NON ABELIAN COHOMOLOGY

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INTRODUCTION

The cohomology of abelian groups and the cohomology of a topological space with values in a sheaf of abelian groups are known for a long time.
Later on one has constructed cohomology theories for nonabelian groups and sheafs of nonabelian groups on a topological space.
All these cohomology theories have two essential properties : they are functorial and with a short exact sequence in the coefficient category they associate an exact cohomology sequence.
Giraud has developed a cohomology theory in the more general case of a sheaf of groups on a site.
This Giraud cohomology contains the preceding ones as special case. Moreover, it has the advantage that in the case of the site of the open sets of a topological space it is not of the Čechtype and consequently not limited to paracompact topological spaces.
However, it has the disadvantage that it is not truly functorial and with a short exact sequence in the coefficient category it does not associate an exact cohomology sequence.
We have tried to eliminate these two imperfections. Before we explain how this can be done we will give a brief review of the Giraud cohomology. In the second section of this paper we eliminate the two defects of the Giraud cohomology and we define the new 2-cohomology.
Most of the properties and theorems are stated without a proof, for the others only a sketch of the proof is given.
For additional information the reader is referred to [1]. The concepts and notations used in this paper are (almost) the same as those used by Giraud in [4].
I would like to thank Professor Alfred Warrinnier who suggested I undertake this study and directed my doctoral thesis in which these results first appeared. I wish also to record my thanks to professor Roger Holvoet who kindly read the manuscript.

1. THE GIRAUD COHOMOLOGY

Let E be a site and A a sheaf of groups on E. H^0 and H^1 are defined as follows :

$H^0(E,A)$ = the set of the sections of A,

$H^1(E,A)$ = the set of classes of A-isomorphic A-torsors on E.

If $u : A \to B$ is a morphism of sheafs of groups on E, then one has the following maps :

$$u^{(0)} : H^0(A) \longrightarrow H^0(B)$$
$$s \longmapsto u \circ s$$
$$u^{(1)} : H^1(A) \longrightarrow H^1(B)$$
$$p = [P] \longmapsto u^{(1)}(p) = [^uP] = [P \overset{A}{\wedge} B_d].$$

For any short exact sequence of sheafs of groups

$$1 \to A \xrightarrow{u} B \xrightarrow{v} C \to 1$$

one has an exact cohomology sequence

$$1 \to H^0(A) \xrightarrow{u^{(0)}} H^0(B) \xrightarrow{v^{(0)}} H^0(C) \xrightarrow{d} H^1(A) \xrightarrow{u^{(1)}} H^1(B) \xrightarrow{v^{(1)}} H^1(C) \qquad (*).$$

Next, one has to define H^2 such that the cohomology sequence (*) can be extended and the extended sequence is still exact. Therefore the obstruction to the relevation of a C-torsor P to B via the epimorphism $v : B \to C$ is analysed. This obstruction is an E-category K(P) whose fibre over an object S of E is the category whose objects are the couples (Q,λ) where Q is a B-torsor on E/S and λ a v-morphism from Q to P^S.
This E-category K(P) is a FIELD because the relevations can be localized and glued together. The field K(P) has the following properties :

1. There exists a refinement $\mathcal{R}$ of E such that $Ob(K(P)_S) \neq \phi$ for all S in $Ob(\mathcal{R})$, i.e. K(P) is locally not empty. Of course, one always has locally a v-morphism from B_d to P.

2. Each S-morphism is an isomorphism, i.e. the fibres are groupoids. This is clear if one realizes that a B-morphism of B-torsors is an ismorphism.

3. Any two objects of $K(P)_S$ are locally isomorphic. Indeed we have seen that two B-torsors are always locally isomorphic.

All this can be expressed more consisely by saying that K(P) is a GERBE on E. Such a gerbe is called trivial if it admits a section. Hence, the triviality of K(P) amounts to the same thing as the relevability of P to B.
So K(P) is useful as an obstruction to the relevation of a torsor.

