - 1. \end{aligned}$$

Hence, $2g_0 = ({}^tR)NK + r_j$. Since Φ_N is a p-group by hypothesis, we find that

$$\log_p(|\Phi_N|)(p-1) = \sum_{q \text{ prime}} \mathrm{ord}_q(|\Phi_M|)(q-1) \leq 2g_0 \leq Z \cdot K + p.$$

(b) Let us assume now that $-Z = R_j$ is such that $p_j = p$, so that $r_j = 1$. Using N, Γ, and R_j, we again construct an associated arithmetical graph $(G, M, \overline{R})$. Let $w_0, w_1, \dots, w_{p-1}$ denote the ordered vertices of a chain of length p. The tree G is obtained from Γ by attaching with a single edge the vertex w_0 to the vertex v_j of Γ. The matrix M has then size $s = n + p$. Assuming that the order of the vertices of G are $v_1, \dots, v_n, w_0, \dots, w_{p-1}$, we set the diagonal of M to be $(-c_1, \dots, -c_n, -1, -2, \dots, -2)$. Set ${}^t\overline{R} := (r_1, \dots, r_n, p, p-1, \dots, 2, 1)$, and note that $M\overline{R} = 0$. Computing in two different ways the determinant of the matrix $M^{n+1,n+1}$ obtained by removing the row and column of M corresponding to the vertex w_0 gives

$$|\det(M^{n+1,n+1})| = |\Phi_M|p^2 = |\Phi_N|p.$$

Finally, we find that

$$\begin{aligned} 2g_0 - 2 &= \textstyle\sum_{i=1}^s \overline{r}_i(\delta_i - 2) = \sum_{i=1}^n r_i(d_i - 2) + r_j - 1 \\ &= ({}^tR)NK - p_j + r_j - 1. \end{aligned}$$

Hence, $2g_0 = ({}^tR)NK - p_j + 2$. Since Φ_N is a p-group by hypothesis, we find that

$$\log_p(|\Phi_N|/p)(p-1) = \sum_{q \text{ prime}} \mathrm{ord}_q(|\Phi_M|)(q-1) \leq 2g_0 = Z \cdot K - p + 2.$$

It follows that $\log_p(|\Phi_N|)(p-1) \leq Z \cdot K + 1$.

(c) Recall that when $p = 2$, Theorem 10.9 shows that either $-Z$ or $-Z/2$ is a column of N^{-1}. We can thus apply (a) and (b) to obtain that $\log_2(|\Phi_N|) \leq Z \cdot K + 2$. Then $2p(Z) + 2 = Z \cdot K + Z^2 + 4 \geq Z \cdot K + 2$, since by hypothesis, a 2-suitable matrix satisfies $Z^2 + 2 \geq 0$. □

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