COMPENDIUM OF KNOWN SMALL WILD $\mathbb{Z}/p\mathbb{Z}$-QUOTIENT SINGULARITIES OF SURFACES

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Contents

1. Introduction

This compendium of small intersection matrices complements the article [13], and we briefly review below the notation used in [13]. Most equations of quotient singularities in this article are found in [14] Theorem 7.7, and [13] Theorem 10.6. Certain equations when $p = 2$ or $p = 3$ are found already in the foundational papers [1] and [18]. Explicit desingularisations can be computed using Magma [3].

Let p be a prime. Let k be an algebraically closed field of characteristic p. Let $A := k[[u, v]]$ denote the formal power series ring in two variables. Assume that $G := \mathbb{Z}/p\mathbb{Z}$ acts on A, and let A^G denote the ring of invariants. We will say that the closed point of $\mathrm{Spec}(A^G)$ is a *cyclic wild quotient singularity*, where the term *wild* refers to the fact that the order of the group G is divisible by the characteristic p.

Let $\pi : X \to \mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ be a resolution of the singularity, so that in particular X is a regular scheme. Let C_i, $i = 1, \dots, n$, denote the irreducible components of the exceptional divisor of π, and form the *intersection matrix*

$$N := ((C_i \cdot C_j)_X)_{1\leq i,j\leq n} \in M_n(\mathbb{Z}),$$

where $(C_i \cdot C_j)_X$ denotes the intersection number of C_i and C_j computed on the regular surface X. Attached to the resolution π is its *dual graph* Γ_N, with vertices $v_1, \dots, v_n$, where v_i and v_j are linked by $(C_i \cdot C_j)_X$ distinct edges when $i \neq j$. Let $\mathrm{Ad}(\Gamma_N)$ denote the adjacency matrix of

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the graph Γ_N. The matrix N has the form $\mathrm{Diag}(c_{11}, \dots, c_{nn}) + \mathrm{Ad}(\Gamma_N)$, where $c_{ii} = (C_i \cdot C_i)_X$ is the self-intersection number of C_i. It is well-known that the matrix N is negative-definite. The following is also known about such matrices N:

(i) When the exceptional divisor of π has smooth components with normal crossings, the components C_i are smooth projective lines and the graph Γ_N is a tree ([10], Theorem 2.8).

(ii) The *discriminant group* $\Phi_N := \mathbb{Z}^n/\mathrm{Im}(N)$ is an elementary abelian p-group ([10], Theorem 2.6), so that in particular $|\Phi_N| = |\det(N)| = p^s$ for some integer $s \geq 0$.

(iii) The *fundamental cycle* $Z \in \mathbb{Z}^n_{>0}$ of N is the minimal positive vector such that NZ is a non-positive vector. The self-intersection $Z \cdot Z := ({}^t Z)NZ$ is such that $|Z \cdot Z| \leq p$ ([10], Theorem 2.4).

Let p be any prime. Motivated by the above theorems, we call an intersection matrix $N \in M_n(\mathbb{Z})$ *p-suitable* if it satisfies the following linear algebraic properties:

(a) There exists a connected tree Γ on n vertices, and integers $c_1, \dots, c_n \geq 2$, such that $N = \mathrm{Diag}(-c_1, \dots, -c_n) + \mathrm{Ad}(\Gamma)$.

(b) The matrix N is negative definite and the group Φ_N is killed by p.

(c) The fundamental cycle Z of N is such that $|Z \cdot Z| \leq p$.

We will say that a p-suitable intersection matrix N *arises from a quotient singularity* if there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity $\mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ with a resolution of singularities $\pi : X \to \mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$ such that all irreducible components C_i of the exceptional divisor E of π are *smooth*, and such that up to a choice of the ordering of the irreducible components C_i, the intersection matrix associated with E is equal to the given matrix N.

We give in this compendium a list of p-suitable intersection matrices N for $p = 2, 3, 5$ and 7, and for small n. In each case we also indicate if the given matrix N is known to arise from a $\mathbb{Z}/p/\mathbb{Z}$-quotient singularity. A more complete study of such matrices is done in [13].

2. Notation

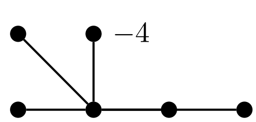
We will use the following conventions to describe intersection matrices. Let $N \in M_n(\mathbb{Z})$ be a p-suitable intersection matrix whose associated graph is a connected tree Γ on n vertices $v_1, \dots, v_n$. Thus by our definition, there exist integers $c_1, \dots, c_n \geq 2$, such that

$$N = \mathrm{Diag}(-c_1, \dots, -c_n) + \mathrm{Ad}(\Gamma).$$

In this article, we will describe N using its tree Γ, and adorn each vertex v_i with the negative integer $-c_i$. We follow the established custom and omit to adorn v_i if the integer $-c_i$ is -2.

Example 2.1. We use the decorated tree Γ on the left in (a) below to represent the 6×6-matrix N on the right after having made a choice of ordering of the vertices of the tree Γ.

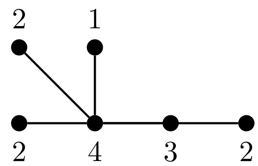
(a)

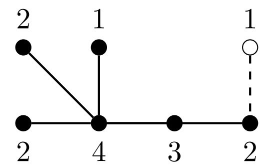


$$N = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Let N be any intersection matrix. Let $Z \in \mathbb{Z}^n_{>0}$ denote the fundamental cycle of N. We represent the vector Z with ${}^t Z := (z_1, \dots, z_n)$ by adorning the vertex v_i of Γ with the positive integer z_i. In the case of the above matrix N, we have ${}^t Z := (4, 2, 2, 1, 3, 2)$, which we record on the left below.

We found it efficient to record the vector NZ on the same drawing as we draw the vector Z. We use the following convention. Let ${}^t(NZ) = (s_1, \ldots, s_n)$, with $s_i \leq 0$ for all $i = 1, \ldots, n$. For each index i such that $s_i \neq 0$, add a white vertex to the graph of Γ, and link it with a dashed line to the vertex v_i. Adorn the new white vertex with the coefficient $|s_i|$. In the example of the matrix N above, we find that ${}^t(NZ) = (0, \ldots, 0, -1)$, which we record in (b) on the right below.

Z 

(b) 

Note that the information provided in the diagram (b) above, namely, the graph Γ, the vector Z, and the vector NZ, allows the recovery of the diagonal elements of the matrix N, and thus this data is sufficient to describe N itself. For the convenience of the reader, we will often include the information of the diagonal of N explicitly, and will provide a pair of diagrams as in (a) and (b) above to describe a matrix N, even if only one diagram would suffice.

The drawing of Z and NZ allows for a quick computation of the self-intersection $|Z^2| := |({}^tZ)NZ|$ by simply multiplying the integers linked by dashed lines, and adding the results of the multiplications together. In the example above, we find that $|Z^2| = 1 \cdot 2 = 2$.

Note that in the given example, NZ is equal, up to a sign, to a standard vector of $\mathbb{Z}^n$. When such is the case and Γ is any tree, the drawing of ${}^tZ = (z_1, \ldots, z_n)$ allows for a quick computation of $|\Phi_N|$. Indeed, let d_i denote the degree in Γ of the vertex v_i. If $NZ = -e_j$, then $|\Phi_N| = z_j \prod_{i=1}^n z_i^{d_i - 2}$ (use [10], Theorem 3.14). For instance, in the example above, we obtain that $|\Phi_N| = 2\frac{4^2}{2 \cdot 2 \cdot 2} = 4$. When the order of Φ_N is not prime, the precise group structure of Φ_N needs to be determined using for instance the Smith Normal form of N.

2.2. When describing an intersection matrix N in later sections, we might also indicate whether N is numerically Gorenstein. Recall that this is a purely linear algebraic condition which can be expressed as follows. Write $N = \mathrm{Ad}(\Gamma_N) - \mathrm{Diag}(c_1, \ldots, c_n)$, with $c_i \geq 2$ for $i = 1, \ldots, n$. Let ${}^tH := (c_1 - 2, \ldots, c_n - 2)$. Since N is invertible, the equation $NK = H$ has a unique solution $K \in \mathbb{Q}^n$. The vector K is called the *canonical cycle* of N.

The $n \times n$ intersection matrix N is *numerically Gorenstein* if $K \in \mathbb{Z}^n$. If a p-suitable intersection matrix arises from a hypersurface quotient singularity, then the matrix N is numerically Gorenstein (see [15], Lemma 10.3). In the explicit example introduced above, the matrix N is numerically Gorenstein because every 2-suitable intersection matrix is numerically Gorenstein ([15], Proposition 10.5).

In later sections, we will draw lists of p-suitable matrices for small p and small n. We will title each paragraph describing a p-suitable intersection matrix N by either **Intersection Matrix** or **Quotient Singularity**. By convention, we use the title **Intersection Matrix** when we do not know whether the p-suitable intersection matrix N described in that section actually arises as a quotient singularity. This is the case in particular for the matrix N described in 2.1, which is also found in Intersection Matrix 4.9. When $p = 2$, this matrix N is the smallest for which we do not know if it arises from a quotient singularity. When we know that a given p-suitable intersection matrix N arises as a quotient singularity, we use the title **Quotient Singularity** and we include a description of the quotient singularity.

3. Paths in small characteristics

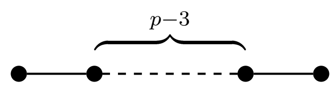
Let $N \in M_n(\mathbb{Z})$ be an intersection matrix whose associated graph is a path on n vertices. It is known that the group Φ_N is always cyclic. If all the diagonal coefficients of N are equal to -2, then we have $|\Phi_N| = n+1$, and in general if the coefficients are at most -2, then $|\Phi_N| \geq n+1$.

Intersection Matrix 3.1. Let p be prime. Consider the p-suitable intersection matrix $N = (-p)$, with $Z^2 = -p$. The graph of N is reduced to a single vertex with no edges.

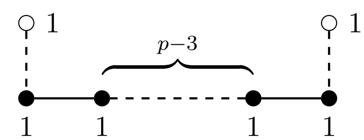
Quotient Singularity 3.2. The only known case where the matrix $N = (-p)$ arises as a quotient singularity is when $p = 2$. See [13], 11.8.

Quotient Singularity 3.3. Let $p \geq 3$ be prime. Let Γ denote the path on $p-1$ vertices.

Diagonal of N



Z



The associated group Φ_N has order p, and $Z^2 = -2$. This intersection matrix arises as a quotient singularity (singularity of type A_{p-1}) (see [15], Theorem 9.4).

Intersection Matrix 3.4. Let $p \leq 7$ be a prime. There are only three additional p-suitable intersection matrices whose associated graph is a path. When $p = 5$, we have

$$N = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}, \quad |\Phi_N| = 5, \quad \text{and } Z^2 = -3.$$

When $p = 7$, we have

$$N = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}, \quad \text{or} \quad N = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix},$$

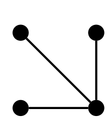
both with $|\Phi_N| = 7$, and $Z^2 = -4$ (resp. -3). None of these matrices are numerically Gorenstein.

4. Small trees in characteristic 2

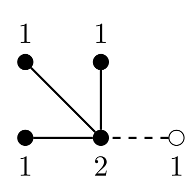
We start by listing below the twelve 2-suitable intersection matrices of size $n \leq 6$ with a graph having at least one node (see also 3.2). All but one are known to arise from quotient singularities. We then provide a partial list of 2-suitable intersection matrices of size $n = 7$. Recall that when $p = 2$, any 2-suitable intersection matrix is automatically numerically Gorenstein ([15], 10.5), and any $\mathbb{Z}/2\mathbb{Z}$-quotient singularity is automatically a hypersurface ([15], 10.1).

Quotient Singularity 4.1. ($n = 4$)

Diagonal of N

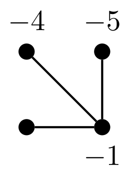


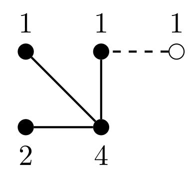
Coefficients of Z



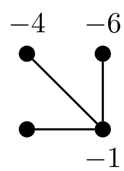
We have $Z^2 = -2$. The associated group Φ_N has order 2^2. This matrix arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2 ([15], 8.3). It is the resolution of the hypersurface singularity $f = z^2 + x^3 + y^3 = 0$. The matrix N^{-1} has only one integer column, which, with the help of [13], Theorem 3.4, can be used to produce the two matrices in 4.3.

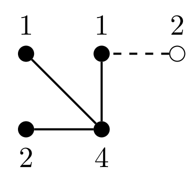
Quotient Singularity 4.2. ($n = 4$) Recall that in our definition of a 2-suitable intersection matrix N, we require that no self-intersection on the diagonal is equal to -1. When we do not impose this restriction, we found the following quotient singularities with a resolution with four smooth components having normal crossings.

(a1) Diagonal of N 

Coefficients of Z 

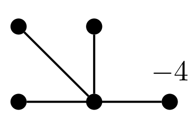
We have $Z^2 = -1$. The associated group Φ_N has order 2. This matrix arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2 ([13], Theorem 7.6). It is the resolution of the hypersurface singularity $f = z^2 + x^5y + y^3 = 0$, and also of $f = z^2 + x^9y + y^5 = 0$. Blowing-up this latter singularity produces the resolution is our next example.

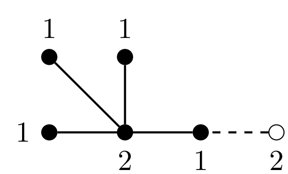
(a2) Diagonal of N 

Coefficients of Z 

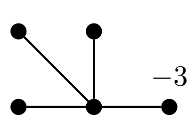
We have $Z^2 = -2$. The associated group Φ_N has order 2^2. This matrix arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2 ([13], Theorem 7.6). It is the resolution of the hypersurface singularity $f = z^2 + x^6y + xy^5 = 0$.

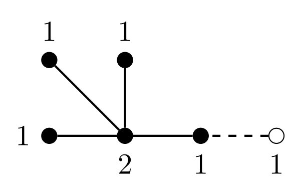
Quotient Singularity 4.3. ($n = 5$) The next two intersection matrices are obtained from the Example 4.1 using [13], Theorem 3.4.

(a1) Diagonal of N 

Coefficients of Z 

We have $Z^2 = -2$. The associated group Φ_N has order 2^3. The matrix N^{-1} has only one integer column, which, with the help of [13], Theorem 3.4, can be used to produce the two matrices in 4.6. This intersection matrix arises as the resolution matrix of the singularity $f := z^p - (aby)^{p-1}z + a^pxy + b^py$ with $a := x$ and $b := y^3$. It is also the resolution of the weighted homogeneous quotient singularity given by $g = z^2 + x^3y + y^7$. This latter equation defines a chart of the blow-up of $h = z^2 + x^3 + y^9$ appearing below.

(a2) Diagonal of N 

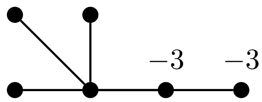
Coefficients of Z 

We have $Z^2 = -1$. The associated group Φ_N has order 2^2. This is the resolution of $z^2+x^3+y^9$. The matrix N^{-1} has two integer columns, which, with the help of [13], Theorem 3.4, can be used to produce the matrices in 4.4 and 4.5.

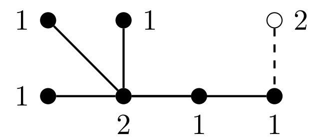
We list below the nine 2-suitable matrices of size $n = 6$. We indicate for each of them the number of possible extensions to 2-suitable matrices of size $n = 7$ that can be obtained using [13], Theorem 3.4. As the reader will note, the total number of 2-suitable matrices of size $n = 7$ is very large already and we will not list them all in this article.

Quotient Singularity 4.4. ($n = 6$) The matrices below are obtained from the matrix 4.3 (a2) in the previous example using [13], Theorem 3.4.

(a1) Diagonal of N

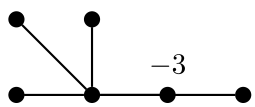


Coefficients of Z

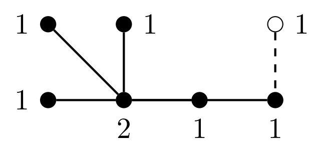


We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This matrix arises from the case $a := x$ and $b := y^6$ in the equation $f := z^p - (abxy)^{p-1}z - a^pxy + b^py = 0$ when $p = 2$. The matrix N^{-1} has a unique integer column, which produces two new 2-suitable matrices with $n = 7$, exhibited in 4.16.

(a2) Diagonal of N



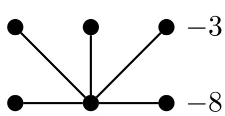
Coefficients of Z



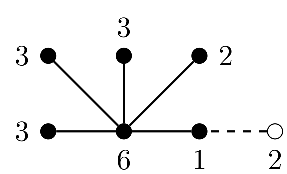
We have $Z^2 = -1$. The associated group Φ_N has order 2^2. This matrix arises as a Brieskorn singularity with $z^2 + x^3 + y^{15}$. The matrix N^{-1} has three integer columns, which produce six new 2-suitable matrices with $n = 7$. Four of these matrices are known to arise from quotient singularity, including 4.15, and two are not (see 4.26).

Quotient Singularity 4.5. ($n = 6$) The matrices below are extensions of the matrix 4.3 (a2).

(a1) Diagonal of N

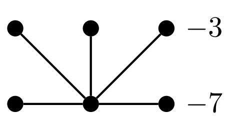


Coefficients of Z

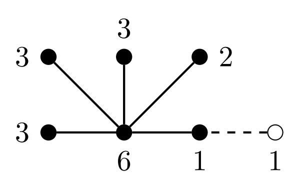


We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This matrix corresponds to the quotient singularity ramified in codimension 1 given by $f := z^2 + x^4y^7z + x^9y + y^{13}$, part of the family $f := z^p - (aby)^{p-1}z - a^pxy + b^py$ with $a = x^4$ and $b = y^6$. The matrix N^{-1} has two integer columns, which produce four new 2-suitable matrices with $n = 7$.

(a2) Diagonal of N

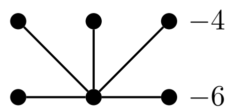


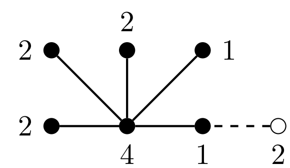
Coefficients of Z



We have $Z^2 = -1$. The associated group Φ_N has order 2^2. This matrix appears with the Brieskorn singularity $x^9 + y^{21} + z^2$. The matrix N^{-1} has three integer columns, which produce six new 2-suitable matrices with $n = 7$. Two of these extensions are associated with new quotient singularities, in 4.18 and 4.17.

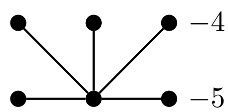
Quotient Singularity 4.6. **(**$n = 6$**)** The first two matrices below are extensions of the matrix 4.3 (a1).