Now we have to examine the relationship between this gerbe and the sheaf of groups A.
There exists a refinement $\mathcal{R}$ of E such that $Ob(K(P)_S \neq \phi$ for each object S of $\mathcal{R}$. For each object S of $\mathcal{R}$ we take any object (Q,λ) in the fibre category $K(P)_S$. The family of sheafs of groups

$$\{\underline{Aut}_S(Q,\lambda) \mid S \in Ob(\mathcal{R})\} \qquad (1)$$

is not coherent in the field of sheafs of groups on E. The interior automorphisms of the sheafs of groups $\underline{Aut}_S(Q,\lambda)$ prevent this family from being coherent. That is the reason why we pass on from the field of sheafs of groups on E to a new field which, roughly speaking, orginates from the former one by annihilating the interior automorphisms. This new field is called the fiel of bands on E. Each object G of the field of sheafs of groups on E determine an object in the field of bands on E. We denote it by lien (G) and we call it the band determined by G.
The field of bands on E is denoted by LIEN(E).
The family (1) determines in the field LIEN(E) a family of bands

$$\{lien(\underline{Aut}_S(Q,\lambda)) \mid S \in Ob(\mathcal{R})\}$$

which is coherent. These local bands can be glued together in LIEN(E) and con sequently they determine a band L on E.
We call L the band of the gerbe K(P).
For $S \in Ob(\mathcal{R})$ and $(Q,\lambda) \in Ob(K(P)_S)$ one has local isomorphisms

$$\underline{Aut}_S(Q,\lambda) \xrightarrow{\sim} A^S$$

and these local isomorphisms are unique up to interior automorphisms of A. Hence in the field of bands on E, one has a global isomorphism

$$lien(\underline{Aut}_S(Q,\lambda)) \xrightarrow{\sim} lien(A)^S. \qquad (2)$$

Since the isomorphisms (2) do not constitute a coherent family, one must not conclude from this that $L \xrightarrow{\sim} lien(A)$ but $L \xrightarrow{\sim} {}^P lien(A)$.
This makes clear why Giraud defines his 2-cohomology with values in a band and not with values in a sheaf of groups.
We now give some definitions and properties concerning bands. It follows from the construction of LIEN(E) that this is a split E-field and moreover, one has a morphism of split E-fields :

$$lien(E) = FAGRSC(E) \longrightarrow LIEN(E)$$
$$A \longmapsto lien(A).$$

Let L be a band on E.

If there exists an isomorphism of bands $a : L \xrightarrow{\sim} lien(A)$ then L is said to be representable and (A,a) is called a representative of the band L.
It can be proved that each band is locally representable. Let $u : L \to M$ be a morphism of bands, (A,a) a representative of L and (B,b) a representative of M. A morphism of sheafs of groups $m : A \to B$ is called a representative of u if $lien(m) \circ a = b \circ u$.
One can prove that every morphism of bands is locally representable.
We have seen that there belongs a band to the gerbe K(P) and this band is, up to an isomorphism, equal to the twisted band ${}^{P}lien(A)$ of lien(A) by the C-torsor P.
More generally, one proves that to each gerbe G there belongs a band L which is unique up to an isomorphism. Then one says that the gerbe G is bound by the band L and G is called an L-gerbe.

Let G be an L-gerbe, H an M-gerbe and $m : G \to H$ a morphism of gerbes.
Then there exists one and only one morphism of gerbes

$$u : L \to M$$

such that for each $S \in Ob(E)$ and each $x \in Ob(G_S)$ the morphism of sheafs of groups $\underline{Aut}_S(x) \to \underline{Aut}_S(m(x))$ is a representative of the restriction of u to S. We express this fact by saying that m is bound by u and we call m a u-morphism.
Moreover, it is proved that a morphism of gerbes which is bound by an isomorphism of bands, is an E-equivalence. Consequently, if G and H are L-gerbes on E and $m : G \to H$ is a id_L-morphism of gerbes, then m is an equivalence. In that case we say that G and H are L-equivalent L-gerbes.

We are now ready to introduce Giraud's 2-cohomology. Let L be a band on a site E. Then $H^2(E,L)$ is the set of the classes of L-equivalent L-gerbes on E.
If A is a sheaf of groups on E then we put

$$H^2(E,A) = H^2(E,lien(A)).$$

To simplify the notation, we shall write $H^2(L)$ instead of $H^2(E,L)$ and $H^2(A)$ instead of $H^2(E,A)$. Let $u : L \to M$ be a morphism of bands on E.
How can we associate a map

$$u^{(2)} : H^2(L) \to H^2(M)$$

with the given morphism u ?
In the situation of the H^1, this was possible via the operation "extension of the structural group".