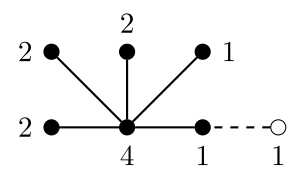
(a1) Diagonal of N 

Coefficients of Z 

We have $Z^2 = -2$. The associated group Φ_N has order 2^4. This matrix corresponds to the quotient singularity ramified in codimension 1 given by $f := z^2 + x^5y^4z + x^{10}y + xy^7$ part of the family $f := z^p - (aby)^{p-1}z - a^pxy + b^py$ with $a = y^3$ and $b = x^5$.

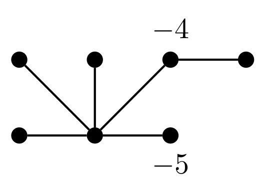
The matrix N^{-1} has one integer column, which produce two new 2-suitable matrices with $n = 7$.

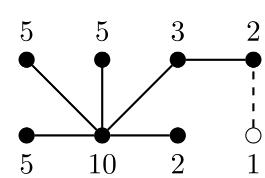
(a2) Diagonal of N 

Coefficients of Z 

We have $Z^2 = -1$. The associated group Φ_N has order 2^3. The matrix N^{-1} has two integer columns, which produce four new 2-suitable matrices with $n = 7$. This matrix is the resolution of $f := z^p - (aby)^{p-1}z - a^pxy + b^py$ with $a := x^7$ and $b = y^3$.

Note that blowing up $z^2 + x^{15}y + y^7$ gives $z^2 + x^{14}y + x^5y^7$ and normalizing $z^2 + x^{10}y + xy^7$, which is the equation for the above matrix.

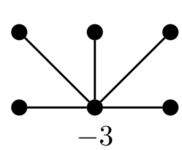
(b) Diagonal of N 

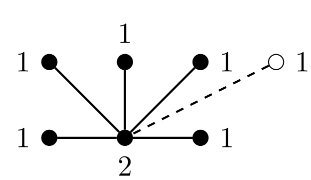
Coefficients of Z 

We have $Z^2 = -2$. The associated group Φ_N has order 2^2. This matrix occurs in the resolution of the hypersurface singularity $f = z^2 + x^{15} + y^{21} = 0$.

Many 2-suitable matrices appear in pairs differing at only one vertex. Our next three examples do not have such a companion matrix,

Quotient Singularity 4.7. ($n = 6$)

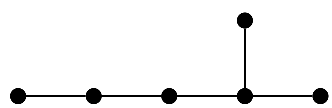
Diagonal of N 

Coefficients of Z 

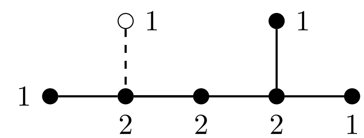
We have $Z^2 = -2$. The associated group Φ_N has order 2^4. This matrix arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2, namely $z^2 + x^5 + y^5 = 0$. It is also the resolution of the quotient singularity $z^2 + x^4y + xy^4 = 0$ (see [15], 8.3). The matrix N^{-1} has one integer column, which produces two new 2-suitable matrices with $n = 7$.

Quotient Singularity 4.8. ($n = 6$) We found only one 2-suitable intersection matrix associated with the graph of the Dynkin diagram D_6, the Dynkin diagram itself.

D_6



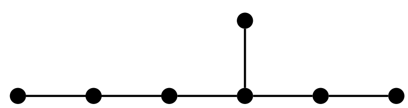
Z



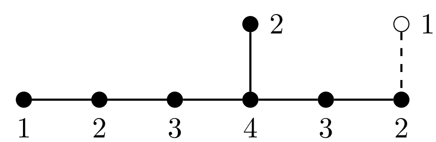
We have $Z^2 = -2$. The associated group Φ_N has order 2^2. This matrix arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2 ([15], 8.5). The matrix N^{-1} has two integer columns, which produce four new 2-suitable matrices with $n = 7$.

The Dynkin diagram E_7 with $n = 7$:

E_7



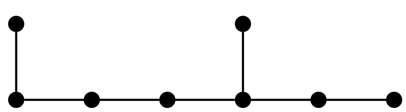
Z



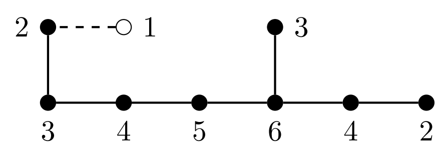
The associated group Φ_N has order 2 and $Z^2 = -2$. This is the only example of a matrix N in our list with $|\det(N)| = 2$. To obtain further examples with $n = 9$, one can start with a matrix having $n = 8$ and determinant 1 listed in Section 8 and extend it using [13], Theorem 3.4.

The Dynkin diagram E_8 with $n = 8$:

E_8



Z

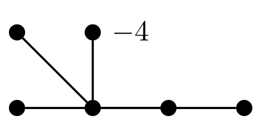


The associated group Φ_N is trivial and $Z^2 = -2$. This is the first example of a matrix N in our list with $|\det(N)| = 1$. Further examples, also with $n = 8$, are listed in Section 8.

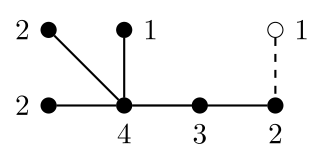
This is a quotient singularity associated with $z^2 + x^3 + y^5$. Blowing up E_8 produces the equation $z^2 + x^3y + y^3$ which resolves into the Dynkin Diagram E_7. Blowing up D_7 produces the equation $z^2 + x^2y + xy^3$ which resolves into the Dynkin Diagram D_6.

Intersection Matrix 4.9. ($n = 6$)

Diagonal of N



Coefficients of Z

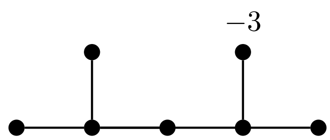


We have $Z^2 = -2$. The associated group Φ_N has order 2^2. The matrix N^{-1} has two integer columns in addition to the vector Z. The integer columns of N^{-1} produce six new 2-suitable matrices with $n = 7$, including the matrices in 4.24 and 4.25. None are known to arise as quotient singularities.

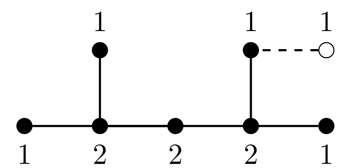
Since the 2-suitable intersection matrices with $n = 7$ are numerous, we do not attempt to list them all and describe below only twenty-four of them. We start with two extensions of the Dynkin diagram D_6.

Intersection Matrix 4.10. ($n = 7$) This intersection matrix is our only example where $Z^2 = -1$ and the matrix is not known to be a quotient singularity, even though it has a companion below in 4.11 with $Z^2 = -2$ which does arise from a quotient singularity.

Diagonal of N



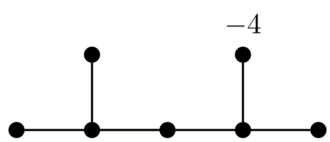
Coefficients of Z



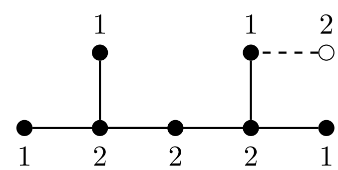
We have $Z^2 = -1$. The associated group Φ_N has order 2^2.

Quotient Singularity 4.11. ($n = 7$)

Diagonal of N



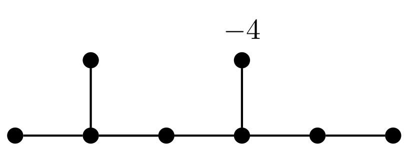
Coefficients of Z



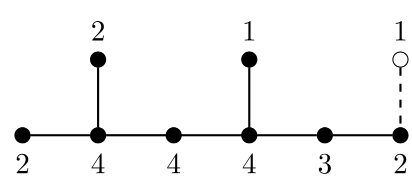
We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This matrix arises as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in characteristic 2. Case $a := x^3 + xy$ and $b := y^3 + x^2y$ of $f := z^p - (abxy)^{p-1}z - a^pxy + b^py$.

Intersection Matrix 4.12. ($n = 8$) The matrix below contains the matrix 4.11, but its group Φ_N is smaller than the corresponding group in 4.11.

Diagonal of N



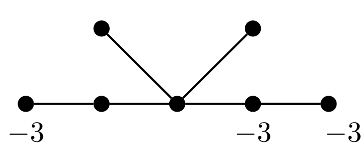
Z



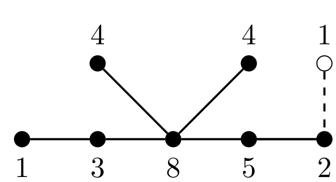
We have $Z^2 = -2$. The associated group Φ_N has order 2^2. This matrix arises as the intersection matrix of the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity where the component of self-intersection -4 is not smooth. It is obtained by resolving $f := z^p - (ab)^{p-1}z - a^px + b^py = 0$ with $a := x^3 + xy^3$ and $b := y^6 + x^2y^2$.

Intersection Matrix 4.13. ($n = 7$)

Diagonal of N



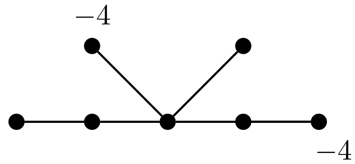
Z



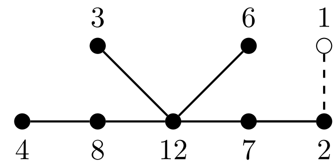
The associated group Φ_N has order 2^2 and $Z^2 = -2$.

Intersection Matrix 4.14. ($n = 7$)

Diagonal of N



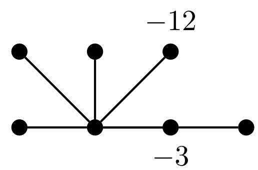
Z



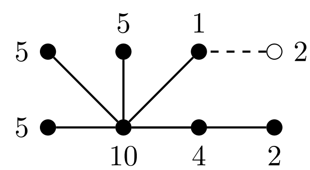
The associated group Φ_N has order 2 and $Z^2 = -2$.

Quotient Singularity 4.15. ($n = 7$) The matrices below are obtained from the matrix 4.4(a2) using [13], Theorem 3.4, with an integer vector which is not the fundamental cycle.

(a1) Diagonal of N

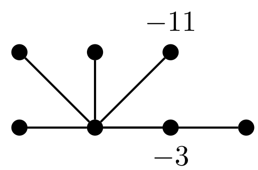


Coefficients of Z

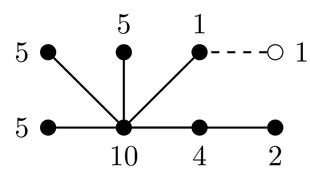


We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This matrix occurs in the resolution of the hypersurface singularity $z^2 + x^7y^{10}z + x^{15}y + y^{19}$, so the case $a = x^7$ and $b = y^9$.

(a2) Diagonal of N



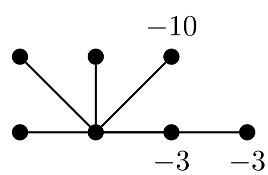
Coefficients of Z



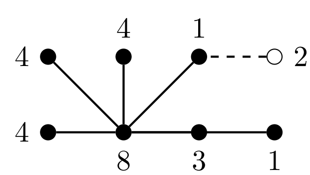
We have $Z^2 = -1$. The associated group Φ_N has order 2^2. This matrix occurs in the resolution of the hypersurface singularity $f = z^2 + x^{15} + y^{33} = 0$.

Quotient Singularity 4.16. ($n = 7$) The first two matrices below are extensions of the matrix 4.4 (a1).

(a1) Diagonal of N

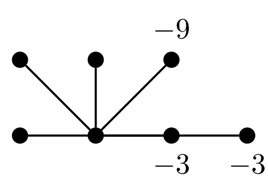


Coefficients of Z

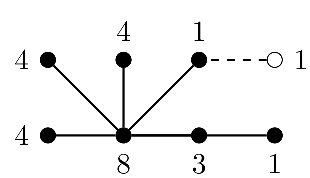


We have $Z^2 = -2$. The associated group Φ_N has order 2^4. This matrix occurs in the resolution of the hypersurface singularity $f = 0$ for $f := z^p - abxyz - a^py + b^pxy$ with $a = x^8$ and $b = y^6$.

(a2) Diagonal of N

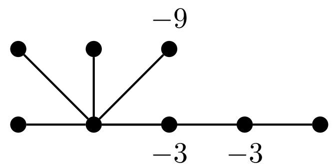


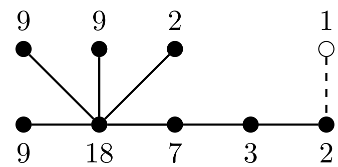
Coefficients of Z



We have $Z^2 = -1$. The associated group Φ_N has order 2^3. This comes from the weighted homogeneous $x^{27}y + y^{13} + z^2$. This is the blow up of $x^{27} + y^{39} + z^2$ below. Blowing up

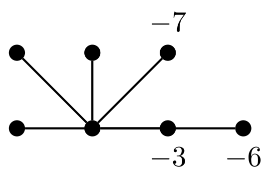
$x^{27}y + y^{13} + z^2$ gives after normalization $z^2 + y^{13}x + x^{16}y$, which is resolved above in (a1).

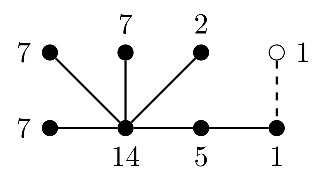
(b) Diagonal of N 

Z 

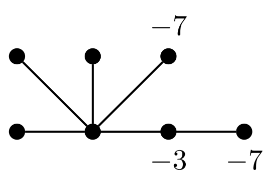
We have $Z^2 = -2$. The associated group Φ_N has order 2^2. This is the resolution of $x^{27}+y^{39}+z^2$. One blow-up gives the weighted homogeneous $x^{27}y + y^{13} + z^2$, whose resolution matrix is the previous matrix.

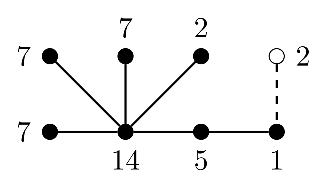
Quotient Singularity 4.17. ($n = 7$) The matrices below are extensions of 4.5 (a2).

(a1) Diagonal of N 

Coefficients of Z 

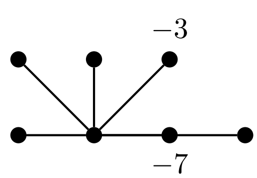
We have $Z^2 = -1$. The associated group Φ_N has order 2^2. This matrix occurs in the resolution of the hypersurface singularity $f = z^2 + x^{21} + y^{51} = 0$.

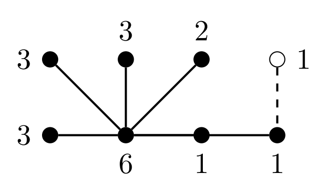
(a2) Diagonal of N 

Coefficients of Z 

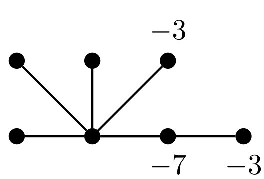
We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This matrix occurs in the resolution of the hypersurface singularity $f = x^{21}y + x^{10}y^{16}z + y^{31} + z^2$. This is the case $a = x^{10}$ and $b = y^{15}$ of the quotient singularity $f = z^p - (aby)^{p-1} - a^p xy + b^p y$.

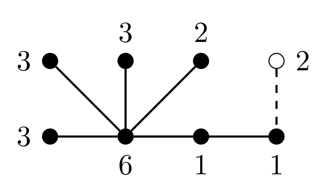
Quotient Singularity 4.18. ($n = 7$) The matrices below are extensions of 4.5 (a2).

(a1) Diagonal of N 

Coefficients of Z 

We have $Z^2 = -1$. The associated group Φ_N has order 2^2. This matrix occurs in the resolution of the hypersurface singularity $f = z^2 + x^9 + y^{39} = 0$.

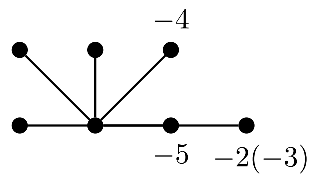
(a2) Diagonal of N 

Coefficients of Z 

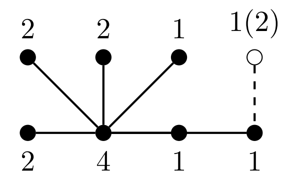
We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This matrix occurs in the resolution of the hypersurface singularity $f = z^2 + x^9y + y^{31} = 0$ obtained by blowup of above.

Intersection Matrices 4.19. **($n = 7$)** The following two matrices are extensions of the matrix in 4.6 (a2).

Diagonal of N



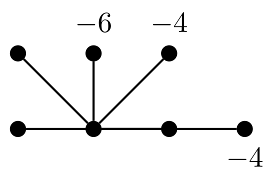
Coefficients of Z



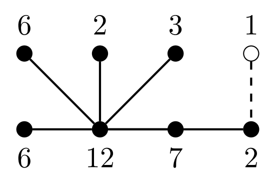
We have $Z^2 = -1$ (resp. -2). The associated group Φ_N has order 2^3 (resp. 2^4).

Intersection Matrix 4.20. **($n = 7$)**

Diagonal of N



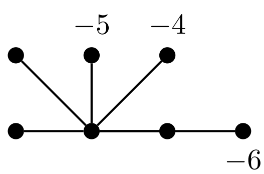
Coefficients of Z



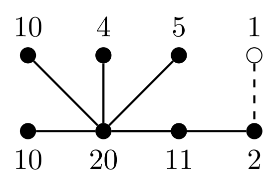
We have $Z^2 = -2$. The associated group Φ_N has order 2^3.

Intersection Matrix 4.21. **($n = 7$)**

Diagonal of N



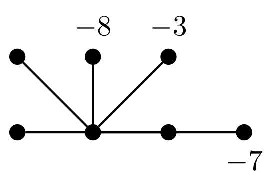
Coefficients of Z



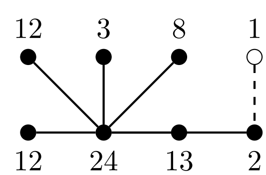
We have $Z^2 = -2$. The associated group Φ_N has order 2^2.

Intersection Matrix 4.22. **($n = 7$)**

Diagonal of N



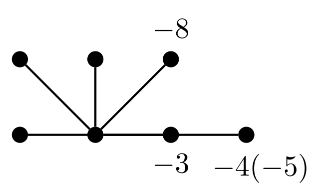
Coefficients of Z



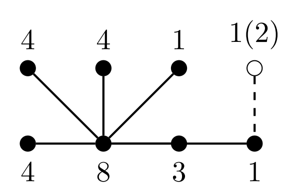
We have $Z^2 = -2$. The associated group Φ_N has order 2^2.

Intersection Matrices 4.23. **($n = 7$)**

Diagonal of N



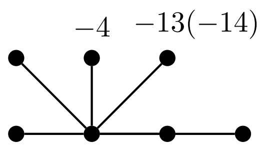
Coefficients of Z



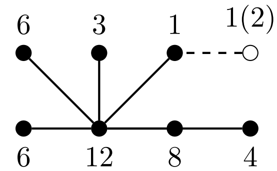
We have $Z^2 = -1$ (resp. -2). The associated group Φ_N has order 2^3 (resp. 2^4).

Intersection Matrices 4.24. ($n = 7$) The following two matrices are extensions of the matrix in 4.9.

Diagonal of N



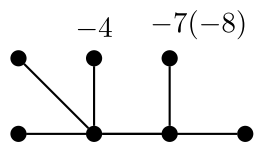
Coefficients of Z



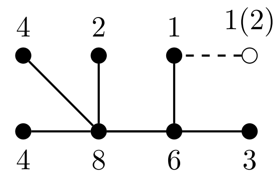
We have $Z^2 = -1$ (resp. -2). The associated group Φ_N has order 2^2 (resp. 2^3).

Intersection Matrices 4.25. ($n = 7$) The following matrices are extensions of the matrix 4.9.

Diagonal of N



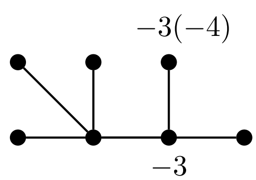
Coefficients of Z



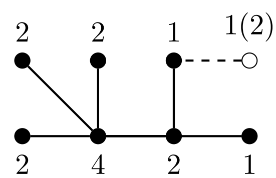
We have $Z^2 = -1$ (resp. -2). The associated group Φ_N has order 2^2 (resp. 2^3).

Intersection Matrices 4.26. ($n = 7$) The following matrices are extensions of the matrix 4.4(a2).

Diagonal of N



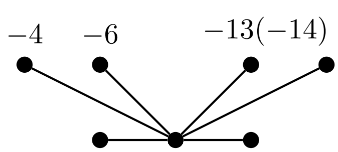
Coefficients of Z



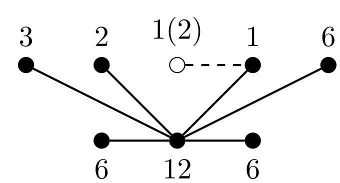
We have $Z^2 = -1$ (resp. -2). The associated group Φ_N has order 2^2 (resp. 2^3).

Intersection Matrices 4.27. ($n = 7$)

Diagonal of N



Coefficients of Z

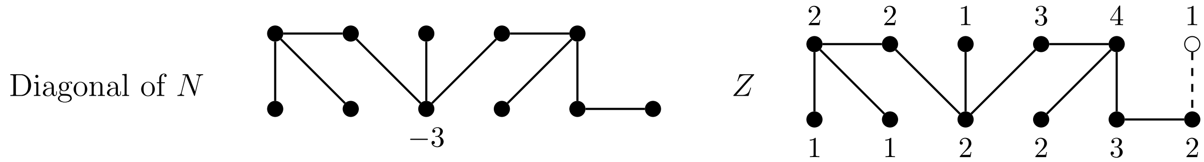


We have $Z^2 = -1$ (resp. -2). The associated group Φ_N has order 2^4 (resp. 2^5).

There are additional structures on the star on 7 vertices. For instance, two such structures are obtained by substituting the triple $[-4, -6, -13]$ above by the triple $[-3, -7, -43]$ or $[-3, -8, -25]$.

The complete list of 2-suitable matrices of size $n = 8$ is long and will not be given here. It includes for instance the matrices listed in Section 8 with determinant 1 and $|Z^2| \leq 2$, and the matrices which can be obtained using [13], Theorem 3.4, with the twenty-four 2-suitable matrices of size $n = 7$ listed in this section. We end with a few graphs which have more than one node.

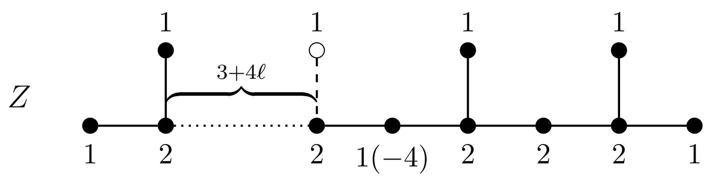
Quotient Singularity 4.28. ($n = 11$ **and three nodes**)



This matrix occurs in the resolution of the hypersurface singularity $f = 0$, where $f := z^p - (aby)^{p-1}z - a^p xy - b^p y$ with $a := x^3 + xy$ and $b := y^2 + x^3 y$. The associated group Φ_N has order 2^3 and $Z^2 = -2$. We do not know of an example of a $\mathbb{Z}/2\mathbb{Z}$ quotient singularity of smaller size whose graph has three nodes.

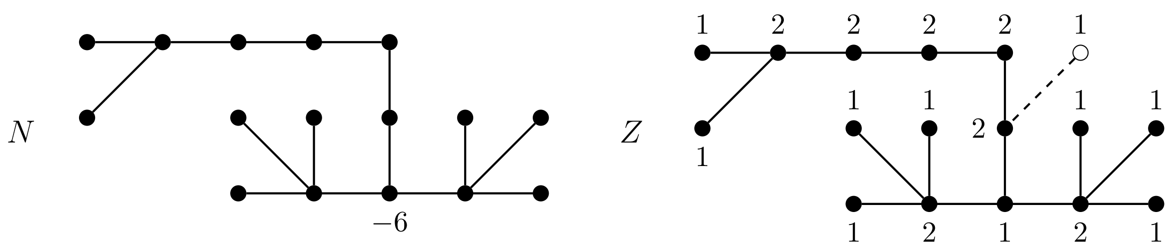
Quotient Singularity 4.29. ($n = 12 + 4\ell$ **and three nodes**)

The matrix N below has a single coefficient on the diagonal smaller than -2, which we indicate below along with the coefficients of Z.



Computations show that this matrix occurs in the resolution of the hypersurface singularity $f = 0$, where $f := z^p - (abxy)^{p-1}z - a^p xy - b^p y$ with $a := y^{2+\ell} + xy$, $\ell \geq 0$, and $b := x^4 + xy$. The associated group Φ_N has order 2^4 and $Z^2 = -2$.

Quotient Singularity 4.30. ($n = 16$ **and four nodes**)



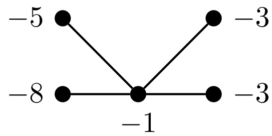
This matrix occurs in the resolution of the hypersurface singularity $f = 0$, where $f := z^p - (ab)^{p-1}z - a^p y - b^p x$ with $a := x^5 + y(x^3 + xy + y^2)$ and $b := y^2(x^3 + xy)$. The associated group Φ_N has order 2^6 and $Z^2 = -2$.

In this example with four nodes, the quotient singularity has a resolution graph with the following additional property: two nodes are linked by one single edge. An example of a quotient singularity with four nodes without this property can be found with $n = 14$, but has been omitted.

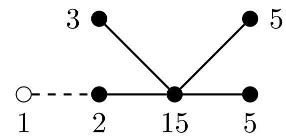
5. Small trees in characteristic 3

Quotient Singularity 5.1. ($n = 5$) Recall that in our definition of a 3-suitable intersection matrix N, we require that no self-intersection on the diagonal is equal to -1. When we do not impose this restriction, we found the following quotient singularities with a resolution with five smooth components having normal crossings.

(a1) Diagonal of N

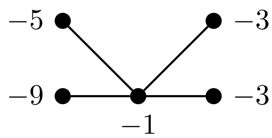


Coefficients of Z

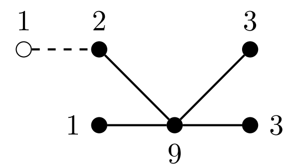


The associated group Φ_N has order 3. We have $Z^2 = -2$. This matrix arises as the resolution matrix of the hypersurface $f := z^3+x^{10}+y^{16} = 0$, which is a quotient singularity in characteristic 3. Blowing up the origin produces the next example.

(a2) Diagonal of N



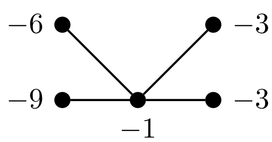
Coefficients of Z



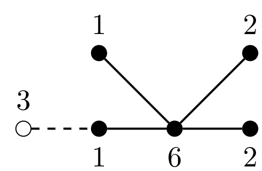
The associated group Φ_N has order 3^2. We have $Z^2 = -2$. This matrix arises as the resolution matrix of the hypersurface $f := z^3 + x^{10}y + y^7 = 0$, which is a quotient singularity in characteristic 3 ([13], Theorem 7.6). Blowing up the origin produces the next example.

Intersection Matrix 5.2.

(a1) Diagonal of N

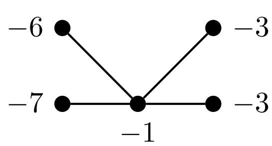


Coefficients of Z

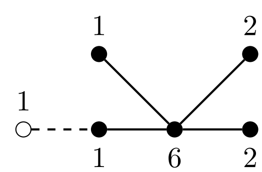


The associated group Φ_N has order 3^3. We have $Z^2 = -3$. This matrix arises as the resolution matrix of the hypersurface $f := z^3 + x^5y + xy^7 = 0$, which is not known to be a quotient singularity in characteristic 3.

(a2) Diagonal of N



Coefficients of Z

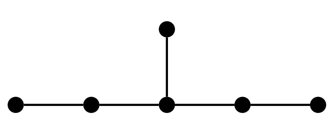


The associated group Φ_N has order 3^2. We have $Z^2 = -1$. See 5.14 for a similar example.

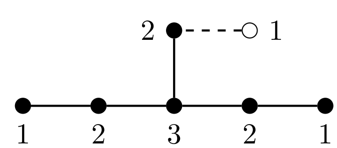
We did not find any 3-suitable intersection matrices N of sizes $n = 3, 4, 5$. We found six 3-suitable intersection matrices when $n = 6$, and they are listed below.

Quotient Singularity 5.3. ($n = 6$) These quotient singularities are part of the families exhibited in [13], 11.7.

(a1) Diagonal of N



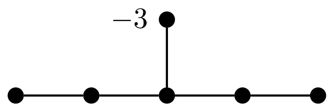
Z



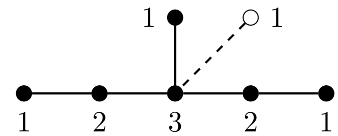
The associated group Φ_N has order 3 and $Z^2 = -2$. This is the Dynkin diagram E_6 (see [15], Theorem 6.3) This is the singularity studied by Peskin. It corresponds to $z^3 + x^2 + y^4$ with

$n = 6$, and gives the extensions $z^3 + x^2 + y^{3j+1}$ with j odd, and $n = 6 + (j-1)/2$. The matrix N^{-1} has only two integer columns, leading to the matrices of size $n = 7$ found in 5.6, 5.7, and 5.10.

(a2) Diagonal of N



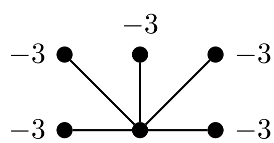
Z



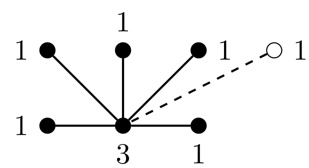
The associated group Φ_N has order 3^2 and $Z^2 = -3$. The matrix N is not numerically Gorenstein. It arises as a quotient singularity (see [15], Example 10.7, and [10], 6.8). The matrix N^{-1} has a unique integer column, which can be use to extend N to the matrices found in 5.9.

Intersection Matrix 5.4. ($n = 6$)

(a1) Diagonal of N

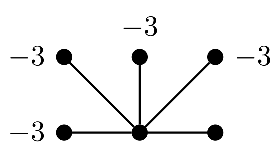


Coefficients of Z

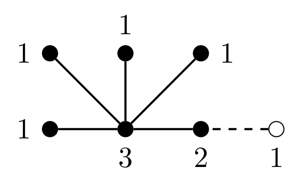


The associated group Φ_N has order 3^4. We have $Z^2 = -3$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^3 + x^5 + y^5 = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 3.

(a2) Diagonal of N



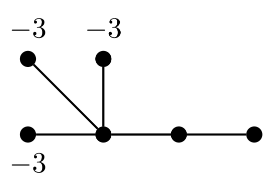
Coefficients of Z



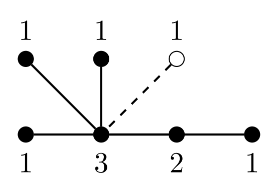
The associated group Φ_N has order 3^3. We have $Z^2 = -2$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^3 + x^4 + y^8 = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 3. Blowing up this singularity produces the singularity $f := z^3 + x^4y + y^5 = 0$ which resolves with the previous intersection matrix.

Intersection Matrix 5.5. ($n = 6$)

(a1) Diagonal of N

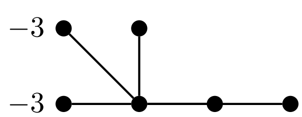


Coefficients of Z

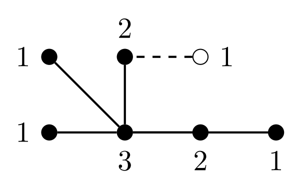


We have $Z^2 = -3$. The associated group Φ_N has order 3^3. The matrix is not numerically Gorenstein.

(a2) Diagonal of N



Coefficients of Z

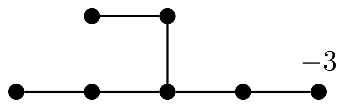


We have $Z^2 = -2$. The associated group Φ_N has order 3^2. The matrix is not numerically Gorenstein.

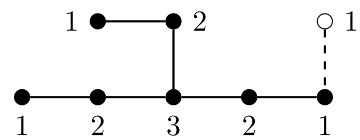
We now turn to describing some 3-suitable intersection matrices of size $n = 7$. Our next two matrices below in 5.6 and 5.7 are obtained from the Dynkin diagram E_6 in 5.3 (a1) above.

Quotient Singularity 5.6. ($n = 7$)

Diagonal of N



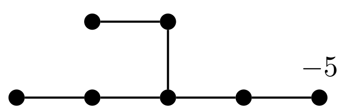
Coefficients of Z



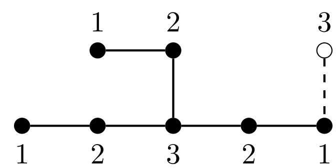
The associated group Φ_N has order 3 and $Z^2 = -1$. This intersection matrix arises from Peskin's singularities with $j = 3$. It is the resolution of $z^3 + x^2 + y^{3j+1}$.

Intersection Matrix 5.7. ($n = 7$) Our next intersection matrix is the companion of 5.6 in the construction of [13], Theorem 3.4.

Diagonal of N



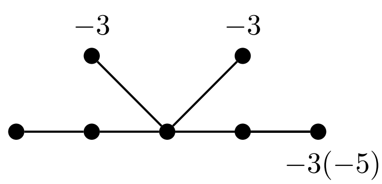
Coefficients of Z



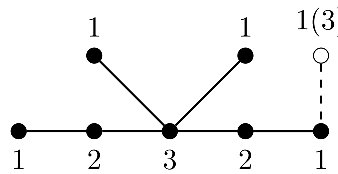
The associated group Φ_N has order 3^2 and $Z^2 = -3$. The matrix is numerically Gorenstein. It is associated for instance with the resolutions of $f = z^3 + x^2y + y^7 = 0$ or $f = z^3 + x^2y + y^4x = 0$. The latter hypersurface singularity is obtained after two blow-ups from $z^3 + x^4 + y^7 = 0$ (see 5.19). See also 5.17 and 5.26.

Intersection Matrices 5.8. ($n = 7$) We present below the two 3-suitable intersection matrices obtained from the matrix 5.5 (a2) using [13], Theorem 3.4.

Diagonal of N



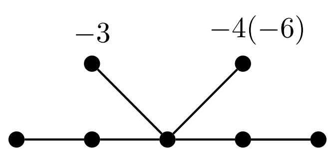
Z



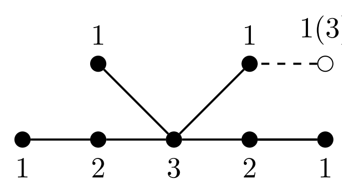
The associated group Φ_N has order 3^2 (resp. 3^3) and $Z^2 = -1$ (resp. -3). Both matrices are not numerically Gorenstein.

Intersection Matrices 5.9. ($n = 7$)

Diagonal of N



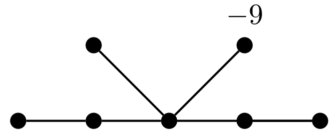
Coefficients of Z



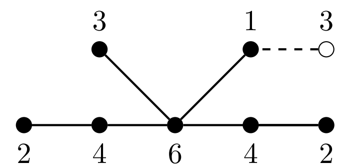
The associated group Φ_N has order 3^2 (resp. 3^3) and $Z^2 = -1$ (resp. -3). Both matrices are not numerically Gorenstein.

Intersection Matrix 5.10. ($n = 7$) The matrices (a1) and (a2) below are extensions of the Dynkin diagram E_6 in 5.3 (a1) using [13], Theorem 3.4.

(a1) Diagonal of N

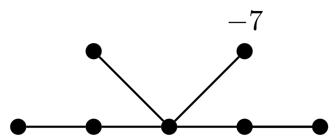


Z

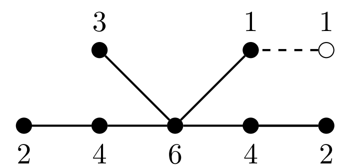


The associated group Φ_N has order 3^2 and $Z^2 = -3$. The matrix is not numerically Gorenstein.

(a2) Diagonal of N



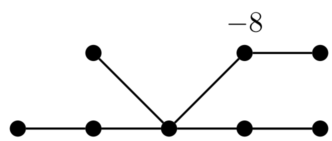
Z



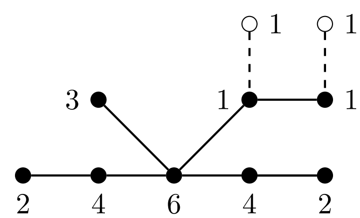
The associated group Φ_N has order 3 and $Z^2 = -1$. The matrix is numerically Gorenstein. It is the intersection matrix associated with the hypersurface singularity $f = z^3 + x^4 + y^{14} = 0$. The local ring $k[[x, y, z]]/(f)$ is not known to be a quotient singularity when $p = 3$.

The blow-up of the singularity $f = 0$, given by $z^3 + x^4y + y^{11} = 0$, gives a 3-suitable intersection matrix which has one additional vertex and is given below. This matrix with $n = 8$ is obtained from the matrix (a2) using the construction of [13], Theorem 3.9.