Can we obtain the desired map in this situation through an analogous operation "extension of the structural band" ?
Giraud proves that this is not possible if the natural morphism $C_m \to C_u$ is not an isomorphism. Hence, with a given morphism of bands $u : L \to M$, we only have a relation between $H^2(L)$ and $H^2(M)$.
Two elements $p \in H^2(L)$ and $q \in H^2(M)$ stand in this relation to each other, if there exist representatives P of p, Q of q and also a u-morphism from P to Q. Then we denote : $p \longrightarrow\!\!\circ\ q$.
The so defined relation between $H^2(L)$ and $H^2(M)$ is denoted as follows :

$$H^2(L) \xrightarrow{u^{(2)}}\!\!\circ\ H^2(M).$$

In order to find the second defect of the Giraud cohomology, we consider a short exact sequence of sheafs of groups on E :

$$1 \to A \xrightarrow{a} B \xrightarrow{b} C \to 1.$$

Given a C-torsor P, we observed that the gerbe K(P) of the relevations of the C-torsor P to B is bound by the band ${}^{P}lien(A)$. So generally, the obstruction K(P) does not determine an element of $H^2(A)$ because $H^2(A)$ is the set of classes of lien(A)-equivalent gerbes.
This shows that $H^2(A)$ is generally not big enough to contain all the obstructions to the relevations of C-torsors to B.
So Giraud is obliged to introduce an adapted set :

$$O(b) = N(b)/_R .$$

In this, N(b) is the set of all triples (K,L,u) with K a gerbe, L its band and $u : L \to lien(B)$ a morphism of bands such that the following sequence is exact :

$$1 \to L \xrightarrow{u} lien(B) \xrightarrow{lien(b)} lien(C) \to 1.$$

An element (K,L,u) stands in the relation R to an element (K',L',u') if there exists a morphism of gerbes $m : K \to K'$ such that $u' \circ \alpha = u$ where $\alpha : L \to L'$ is the morphism of bands which binds m. Then one has the following cohomology sequence :

$$1 \to H^\circ(A) \to \dots \to H^1(B) \xrightarrow{b^{(1)}} H^1(C) \xrightarrow{d} O(b) \xrightarrow{a^{(2)}}\!\!\circ\ H^2(B) \xrightarrow{b^{(2)}} H^2(C).$$

2. THE NEW 2-COHOMOLOGY

We start with a new examination of the obstruction to the relevation of a torsor. Let $1 \to A \xrightarrow{u} B \xrightarrow{v} C \to 1$ be a short exact sequence of sheafs of groups on a site E.

The investigation of the obstruction to the relevation of a C-torsor P to B carried out by Giraud has shown that this obstruction is a gerbe K(P). But this result is incomplete.

To start with, there belongs a morphism of gerbes on E to K(P):

$$\nu : K(P) \longrightarrow \mathrm{TORSC}(E, \underline{\mathrm{Int}}(B)).$$

For each $(Q,\lambda) \in \mathrm{Ob}(K(P)_S)$, $S \in \mathrm{Ob}(E)$, let $\nu(Q,\lambda)$ be defined as follows :

$$\nu(Q,\lambda) = Q \overset{B}{\wedge} \underline{\mathrm{Int}}(B).$$

The action of ν on the morphisms is defined using the functoriality of the operation contracted product.

Next, for each object $\nu(Q,\lambda)$ of $K(P)_S$, $S \in \mathrm{Ob}(E)$, one has a natural isomorphism

$$k_{(Q,\lambda)} : \underline{\mathrm{Aut}}_S(Q,\lambda) \xrightarrow{\sim} \nu(Q,\lambda) \overset{\underline{\mathrm{Int}}(B)}{\wedge} A$$

because $\underline{\mathrm{Aut}}_S(Q,\lambda)$ as well as $\nu(Q,\lambda) \overset{\underline{\mathrm{Int}}(B)}{\wedge} A$ is the twisted group of A by the $\underline{\mathrm{Int}}(B)$-torsor $\nu(Q,\lambda)$.

Since these isomorphisms are compatible with the restriction they determine an isomorphism of morphisms of fields :

$$k : \mathrm{Aut}(K(P)) \xrightarrow{\simeq} (- \overset{\underline{\mathrm{Int}}(B)}{\wedge} A) \circ \nu$$

where $- \overset{\underline{\mathrm{Int}}(B)}{\wedge} A$ denotes the E-functor from $\mathrm{TORSC}(E,\underline{\mathrm{Int}}(B))$ to FAGRSC(E) which associates with a $\underline{\mathrm{Int}}(B)$-torsor P the sheaf of groups $P \overset{\underline{\mathrm{Int}}(B)}{\wedge} A$.

The triple $(K(P),\nu,k)$ satisfies the following condition. Let S be an object of E and (Q,λ) an object of the fibre category $K(P)_S$.