(b) Diagonal of N



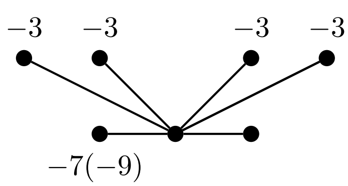
Z



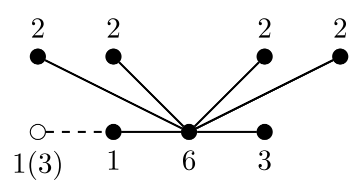
The associated group Φ_N has order 3^2 and $Z^2 = -2$. The same graph also supports the two extensions of the matrix (a2) obtained using its fundamental cycle and [13], Theorem 3.4.

Intersection Matrices 5.11. ($n = 7$) The two 3-suitable intersection matrices below on the star on 7 vertices are extensions of the matrix 5.4 (a2).

Diagonal of N



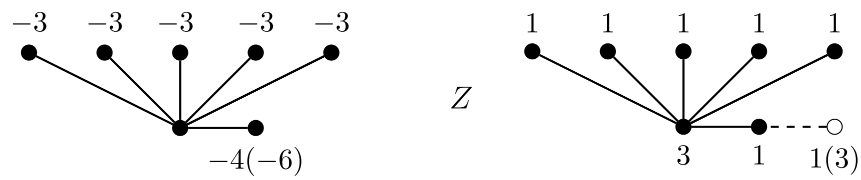
Z



The associated group Φ_N has order 3^3 (resp. 3^4) and $Z^2 = -1$ (resp. -3). Both matrices are numerically Gorenstein. The matrix with $Z^2 = -1$ is associated with the resolution of $z^3 + x^8 + y^{28} = 0$, which is not known to arise from a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity.

Intersection Matrices 5.12. ($n = 7$) The two 3-suitable intersection matrices below are extensions of the matrix 5.4 (a1).

Diagonal of N $\quad$ Z

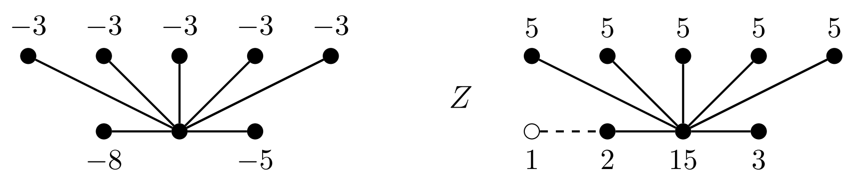


The associated group Φ_N has order 3^4 (resp. 3^5) and $Z^2 = -1$ (resp. -3). Both matrices are numerically Gorenstein. The matrix with $Z^2 = -3$ is associated with the resolution of $z^3 + x^5y + y^{11} = 0$, which is not known to arise from a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity.

The complete list of 3-suitable matrices of size $n = 8$ is long and will not be given here. It includes for instance the matrices listed in Section 8 with determinant 1, and the matrices which can be obtained using [13], Theorem 3.4, with the ten 3-suitable matrices of size $n = 7$ listed above. We list below some 3-suitable intersection matrices of size $n = 8$ and $n = 9$ which are known to arise as quotient singularities.

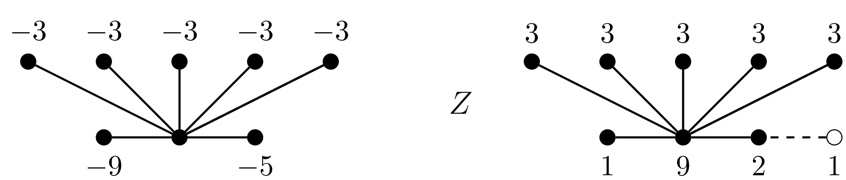
Quotient Singularity 5.13. ($n = 8$)

(a1) Diagonal of N $\quad$ Z



The associated group Φ_N has order 3^4 and $Z^2 = -2$. The matrix appears as a quotient singularity with $z^3 + x^{25} + y^{40}$. Quotient singularity: $z^3 + x^{25} + y^{40} + (x^8y^{13})^2z$. Blow up: $z^3 + x^{25}y^{22} + y^{37} + x^{16}y^{40}z$. Normalize $z^3 + x^{25}y + y^{16} + x^{16}y^{26}z$. This is almost the next equation.

(a2) Diagonal of N $\quad$ Z

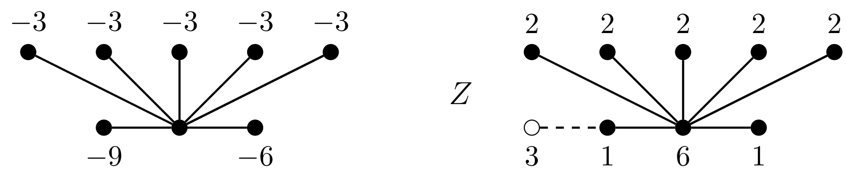


The associated group Φ_N has order 3^5 and $Z^2 = -2$. This matrix corresponds to the quotient singularity ramified in codimension 1 given by $f := z^p - (abxy)^{p-1}z - a^pxy + b^py$ with $a = x^8$ and $b = y^5$. It also corresponds to the resolution of $f := z^3 + x^{25}y + y^{16} = 0$.

These two examples with $n = 8$ (and a second blow-up giving the matrix (a1) in 5.14) can be continued with a series of three examples with $n = 11$ and the singularity $z^3 + x^{40} + y^{64} = 0$. In 5.1 and 5.2 we find three examples with $n = 5$ and $z^3 + x^{10} + y^{16} = 0$. The general case might have $n = 5 + 3\ell$ components and equation $z^3 + x^{5(2+3\ell)} + y^{8(2+3\ell)} = 0$.

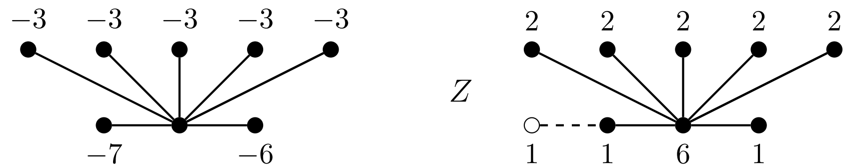
Intersection Matrix 5.14. ($n = 8$)

(a1) Diagonal of N



The associated group Φ_N has order 3^6 and $Z^2 = -3$. This matrix is numerically Gorenstein and corresponds to the resolution of $f := z^3 + x^{11}y + y^{16}x = 0$. This singularity is obtained after one blow up from the quotient singularity in 5.13 (a2). The singularity $f := z^3 + x^{11}y + y^{16}x = 0$ is not known to be a quotient singularity when $p = 3$.

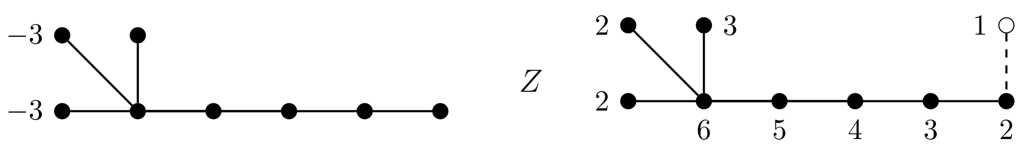
(a2) Diagonal of N



The associated group Φ_N has order 3^5 and $Z^2 = -1$. This matrix is numerically Gorenstein.

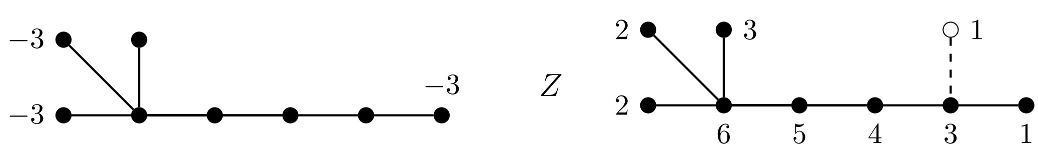
Quotient Singularity 5.15. ($n = 8$)

(a1)



The associated group Φ_N has order 3 and $Z^2 = -2$. This matrix is describes the resolution of the quotient singularity $z^3 + x^4 + y^{10} + (xy^3)^2 z$ (resp. $z^3 + x^4 + y^{10}$). Blow-up this equation to get $z^3 + x^4y + y^7 + x^2y^6z$ (resp. $z^3 + x^4y + y^7$). The resolution of the blow-up is associated with our next matrix.

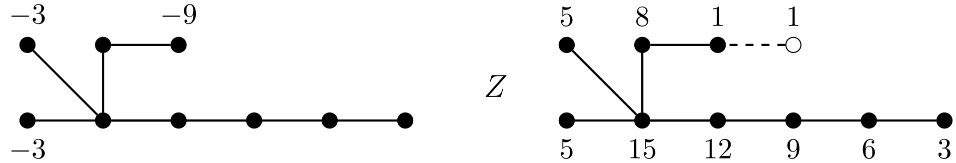
(a2)



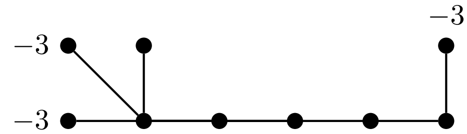
The associated group Φ_N has order 3^2 and $Z^2 = -3$. The equation of the quotient singularity is $z^3 - x^2y^6z - x^4y - y^7 = 0$. This is obtained as the case $a := x$ and $b := y^2$ in the singularity $f := z^p - (abxy)^{p-1}z - a^pxy + b^py = 0$.

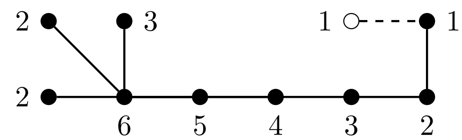
We now use the construction in [13], Theorem 3.4, with two different columns of the inverse of the matrix (a_1), to get the resolutions of $z^3 + x^{34} + y^{10}$ in (a_3) and $z^3 + x^4 + y^{22}$ in (a_4). Both have $n = 9$.

(a3)



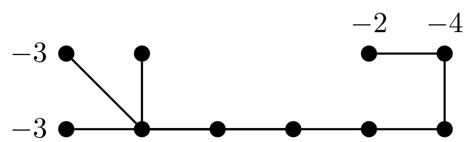
The associated group Φ_N has order 3 and $Z^2 = -1$.

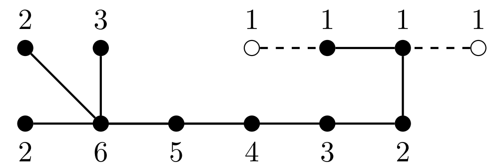
(a4) 

Z 

The associated group Φ_N has order 3 and $Z^2 = -1$.

We now use the construction in [13], Theorem 3.9, with the fundamental vector of the matrix (a_4), to obtain the matrix of the resolution of the blow-up of $z^3 + x^4 + y^{22}$, with equation $z^3 + x^4y + y^{19}$. This new matrix has $n = 10$.

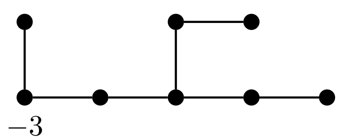
(a5) 

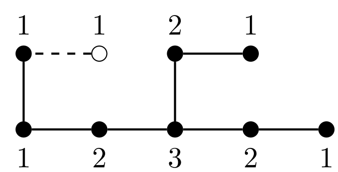
Z 

The associated group Φ_N has order 9 and $Z^2 = -2$.

Our next five examples of intersection matrices all share the same graph.

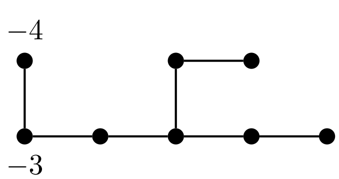
Quotient Singularity 5.16. ($n = 8$) The intersection matrix below arises from the Peskin singularity with $j = 5$, with $z^3 + x^2 + y^{3j+1}$. It arises from the matrix in 5.6 using [13], Theorem 3.4.

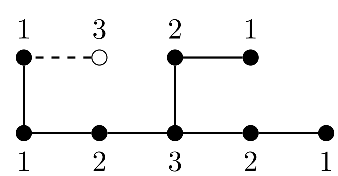
Diagonal of N 

Z 

The associated group Φ_N has order 3 and $Z^2 = -1$. This matrix can be modified in two different ways using [13], Theorem 3.4, and [13], Theorem 3.9. We only represent one modification below.

Intersection Matrix 5.17. ($n = 8$) The intersection matrix below is the companion to the matrix in the previous example and in particular also arises from the matrix in 5.6 using [13], Theorem 3.4.

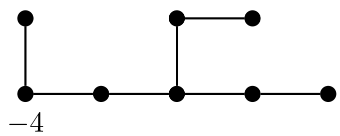
Diagonal of N 

Z 

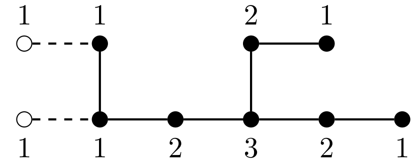
The associated group Φ_N has order 3^2 and $Z^2 = -3$. The matrix N^{-1} has a single integer column, associated with the node. This matrix is numerically Gorenstein, and is obtained from the resolution of $f = z^3 + x^2y + y^{13} = 0$ or $f = z^3 + x^2y + y^7x = 0$. Neither of these equation is known to define a quotient singularity. The latter is the blow-up of $g = z^3 + x^7y + y^7 = 0$, which is a quotient singularity when $p = 3$ (itself the blow-up of $z^3 + x^7 + y^{13} = 0$).

Intersection Matrix 5.18. $(n = 8)$

Diagonal of N



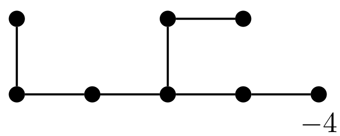
Z



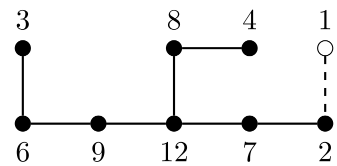
The associated group Φ_N has order 3^2 and $Z^2 = -2$. The matrix is not numerically Gorenstein. The matrix N^{-1} has a single integer column, associated with the node. This matrix is obtained from the matrix 5.6 using the construction of [13], Theorem 3.9.

Quotient Singularity 5.19. $(n = 8)$

(a1) Diagonal of N



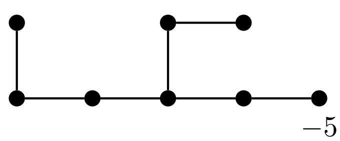
Z



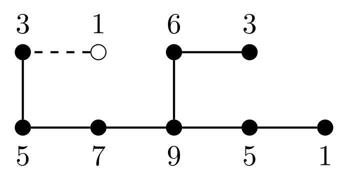
The associated group Φ_N is trivial and $Z^2 = -2$. This matrix arises from the resolution of the hypersurface singularity given by $f = z^3 + x^4 + y^7 = 0$. It arises as a quotient singularity when $p = 3$. (See for instance [15], Theorem 7.1).

Blowing up the surface $f = 0$ at the origin produces a chart with equation $z^3 + x^4y + y^4 = 0$. Its resolution has the matrix N below.

(a2) Diagonal of N



Z

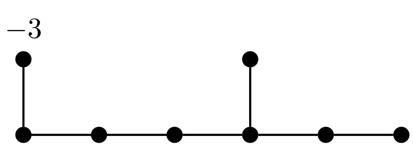


The associated group Φ_N has order 3 and $Z^2 = -3$. This matrix arises from the resolution of a quotient singularity since $z^3 + x^4y + y^4 = 0$ defines a quotient singularity (see [13], 10.6). One more blow-up of the origin produces the hypersurface $z^3 + x^2y + xy^4 = 0$, which is not known to be a quotient singularity. The intersection matrix of its resolution is in 5.7.

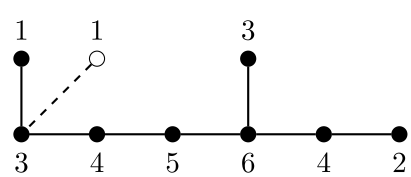
Intersection Matrix 5.20. $(n = 8)$ The Dynkin diagram E_8 has $|\Phi_N| = 1$ and $|Z^2| = 2$ (see 4.8). This intersection matrix arises as a generalized $\mathbb{Z}/3\mathbb{Z}$-quotient singularity in the work of Peskin.

Intersection Matrix 5.21. $(n = 8)$

N

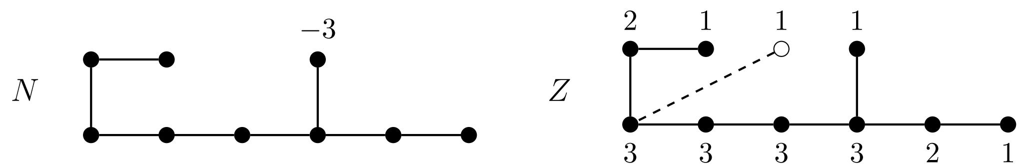


Z



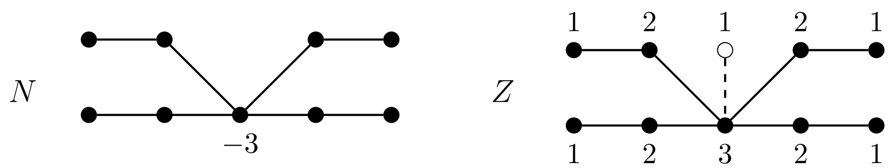
The associated group Φ_N has order 3 and $Z^2 = -3$. The matrix is not numerically Gorenstein.

Quotient Singularity 5.22. **($n = 9$).** The graph below is the graph of the extended Dynkin diagram $\tilde{E}_8$.



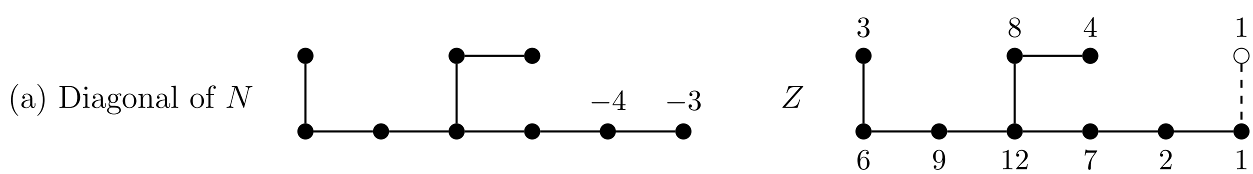
The associated group Φ_N has order 3^2 and $Z^2 = -3$. The matrix is not numerically Gorenstein. It arises from the graph $\Gamma(3, 2, 1)$ using [13], Theorem 9.8. This intersection matrix arises from a quotient singularity, as seen in [11], Theorem 6.8.

Quotient Singularity 5.23. **($n = 9$).**

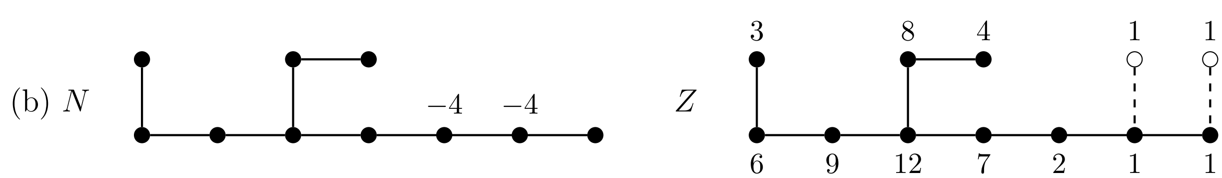


The associated group Φ_N has order 3^3 and $Z^2 = -3$. The matrix arises from the graph $\Gamma(3, 2, 2, 2)$ using [13], Theorem 9.8. This intersection matrix arises from a quotient singularity, as seen in [15], Theorem 9.2. It is associated with the resolution of $f = z^3 + x^4 + y^4$.

Quotient Singularity 5.24. ($n = 9$) The matrix arises from the resolution of the hypersurface singularity given by $f = z^3 + x^4 + y^{19}$ and is an extension of the matrix 5.19(a1).

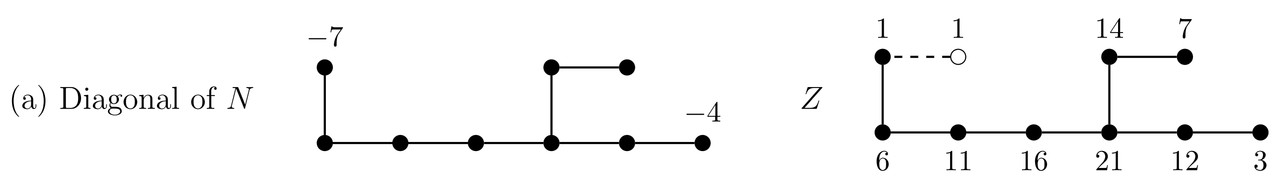


The associated group Φ_N is trivial and $Z^2 = -1$. We now make a blow-up of the above singularity, with new equation $f = z^3 + x^4y + y^{16}$ and get the resolution matrix below, obtained from the matrix (a) using [13], Theorem 3.9.



The associated group Φ_N has order 3 and $Z^2 = -2$. The matrix N^{-1} has four integer columns.

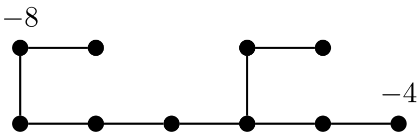
Quotient Singularity 5.25. ($n = 9$) The matrix arises from the resolution of the hypersurface singularity given by $f = z^3 + x^7 + y^{25}$ and is an extension of the matrix 5.19(a1).



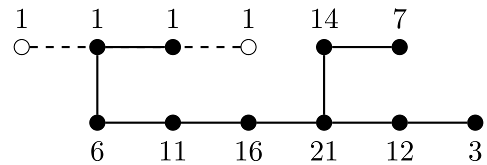
The associated group Φ_N is trivial and $Z^2 = -1$.

($n = 10$). We now make a blow-up of the above singularity, with new equation $f = z^3 + x^7 y + y^{19}$ and get the resolution matrix below, obtained from the matrix (a) using [13], Theorem 3.9.

(b) Diagonal of N



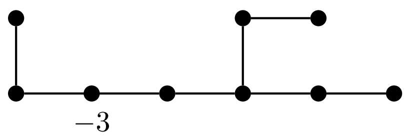
Z



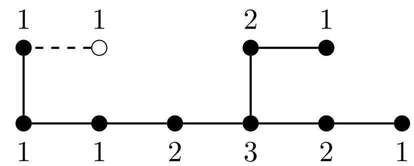
The associated group Φ_N has order 3 and $Z^2 = -2$.

Quotient Singularity 5.26. ($n = 9$) The matrix arises from the resolution of the hypersurface singularity given by $z^3 + x^2 + y^{22} = 0$ (Peskin's singularity with $j = 7$).

Diagonal of N



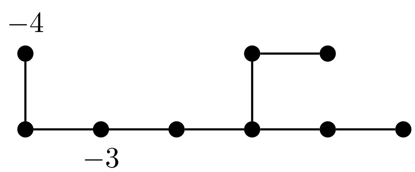
Z



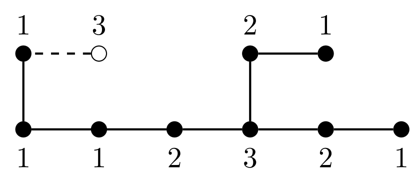
The associated group Φ_N has order 3 and $Z^2 = -1$. As in 5.16 and 5.17, this quotient singularity has a companion found below.

Intersection Matrix 5.27. ($n = 9$) The matrix arises from the resolution of the hypersurface singularity given by $z^3 + x^2 y + x y^{10} = 0$. It is obtained after two blow-ups from $z^3 + x^{10} + y^{19} = 0$.

Diagonal of N



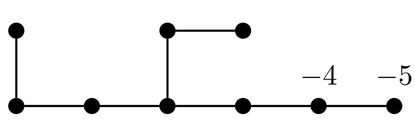
Z



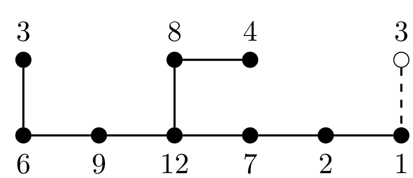
The associated group Φ_N has order 3^2 and $Z^2 = -3$.

Intersection Matrix 5.28. ($n = 9$) The matrix (a) in 5.24 above with $Z^2 = -1$ also has the companion

Diagonal of N



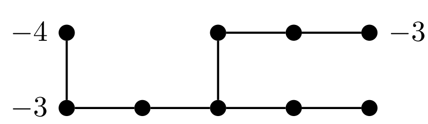
Z



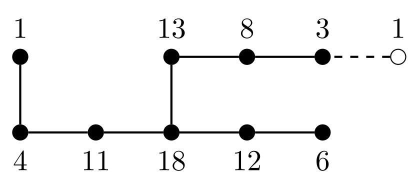
The associated group Φ_N has order 3 and $Z^2 = -3$. This matrix is not numerically Gorenstein.

Quotient Singularity 5.29. ($n = 9$)

(a1) Diagonal of N

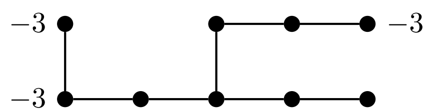


Z

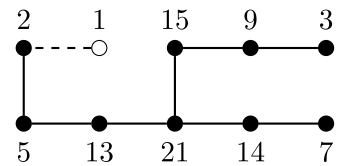


The associated group Φ_N has order 3 and $Z^2 = -3$. The matrix arises from the quotient singularity ramified in codimension 1 given by $f := z^3 + x^7 + xy^7$. ($f := z^p + y^{pr+1}x + x^{ps+1}$ in [13], 10.6). The matrix (a_1) has a companion matrix:

(a2) Diagonal of N



Z

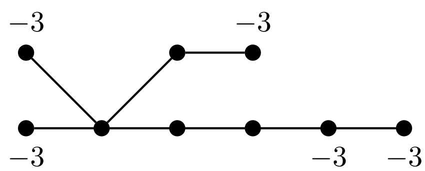


The associated group Φ_N is trivial and $Z^2 = -2$. The matrix is numerically Gorenstein and is associated with the resolution of the $\mathbb{Z}/3\mathbb{Z}$ quotient singularity given by $h := z^3 + x^{13} + y^7$ (of the form $h := z^p + x^{4p+1} + y^{2p+1}$).

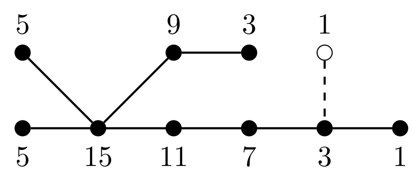
Note that if one starts with the quotient singularity $h := z^3 + x^{13} + y^7$ and blow-up, one gets $z^3 + x^{10} + x^4y^7$, which normalizes to: $z^3 + x^7 + xy^7$. This is the equation of the previous singularity.

Quotient Singularity 5.30. ($n = 9$)

(a1) N

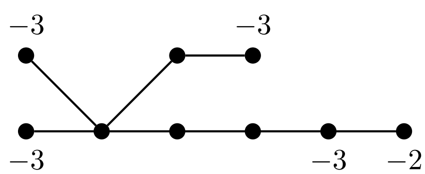


Z

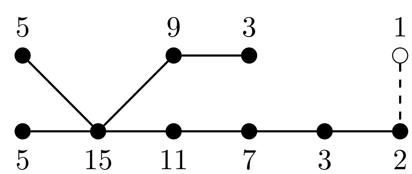


The associated group Φ_N has order 3^2 and $Z^2 = -3$. The matrix arises from the quotient singularity ramified in codimension 1 given by $f := z^3 - x^{10}y - x^6y^{10}z - y^{13}$, obtained from the general equation $f := z^p - (aby)^{p-1}z - a^pxy - b^py$ with $a := x^2$ and $b := y^4$. This matrix has a companion matrix

(a2) N



Z

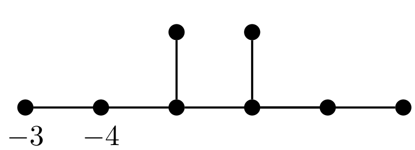


The associated group Φ_N has order 3 and $Z^2 = -2$. This matrix is associated with the $\mathbb{Z}/3\mathbb{Z}$-quotient singularity given by $f = z^3 + x^{10} + y^{22} = 0$. One blow-up of this singularity produces a chart with $g = z^3 + x^{10}y + y^{13} = 0$.

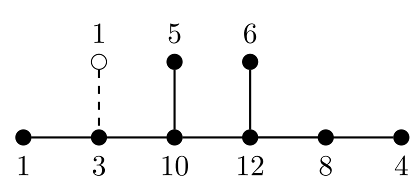
We end with examples of graphs with two nodes. Such examples with $n = 8$ vertices are exhibited in 8.4.

Intersection Matrix 5.31. ($n = 8$).

Diagonal of N



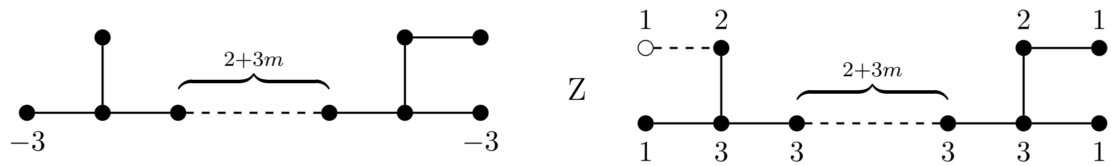
Z



The associated group Φ_N has order 3 and $Z^2 = -3$. The matrix is not numerically Gorenstein. Using the vector Z, we can extend the matrix to a 3-suitable matrix of size $n = 9$ with three nodes.

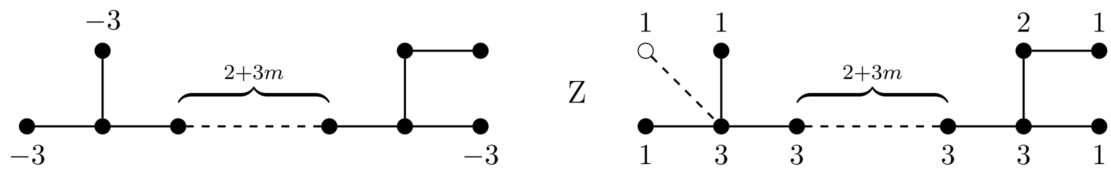
Intersection Matrix 5.32. ($n = 9 + 3m$)

(a1) N



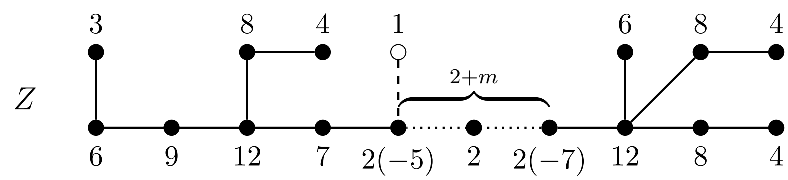
The associated group Φ_N has order 3^2 and $Z^2 = -2$. The matrix is not numerically Gorenstein. The matrices in 5.5 can be considered to be the case $m = -1$ in this family, with a star-shaped graph.

(a2) N



The associated group Φ_N has order 3^3 and $Z^2 = -3$. The matrix is not numerically Gorenstein.

Intersection Matrix 5.33. ($n = 15 + m, m \geq 0$) A similar example is found in [13], 5.2.



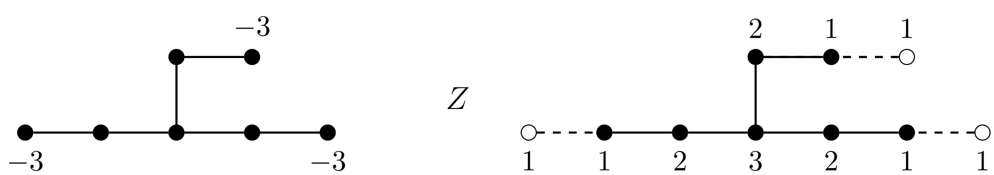
The associated group Φ_N has order 3 and $Z^2 = -2$. One can also consider the case $m = -1$, with now a single vertex with coefficient -10 on the diagonal of N.

6. Small trees in characteristic 5

When $p = 5$, we did not find examples of 5-suitable intersection matrices of size $n \leq 6$ whose graph has at least one node. We present below the five examples that we found with $n = 7$. None of them are known to arise from a $\mathbb{Z}/5\mathbb{Z}$-quotient singularity.

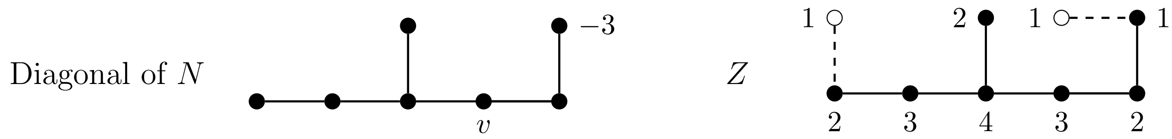
Intersection Matrix 6.1. ($n = 7$)

Diagonal of N



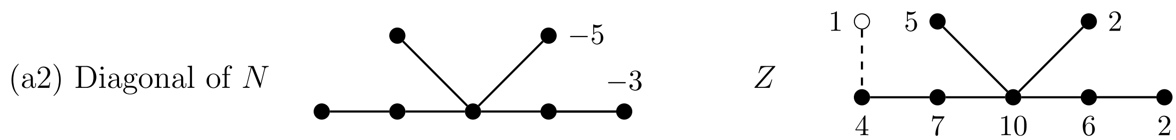
The associated group Φ_N has order 5^2 and $Z^2 = -3$. The matrix is numerically Gorenstein, and in this example, $K = -Z$. This matrix arises as the resolution matrix of the hypersurface $f := z^5 + x^3 + y^3 = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 5.

Intersection Matrix 6.2. ($n = 7$)



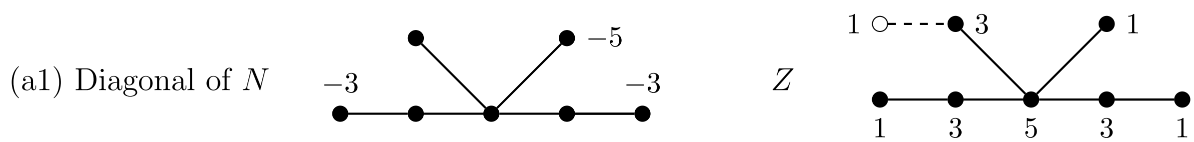
The associated group Φ_N has order 5 and $Z^2 = -3$. The matrix is not numerically Gorenstein. The matrix N^{-1} has a unique integer vector corresponding to the initial vertex v on the longest terminal chain. The graph is the graph of the Dynkin diagram E_7.

Intersection Matrix 6.3. ($n = 7$)



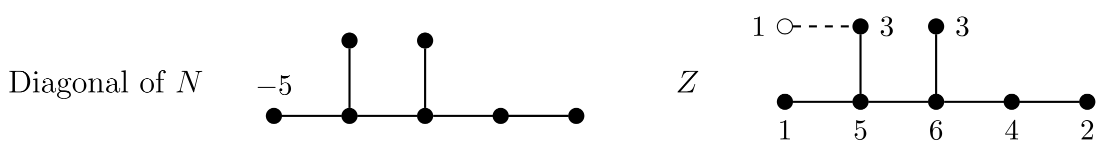
The associated group Φ_N has order 5 and $Z^2 = -4$. The matrix is not numerically Gorenstein. The matrix N^{-1} has two integer columns.

Intersection Matrix 6.4. ($n = 7$)



The associated group Φ_N has order 5^2 and $Z^2 = -3$. The matrix is not numerically Gorenstein. The matrix N^{-1} has four integer columns.

Intersection Matrix 6.5. ($n = 7$)



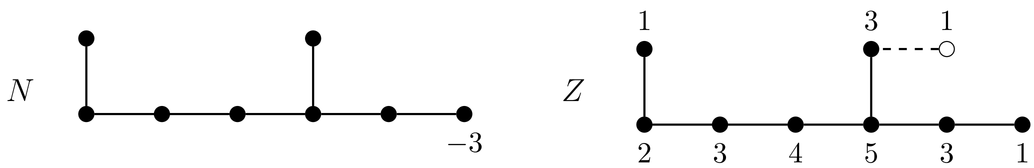
The associated group Φ_N has order 5 and $Z^2 = -3$. The matrix is not numerically Gorenstein. The matrix N^{-1} has two integer columns.

The list of 5-suitable matrices of size $n = 8$ is long. It includes for instance the matrices listed in Section 8 with determinant 1, and the six matrices that can be obtained from the last three 5-suitable matrices of size $n = 7$ listed above using [13], Theorem 3.4.

The smallest known examples of a 5-suitable intersection matrix arising as a $\mathbb{Z}/5\mathbb{Z}$-quotient singularity have size $n = 8$ and we present two of them below.

Quotient Singularity 6.6. ($n = 8$) The Dynkin diagram E_8, represented in 4.8, occurs as a $\mathbb{Z}/5\mathbb{Z}$-quotient singularity (see [2])).

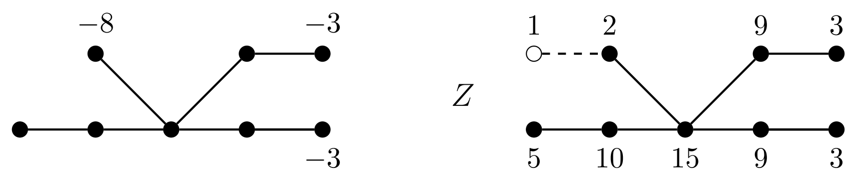
Intersection Matrix 6.7. ($n = 8$) The graph below is the graph of the Dynkin diagram E_8 (see also 7.9).



The associated group Φ_N has order 5 and $Z^2 = -3$. The matrix is not numerically Gorenstein.

Quotient Singularity 6.8. ($n = 8$) This example is the case $p = 5$ in [13], Theorem 8.1.

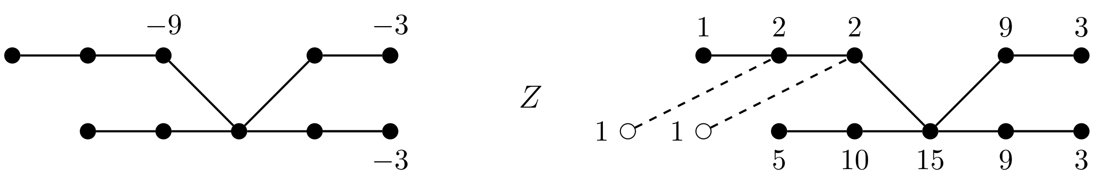
(a) Diagonal of N



The associated group Φ_N has order 5 and $Z^2 = -2$. The matrix arises from $z^5 + x^6 + y^{16}$.

The matrix below with ($n = 10$) is obtained from (a) using the construction of [13], Theorem 3.9.

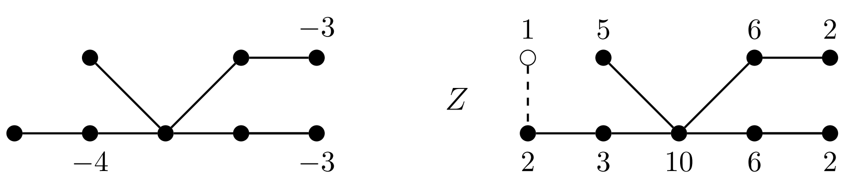
(b) N



The associated group Φ_N has order 5^2 and $Z^2 = -4$. The matrix is numerically Gorenstein.

Intersection Matrix 6.9. ($n = 8$)

Diagonal of N



The associated group Φ_N has order 5 and $Z^2 = -2$. The matrix is numerically Gorenstein.

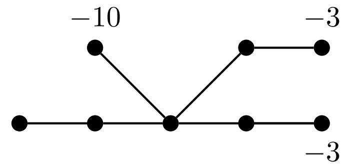
Intersection Matrix 6.10. ($n = 8$)

Diagonal of N

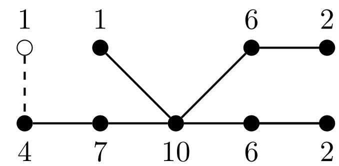
The associated group Φ_N has order 5 and $Z^2 = -4$. The matrix is numerically Gorenstein, associated with $z^5 + x^4 + y^6 = 0$.

Intersection Matrix 6.11. ($n = 8$)

Diagonal of N



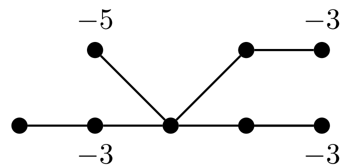
Z



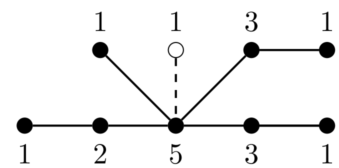
The associated group Φ_N has order 5^2 and $Z^2 = -4$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.12. ($n = 8$)

Diagonal of N



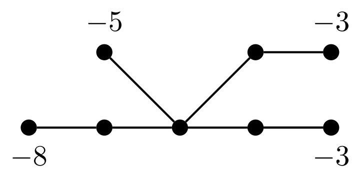
Z



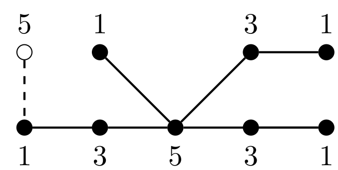
The associated group Φ_N has order 5^3 and $Z^2 = -5$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.13. ($n = 8$)

Diagonal of N



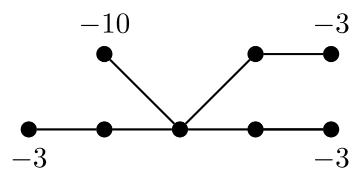
Z



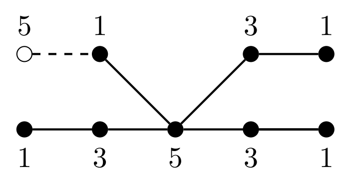
The associated group Φ_N has order 5^3 and $Z^2 = -5$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.14. ($n = 8$)

Diagonal of N



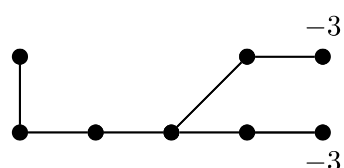
Z



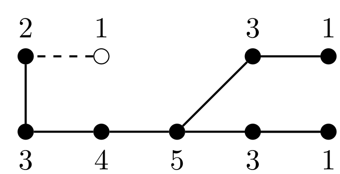
The associated group Φ_N has order 5^3 and $Z^2 = -5$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.15. ($n = 8$)

(a) Diagonal of N



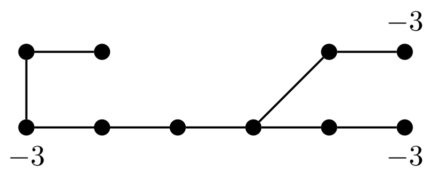
Z



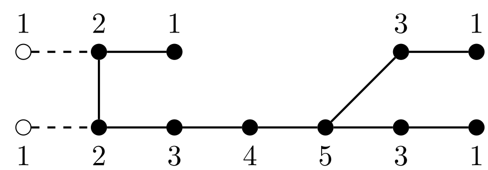
The associated group Φ_N has order 5 and $Z^2 = -2$. The matrix is numerically Gorenstein, associated with the singularity $z^5 + x^2 + y^8 = 0$. This intersection matrix is expected to arise as a generalized quotient singularity.

We describe below the matrix obtained from (a) using [13], Theorem 3.9, and note that this new matrix is not numerically Gorenstein.

(b) N



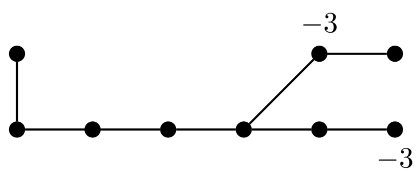
Z



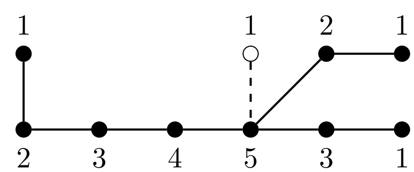
The associated group Φ_N has order 5^2 and $Z^2 = -4$. The matrix is not numerically Gorenstein.

Quotient Singularity 6.16. $(n = 9)$

Diagonal of N



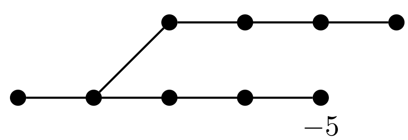
Z



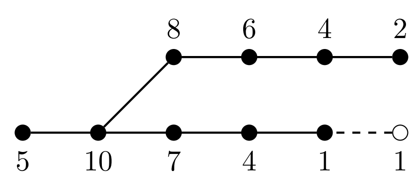
The associated group Φ_N has order 5^2 and $Z^2 = -5$. The matrix is not numerically Gorenstein. It arises as a $\mathbb{Z}/5\mathbb{Z}$ quotient singularity ([11] Theorem 6.8, and [17], Corollary 7.13).

Intersection Matrix 6.17. $(n = 9)$

N



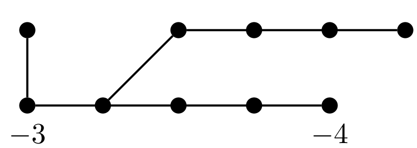
Z



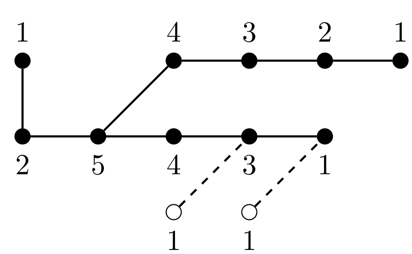
The associated group Φ_N is trivial and $Z^2 = -1$. The matrix is numerically Gorenstein. It is associated with the resolution of $f = z^5 + x^2 + y^{13}$. This matrix contains the Dynkin diagram E_8 and is obtained from E_8 (associated with the resolution of $z^5 + x^2 + y^3 = 0$) by the construction in [13], Theorem 3.4.

Intersection Matrix 6.18. $(n = 10)$

N

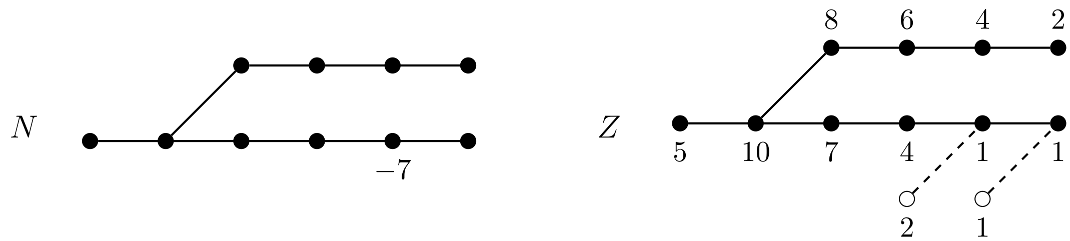


Z



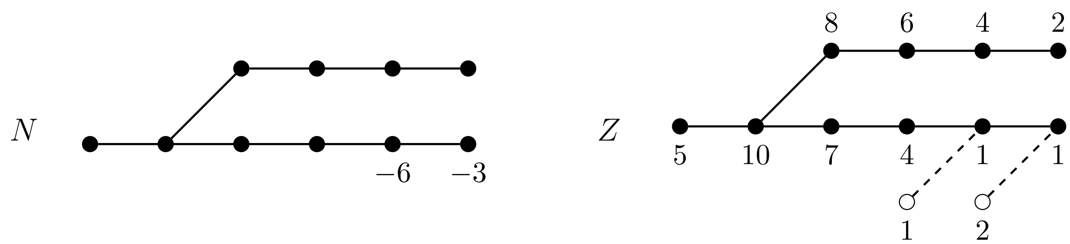
The associated group Φ_N has order 5^2 and $Z^2 = -4$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.19. ($n = 10$)



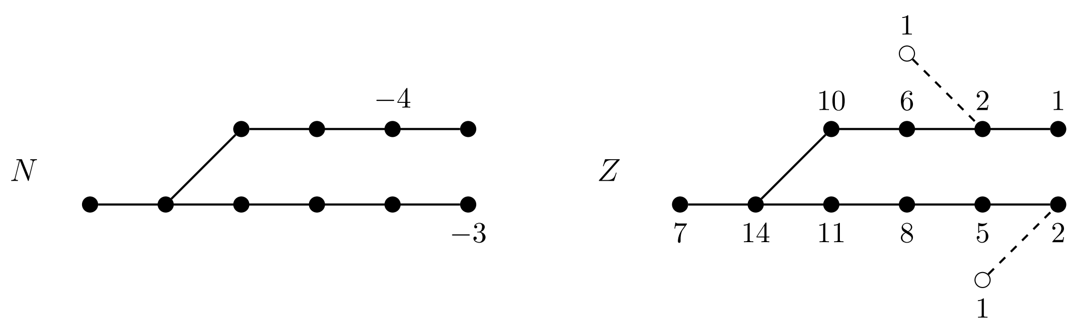
The associated group Φ_N has order 5 and $Z^2 = -3$. The matrix is numerically Gorenstein. It is associated with the resolution of $f = z^5 + x^2y + y^8$.

Intersection Matrix 6.20. ($n = 10$)



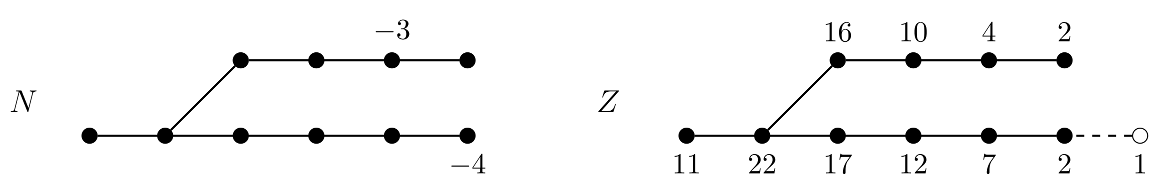
The associated group Φ_N has order 5 and $Z^2 = -3$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.21. ($n = 10$)



The associated group Φ_N has order 5 and $Z^2 = -4$. The matrix is not numerically Gorenstein.

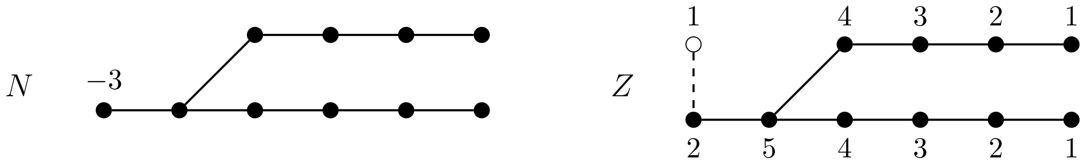
Intersection Matrix 6.22. ($n = 10$)



The associated group Φ_N is trivial and $Z^2 = -2$. The matrix is numerically Gorenstein.

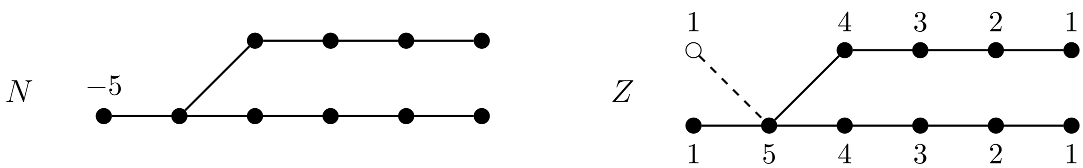
Our next two singularities are part of the families exhibited in [13], 11.7.

Quotient Singularity 6.23. ($n = 10$)



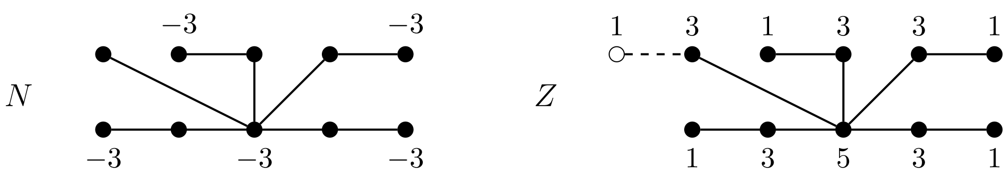
The associated group Φ_N has order 5 and $Z^2 = -2$. The matrix is numerically Gorenstein. It arises as a quotient singularity, for instance in the resolution of $z^5 + x^2 + y^6 = 0$ (see [15], Theorem 5.3).

Quotient Singularity 6.24. ($n = 10$)



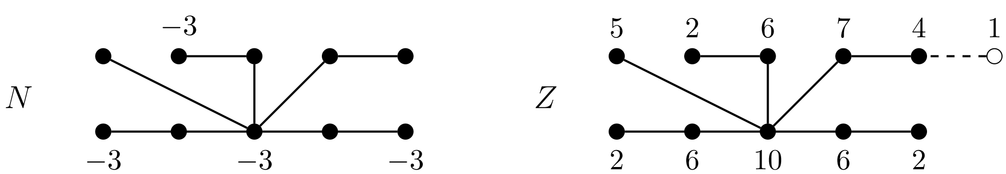
The associated group Φ_N has order 5^2 and $Z^2 = -5$. The matrix is not numerically Gorenstein. It is associated with a $\mathbb{Z}/5\mathbb{Z}$-quotient singularity in [12], Theorem 1.1.

Intersection Matrix 6.25. ($n = 10$)



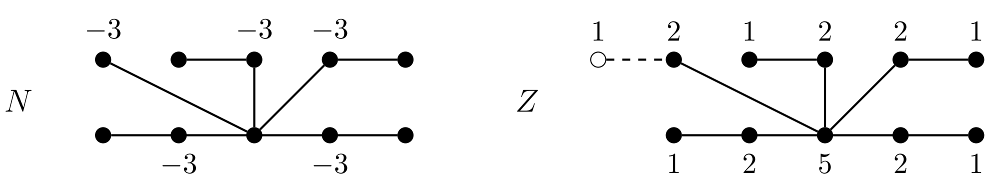
The associated group Φ_N has order 5^3 and $Z^2 = -3$. The matrix is numerically Gorenstein and is associated with the resolution of $z^5 + x^4 + y^8 = 0$.

Intersection Matrix 6.26. ($n = 10$) A change at one vertex from the previous matrix.



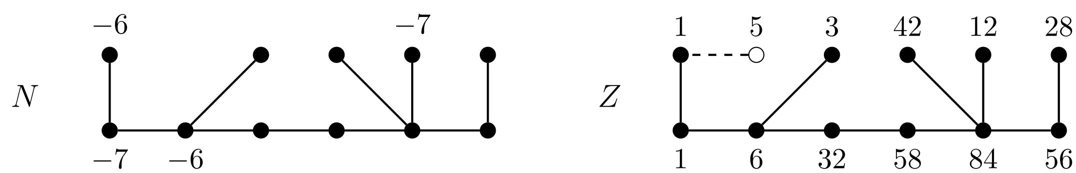
The associated group Φ_N has order 5^2 and $Z^2 = -4$. The matrix is numerically Gorenstein.

Intersection Matrix 6.27. ($n = 10$)



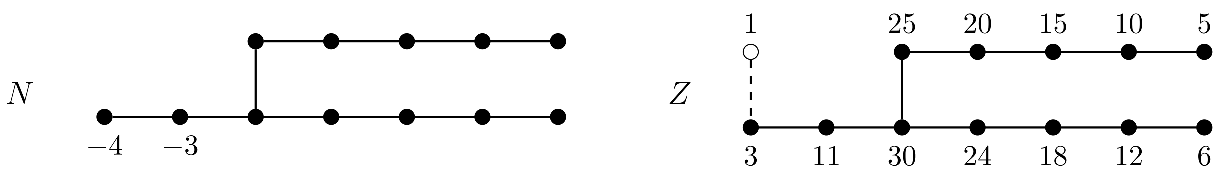
The associated group Φ_N has order 5^3 and $Z^2 = -2$. The matrix is not numerically Gorenstein.

Intersection Matrix 6.28. ($n = 11$) The 5-suitable matrix N below is such that N^{-1} has no integer coefficient.



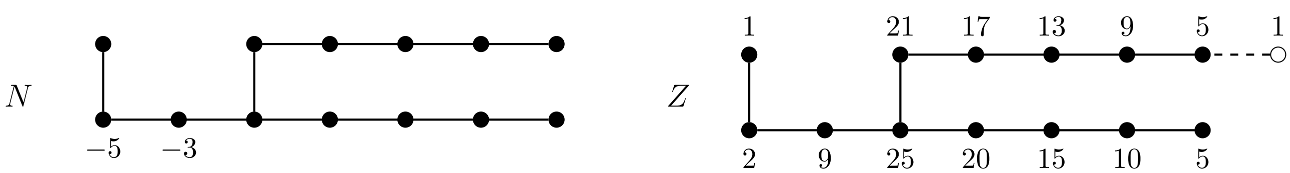
The associated group Φ_N has order 5 and $Z^2 = -5$. The matrix is not numerically Gorenstein.

Quotient Singularity 6.29. ($n = 12$)



The associated group Φ_N is trivial and $Z^2 = -3$. The matrix is the resolution of the singularity $z^5 + x^6 + y^{11} = 0$.

Quotient Singularity 6.30. ($n = 13$)

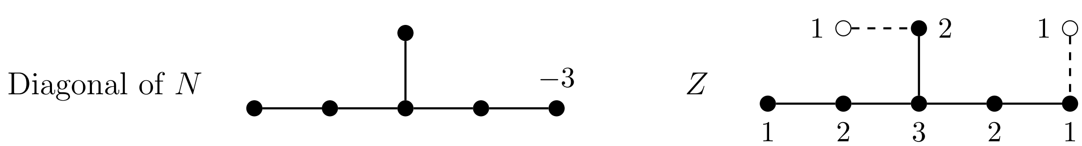


The associated group Φ_N has order 5 and $Z^2 = -5$. The matrix is the resolution of the singularity $z^5 + x^6y + y^6 = 0$, which is the blow-up of the singularity $z^5 + x^6 + y^{11} = 0$ of the previous example.

7. Small trees in characteristic 7

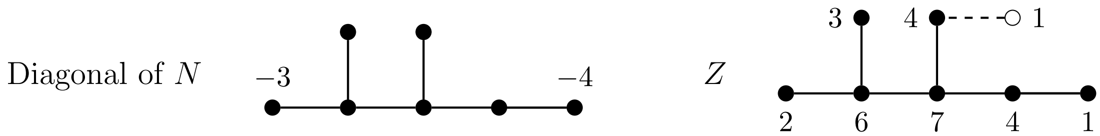
When $p = 7$, we did not find examples of 7-suitable intersection matrices of size $n \leq 5$ whose graph has at least one node. We present below the only example that we found with $n = 6$, and the two examples that we found with $n = 7$. None of these examples are known to arise from a $\mathbb{Z}/7\mathbb{Z}$-quotient singularity. None of the examples in this section have $Z^2 = -5$.

Intersection Matrix 7.1. ($n = 6$)



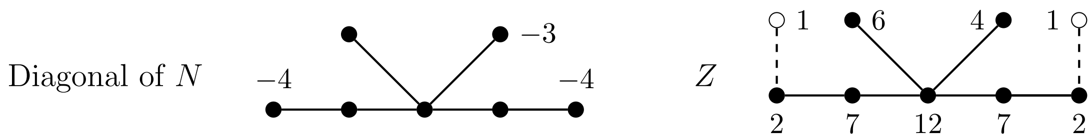
The associated group Φ_N has order 7 and $Z^2 = -3$. The matrix is not numerically Gorenstein. The matrix N^{-1} has no integer entry.

Intersection Matrix 7.2. ($n = 7$)



The associated group Φ_N has order 7 and $Z^2 = -4$. The matrix is not numerically Gorenstein. The matrix N^{-1} has two integer columns.

Intersection Matrix 7.3. ($n = 7$)

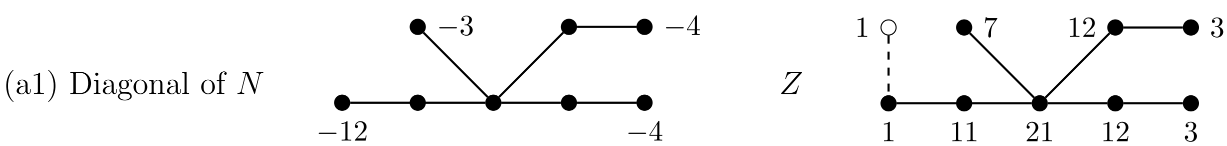


The associated group Φ_N has order 7 and $Z^2 = -4$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^7 + x^6 + y^4 = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 7.

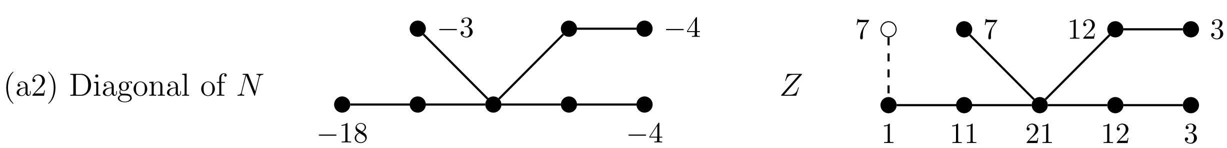
The matrix N^{-1} has three integer columns which let us use [13], Theorem 3.4, to obtain the matrices 7.4, 7.5, and 7.6 below.

The complete list of 7-suitable matrices of size $n = 8$ is long and will not be given here. It includes for instance the eight matrices listed in Section 8 with determinant 1, as well as many 7-suitable matrices on these same graphs. For instance, the graph in 8.3 supports at least six additional 7-suitable intersection matrices given below in 7.7, and in 7.4 and 7.5. The graph 5.19, which occurs as the desingularization of $z^7 + x^3 + y^4 = 0$, has $|\Phi_N| = 1$ and $n = 8$. This latter hypersurface singularity is not known to occur as a $\mathbb{Z}/7\mathbb{Z}$-quotient singularity.

Intersection Matrix 7.4. ($n = 8$)



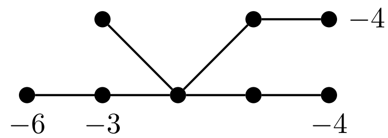
The associated group Φ_N has order 7 and $Z^2 = -1$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^7 + x^6 + y^{46} = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 7.



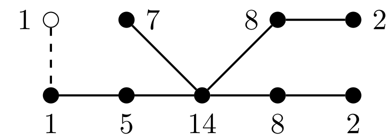
The associated group Φ_N has order 7^2 and $Z^2 = -7$. The matrix is numerically Gorenstein.

Intersection Matrix 7.5. ($n = 8$)

(a1) Diagonal of N

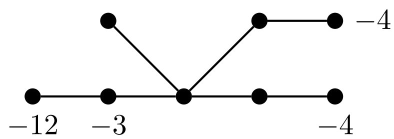


Z

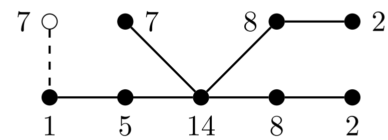


The associated group Φ_N has order 7 and $Z^2 = -1$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^7 + x^4 + y^{34} = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 7.

(a2) Diagonal of N



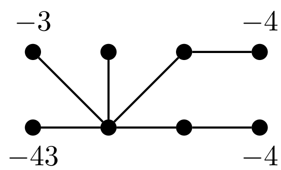
Z



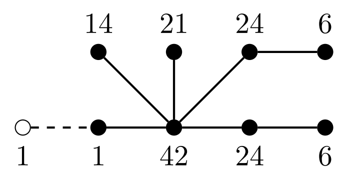
The associated group Φ_N has order 7^2 and $Z^2 = -7$. The matrix is not numerically Gorenstein.

Intersection Matrix 7.6. ($n = 8$)

(a1) Diagonal of N

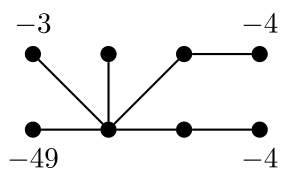


Z

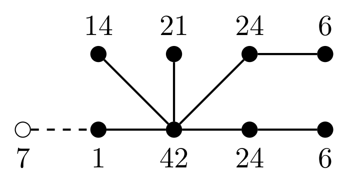


The associated group Φ_N has order 7 and $Z^2 = -1$. The matrix is numerically Gorenstein.

(a2) Diagonal of N



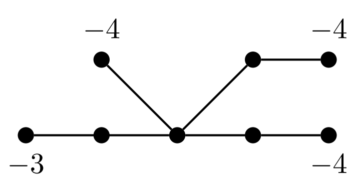
Z



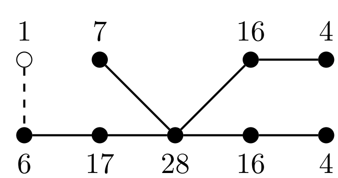
The associated group Φ_N has order 7^2 and $Z^2 = -7$. The matrix is not numerically Gorenstein.

Intersection Matrix 7.7. ($n = 8$)

(a) Diagonal of N

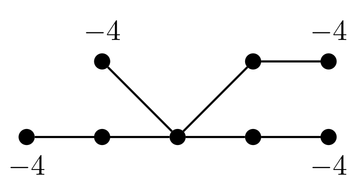


Z

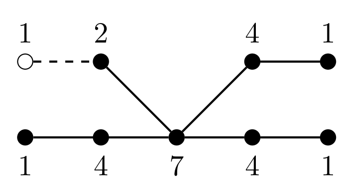


The associated group Φ_N has order 7 and $Z^2 = -6$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^7 + x^8 + y^{10} = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 7.

(b) Diagonal of N

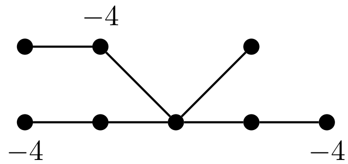


Z

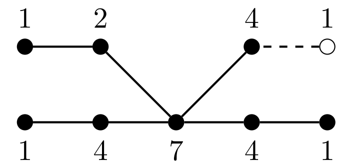


The associated group Φ_N has order 7^2 and $Z^2 = -2$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^7 + x^3 + y^{12} = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 7.

(c) Diagonal of N



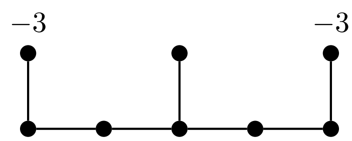
Z



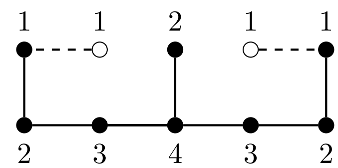
The associated group Φ_N has order 7^2 and $Z^2 = -4$. The matrix is not numerically Gorenstein.

Intersection Matrix 7.8. ($n = 8$)

Diagonal of N



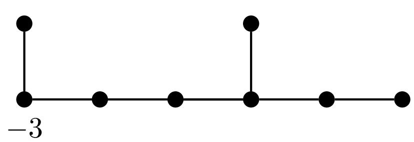
Z



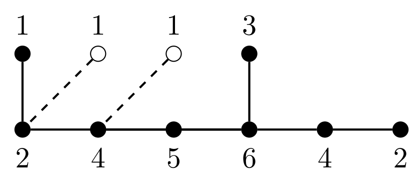
The associated group Φ_N has order 7 and $Z^2 = -2$. The matrix is numerically Gorenstein. This matrix arises as the resolution matrix of the hypersurface $f := z^7 + x^4 + y^2 = 0$. It is not known that the local ring $k[[x, y, z]]/(f)$ is a quotient singularity in characteristic 7.

Intersection Matrix 7.9. ($n = 8$) On the graph of E_8 (see also 6.7):

N



Z

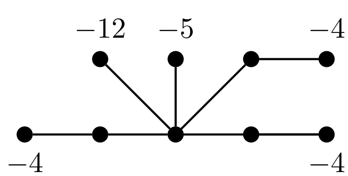


The associated group Φ_N has order 7 and $Z^2 = -6$. The matrix N^{-1} has no integer coefficient.

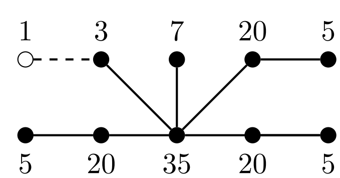
The smallest known example of a 7-suitable intersection matrix arising as a $\mathbb{Z}/7\mathbb{Z}$-quotient singularity is the Brieskorn singularity of size $n = 9$ in our next example.

Quotient Singularity 7.10. ($n = 9$)

(a) Diagonal of N



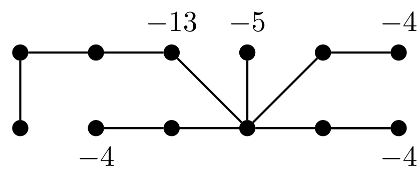
Z



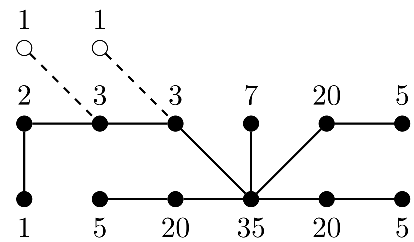
The associated group Φ_N has order 7^2 and $Z^2 = -3$. The matrix arises from the resolution of $z^7 + x^{15} + y^{36} = 0$. Using the vector Z and the matrix N, one obtains using [13], Theorem 3.4, the resolution of the quotient singularity $z^7 + x^{15} + y^{141} = 0$.

Our next matrix (b) below is obtained from N using the construction in [13], Theorem 3.9.

(b) N



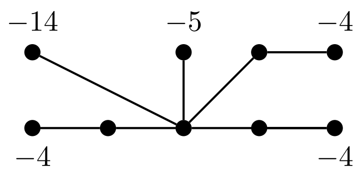
Z



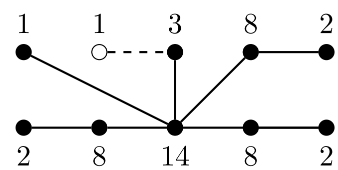
This matrix has $n = 12$. The associated group Φ_N has order 7^3 and $Z^2 = -6$. The matrix N is numerically Gorenstein.

Intersection Matrix 7.11. ($n = 9$) A modification of the matrix 7.10(a) at one vertex.

Diagonal of N



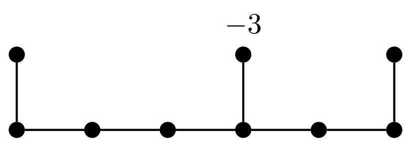
Z



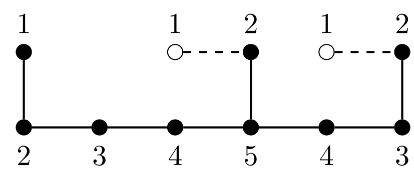
The associated group Φ_N has order 7^3 and $Z^2 = -3$. This matrix is not numerically Gorenstein.

Intersection Matrix 7.12. ($n = 9$)

Diagonal of N



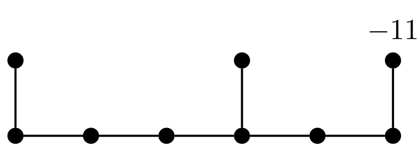
Z



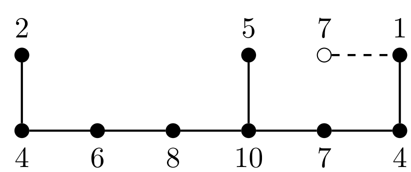
The associated group Φ_N has order 7 and $Z^2 = -4$. The matrix is not numerically Gorenstein. The matrix N^{-1} has no integer entry.

Intersection Matrix 7.13. ($n = 9$) This matrix appears in [13], 6.4. It contains E_8 as a minor. The same method produces an additional seven 7-suitable matrices with discriminant group of size 7 which contain E_8 as a minor. We omit these other examples.

N



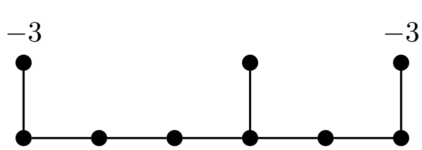
Z



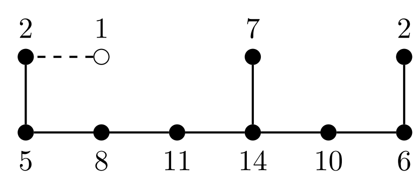
The associated group Φ_N has order 7 and $Z^2 = -7$. The matrix N is not numerically Gorenstein.

Intersection Matrix 7.14. ($n = 9$)

N

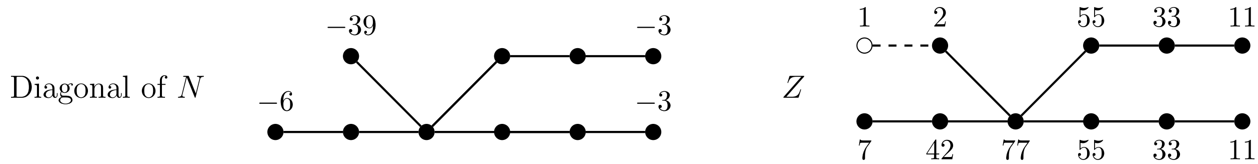


Z



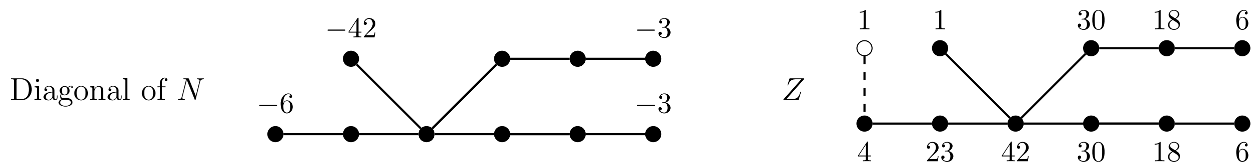
The associated group Φ_N is trivial and $Z^2 = -2$. This matrix is expected to occur as a resolution of a generalized quotient singularity. It is associated with the resolution of the hypersurface singularity given by $f = z^7 + x^2 + y^9 = 0$.

Quotient Singularity 7.15. $(n = 10)$ For completeness, we add here explicitly the case $p = 7$ in [13], Theorem 8.1.



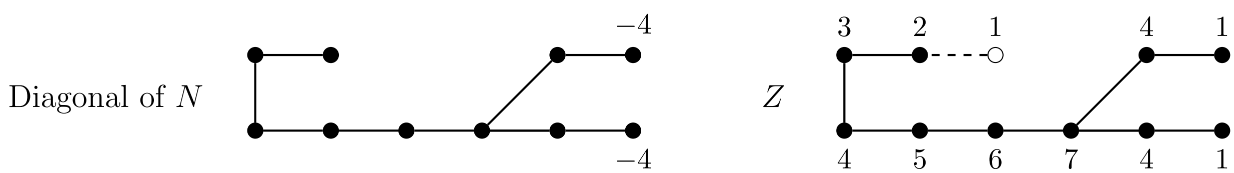
The associated group Φ_N has order 7 and $Z^2 = -2$. This matrix is numerically Gorenstein and is associated with the quotient singularity $z^7 + x^{22} + y^{78} = 0$.

Intersection Matrix 7.16. $(n = 10)$ A change at one vertex from the previous matrix.



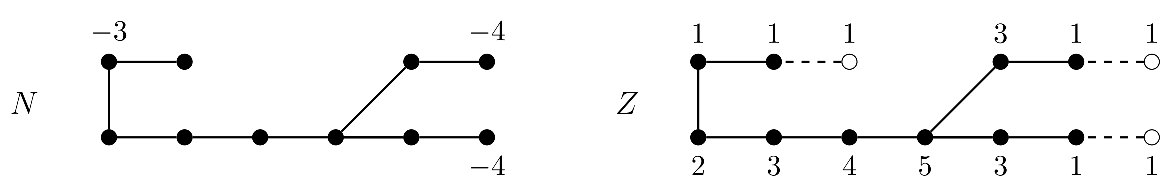
The associated group Φ_N has order 7^2 and $Z^2 = -4$. This matrix is not numerically Gorenstein.

Intersection Matrix 7.17. $(n = 10)$



The associated group Φ_N has order 7 and $Z^2 = -2$. This matrix is numerically Gorenstein and is associated with the singularity $z^7 + x^2 + y^{12} = 0$. This matrix is expected to be associated with the resolution of a generalized $\mathbb{Z}/7\mathbb{Z}$-quotient singularity.

Intersection Matrix 7.18. $(n = 10)$ A change at one vertex from the previous matrix.



The associated group Φ_N has order 7^2 and $Z^2 = -3$. This matrix is numerically Gorenstein.

8. Smallest intersection matrices of determinant 1

Much has been written on the intersection matrices of determinant 1. We refer the reader to [4]-[8] for further information. In this section, we only list the four trees of smallest size which support an intersection matrix N with $|\Phi_N| = 1$, along with all the possible associated intersection matrices of determinant 1 with self-intersections at most -2. These minimal trees on $n = 8$ vertices are listed in [4], page 520.

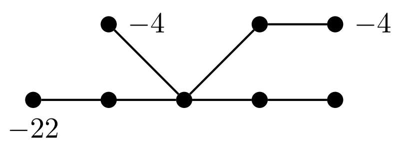
There are a total of nine known different intersection matrices N with $n = 8$ and trivial group Φ_N. We note that in each example, both matrices N and N^{-1} have a diagonal consisting only of even integers.

Quotient Singularity 8.1. The first graph in the list [4] is the graph associated with the Dynkin diagram E_8. The Dynkin diagram arises as a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity when $p = 2$ (see 4.8) and when $p = 5$ (see 6.6). When $p = 3$, it arises as a generalized $\mathbb{Z}/p\mathbb{Z}$-quotient singularity (see 5.20).

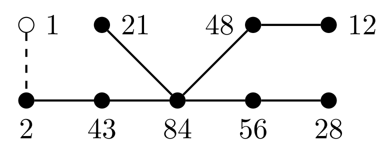
Quotient Singularity 8.2. The second graph in [4] is associated with a 3-suitable intersection matrix N described in 5.19. This matrix arises as a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity.

Intersection Matrix 8.3. The third graph in [4].

(a) Diagonal of N

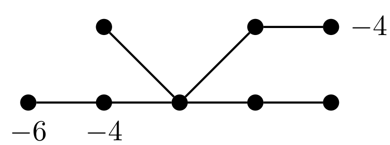


Z

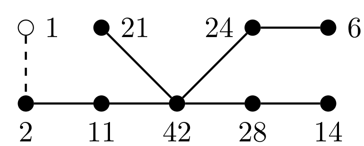


The associated group Φ_N is trivial and $Z^2 = -2$.

(b) Diagonal of N

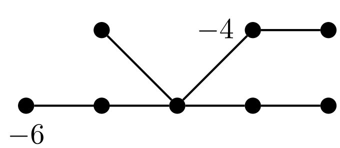


Z

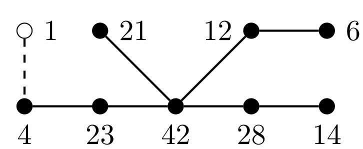


The associated group Φ_N is trivial and $Z^2 = -2$.

(c) Diagonal of N



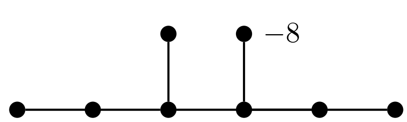
Z



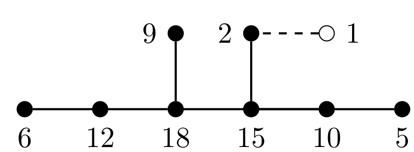
The associated group Φ_N is trivial and $Z^2 = -4$.

Intersection Matrix 8.4. The fourth graph in [4].

(a) Diagonal of N

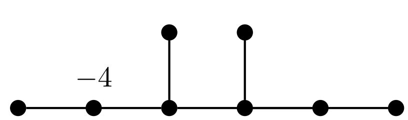


Z

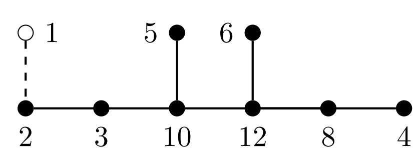


The associated group Φ_N is trivial and $Z^2 = -2$.

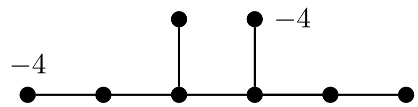
(b) Diagonal of N

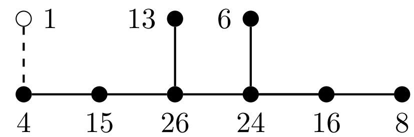


Z

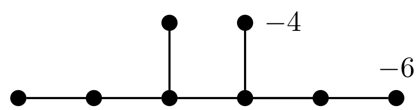


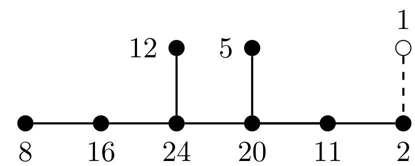
The associated group Φ_N is trivial and $Z^2 = -2$. The above graph has two vertices of degree 2. We can extend the examples (a) and (b) at each of these vertices to obtain p-suitable matrices for any p of size $n = 9$ (resp. $n = 10$) with an associated graph having three (resp. four) nodes. We can also modify the graph (b) and obtain a new 3-suitable matrix in 5.31.

(c) Diagonal of N 

Z 

The associated group Φ_N is trivial and $Z^2 = -4$.

(d) Diagonal of N 

Z 

The associated group Φ_N is trivial and $Z^2 = -2$.

9. Reduction of elliptic curves

Keep the notation of [13], 9.1.

9.1. Consider a curve X/K with a smooth model $\mathcal{Y}/\mathcal{O}_L$ over a Galois extension L/K of degree p. The special fiber $\mathcal{Y}_k$ is thus endowed with an automorphism σ of degree p. Let us assume that we are in the most special situation where σ has exactly one fixed point. When $p = 2$ and X/K is an elliptic curve, the elliptic curve $\mathcal{Y}_k/k$ is then supersingular. The quotient $\mathcal{Z}/\mathcal{O}_K$ has then an irreducible special fiber of multiplicity p, with only a single singular point on it. Resolving this $\mathbb{Z}/p\mathbb{Z}$-quotient singularity produces the regular model $\mathcal{X}'/\mathcal{O}_K$. This model need not be minimal, as the strict transform of $\mathcal{Z}_k$ in $\mathcal{X}'$ might be of self-intersection -1 in $\mathcal{X}'$ and thus be contractible. Thus one can consider the morphism $\mathcal{X}' \to \mathcal{X}_0$ to the regular minimal model[1] of X/K.

When $p = 2$ and $\mathcal{Y}_k$ is supersingular, let us assume in addition that the reduction of X/K is of Kodaira type I_0^*. The quotient construction produces $\mathcal{Z}$ with its unique singularity, and in turn we obtain the morphisms $\mathcal{X}' \to \mathcal{Z}$ and $\mathcal{X}' \to \mathcal{X}_0$. Knowing only that the special fiber of $\mathcal{X}_0$ is of type I_0^* still allows for infinitely many different intersection matrices for the resolution $\mathcal{X}' \to \mathcal{Z}$, as we now explain.

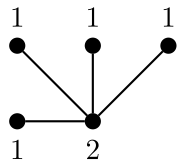
Recall that a regular model $\mathcal{X}/\mathcal{O}_K$ produces a triple (Γ, M, R) called an *arithmetical graph* as follows. The matrix M is the intersection matrix associated with the reduced curve $\mathcal{X}_k^{red}$ on the regular scheme $\mathcal{X}$. This curve has an associated connected graph Γ. Letting s denote the number of irreducible components of $\mathcal{X}_k$, we have a vector $R \in \mathbb{Z}^s_{>0}$ such that $MR = 0$. In general, for instance when X/K has a K-rational point, R is simply the vector of the multiplicities of the components of $\mathcal{X}_k$. We describe the triple (Γ, M, R) by giving the graph Γ and adorning each of its vertices with the corresponding coefficient of R. Since $MR = 0$, this data uniquely determines M.

9.2. The Kodaira type I_0^* is an arithmetical graph given by the data in (a) on the left below. The arithmetical graph in (b) on the right below is the arithmetical graph associated with the

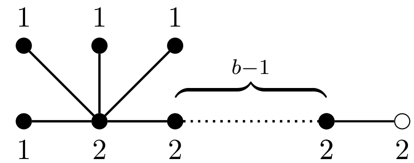
[1]In the case of elliptic curves, the possible reduction types of minimal regular models are completely classified and are labeled by a Kodaira type in $\{\mathrm{I}_0, \mathrm{I}_\mathrm{n}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_\mathrm{n}^*, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*\}$. The type I_0^* is described in 9.2.

special fiber of a sequence of b blow-ups $\beta : \mathcal{X}' \to \mathcal{X}_0$, starting with the blow-up of a regular point on the component of multiplicity 2. The component in white below is the last exceptional divisor in the sequence of blow-ups, and has self-intersection -1 on $\mathcal{X}'$.

(a)



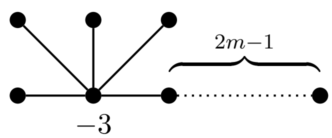
(b)



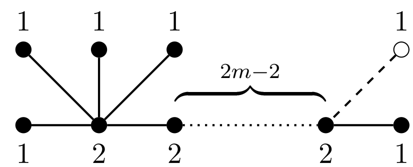
Removing this last white component leaves us with a graph Γ_N, which could potentially be the graph of the resolution of a quotient singularity coming from the resolution $\mathcal{X}' \to \mathcal{Z}$. We represent below the intersection matrix N associated with Γ_N. It turns out that for N to be such that Φ_N is killed by 2, we need $b = 2m$ to be even. (When b is odd, we still have $|\Phi_N| = 2^4$, but $\Phi_N = \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$.)

Intersection Matrix 9.3. ($n = 2m + 4$)

Diagonal of N



Z'

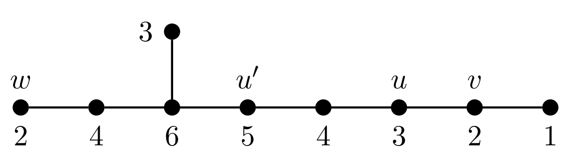


We have $(Z')^2 = -2$, which implies that the fundamental cycle Z is such that $|Z^2| \leq 2$. It is quite likely that $Z = Z'$. The associated group Φ_N has order 2^4. The intersection matrix N so constructed from the reduction type I_0^* is thus 2-suitable for any m.

The case $m = 1$ is described in 4.7, and arises as a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity (see also [9], Theorem C (iii)). The case $m = 2$ also arises as a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. It is the matrix of the resolution of $f = z^p - (abxy)^{p-1}z - a^p xy - b^p y = 0$ with $a := y^3 + xy$ and $b := x^2$.

10. Extended E_8

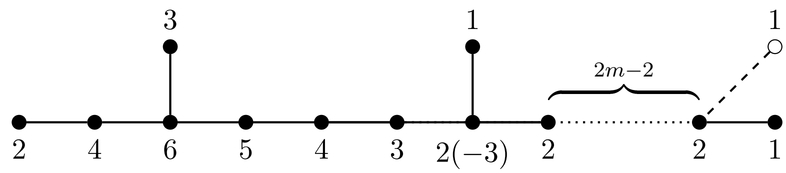
Example 10.1. The extended Dynkin diagram $\tilde{E}_8$, also called the Kodaira reduction type II*, is an arithmetical graph given by the following data:



This arithmetical graph has two vertices v and w of multiplicity 2, and both allow us to use [13], Theorem 9.5, to obtain two new families of 2-suitable intersection matrices. We describe these two families below. In case of v, removing the vertex v leaves a disjoint union of the graphs A_1 and E_7. Both associated groups are $\mathbb{Z}/2\mathbb{Z}$, so that the hypothesis of Theorem 9.5 is satisfied. In the case of w, removing the vertex w leaves the Dynkin diagram D_8, with group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so that Theorem 9.5 can also be applied. This theorem can also be applied with the vertices u or u'.

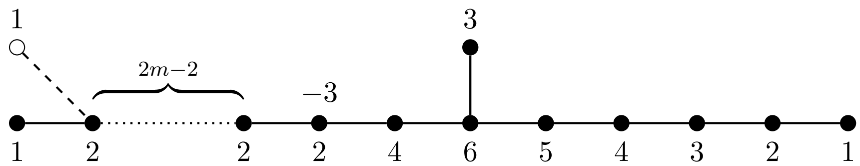
Let $m \geq 1$. The matrices below have $n = 2m + 8 \geq 10$ vertices. Each has only one diagonal coefficient different from -2. We specify the matrix by giving a vector Z' along with NZ'.

(a) Z'



The associated group Φ_N has order 2^2 and $Z'^2 = -2$.

(b) Z'

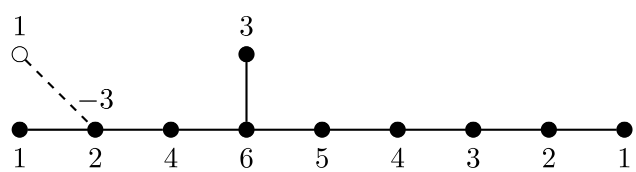


The associated group Φ_N has order 2^2 and $Z'^2 = -2$.

We do not know if both types (a) and (b) of intersection matrices above appear in the context of the resolution of quotient singularities on a model of an elliptic curve. One of them must, since there are examples of elliptic curves E/K with additive reduction of type II* over $\mathcal{O}_K$ and potentially good supersingular reduction after a quadratic extension L/K. For instance, let $K = \mathbb{F}_2(t)$. The elliptic curve E/K given by $y^2 + y = x^3 + t^{-s}$, $s \geq 1$ odd, achieves good reduction over the extension L/K given by the polynomial $z^2 + z + t^{-s}$. It has reduction of type II* modulo (t) when $s = 1 + 6r$ (see [19], 1.2).

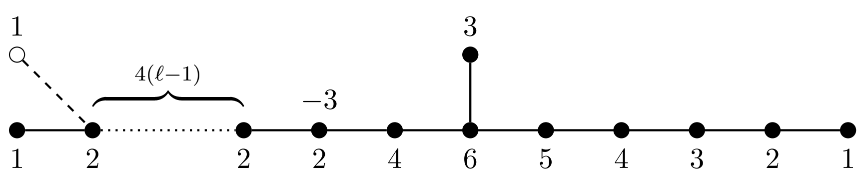
Quotient Singularity 10.2. ($n = 6 + 4\ell$, $\ell \geq 1$) Computations indicate that the matrix N below arises as a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity with equation $f := z^2 + abxyz + a^2xy + b^2y = 0$ and $a = y^{3\ell-1}$, $b = x^2$, at least for $\ell \leq 6$. We start with the case $\ell = 1$.

Z'



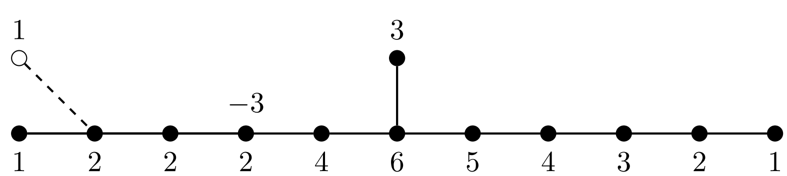
The associated group Φ_N has order 2^2 and $Z'^2 = -2$. The case $\ell > 1$:

(b) Z'



Quotient Singularity 10.3. ($n = 12$) The matrix N below arises as a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity with equation $f := z^2 + abxyz + a^2xy + b^2y = 0$ and $a = y^2$, $b = x^5$.

Z'



The associated group Φ_N has order 2^2 and $Z'^2 = -2$.

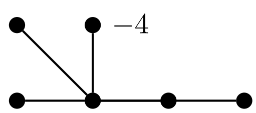
11. Smallest missing 2-suitable matrix

Remark 11.1. Consider the arithmetical graph (G, M, R) on the left below. This reduction type of a curve of genus 2 is denoted by [IV] on page 155 of [16]. Assume given a regular model with reduction type [IV]. Blowing up a closed point of the model on the interior of the component C produces a special fiber with associated arithmetical graph (G', M', R') depicted on the right below. The exceptional component of the blow-up is in white.

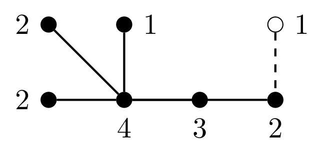
Removing from M' the row and column corresponding to the exceptional divisor (the white component) produces the intersection matrix 4.9 recalled below in 11.2. It is natural to wonder if the matrix 4.9 could occur as the resolution of a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity in the context of models of curves of genus 2. The matrix 4.9 is the only 2-suitable matrix with $n \leq 6$ for which we cannot decide whether it arises from a $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. See 4.10 for an example with $n = 7$.

11.2. Intersection matrix 4.9 ($n = 6$)

Diagonal of N



Coefficients of Z

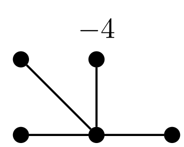


We have $Z^2 = -2$. The associated group Φ_N has order 2^2.

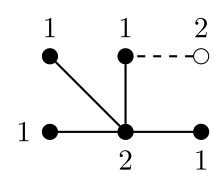
Removing one vertex to the graph produces a known quotient singularity.

11.3. Quotient Singularity 4.3 ($n = 5$)

Diagonal of N



Coefficients of Z



We have $Z^2 = -2$. The associated group Φ_N has order 2^3. This intersection matrix arises as the resolution matrix of the quotient singularity given by $g = z^2 + xy^4z + x^3y + y^7$.

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