For each element $x \in \nu(Q,\lambda) \overset{\underline{\mathrm{Int}}(B)}{\wedge} A$ and each local representative (σ,a) of x we have :

$$\nu(k^{-1}_{(Q,\lambda)}(x))(\sigma) = \sigma \circ \mathrm{int}(a).$$

This condition expresses the commutativity of the following diagram :

$$\begin{array}{ccc}
\underline{\mathrm{Aut}}_S(Q,\lambda) & \xrightarrow{\mathrm{Aut}(\nu)_{(Q,\lambda)}} & \underline{\mathrm{Aut}}_S(\nu(Q,\lambda)) \\
{\scriptstyle k_{(Q,\lambda)}}\ \wr\downarrow & & \downarrow\wr\ {\scriptstyle t_{\nu(Q,\lambda)}} \\
\nu(Q,\lambda) \overset{\underline{\mathrm{Int}}(B)}{\wedge} A & \xrightarrow[\mathrm{id}\,\wedge\,\rho]{} & \nu(Q,\lambda) \overset{\underline{\mathrm{Int}}(B)}{\wedge} \underline{\mathrm{Int}}(B)
\end{array}$$

where $t_{\nu(Q,\lambda)}$ is the canonical isomorphism between two sheafs of groups which both are the twisted group of $\underline{\mathrm{Int}}(B)$ by $\nu(Q,\lambda)$ and $\rho : A \to \underline{\mathrm{Int}}(B)$ is defined by $\rho(a) = \mathrm{int}(a)$.

So the obstruction to the relevation of C-torsor P to B is a triple

$$(K(P),\nu,k)$$

satisfying the foregoing condition.

Let $\Phi = (A,\rho,\Pi,\phi)$ be a sheaf of crossed groups on the site E. We shall frequently use the abbreviated notation (A,Π). The result of the analysis of the obstruction to the relevation of a torsor suggests the following definition of (A,Π)-gerbe.

<u>Definition</u> : A <u>(A,Π)-gerbe on E</u> is a triple

$$(G,\mu,j)$$

where G is a gerbe on E, $\mu : G \longrightarrow \mathrm{TORSC}(E,\Pi)$ a morphism of gerbes on E, and $j : \mathrm{Aut}(G) \longrightarrow (- \overset{\Pi}{\wedge} A) \circ \mu$ an ismorphism of morphisms of fields on E.

These data must satisfy the following condition :

For each object S of E and each $x \in \mathrm{Ob}(G_S)$ the following diagram commutes :

$$\begin{array}{ccc}
\underline{\mathrm{Aut}}_S(x) & \xrightarrow{\mathrm{Aut}(\mu)_x} & \underline{\mathrm{Aut}}_S(\mu(x)) \\
{\scriptstyle j_x}\,\wr\downarrow & & \downarrow\wr\ {\scriptstyle t_{\mu(x)}} \\
\mu(x) \overset{\Pi}{\wedge} A & \xrightarrow[\mathrm{id}\,\wedge\,\rho]{} & \mu(x) \overset{\Pi}{\wedge} \Pi
\end{array}$$

<u>Remark</u> : This condition means that the morphism $\mathrm{Aut}(\mu)_x$ is identified, via the isomorphisms j_x and $t_{\mu(x)}$, with the twisted morphism

$$^{\mu(x)}\rho = \mathrm{id} \wedge \rho$$

<u>Definition</u> : Let $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ be a morphism of sheafs of crossed groups, (G,μ,j) a (A,Π)-gerbe and (G',μ',j') a (A',Π')-gerbe on E.

A <u>(f,φ)-morphism</u> from (G,μ,j) to (G',μ',j') is an ordered pair

$$(\lambda,i)$$

where $\lambda : G \longrightarrow G'$ is a morphism of gerbes on E and $i : \mathrm{TORSC}(E,\varphi) \circ \mu \xrightarrow{\sim} \mu' \circ \lambda$ an isomorphism of morphisms of gerbes such that the following diagram commutes :

$$\begin{array}{ccc}
Aut(G) & \xrightarrow{Aut(\lambda)} & Aut(G') \circ \lambda \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j' * \lambda} \\
(- \overset{\Pi}{\wedge} A) \circ \mu & \xrightarrow[\omega]{} & ((- \overset{\Pi'}{\wedge} A') \circ \mu') \circ \lambda
\end{array}$$

where for each object x of G_S, $S \in Ob(E)$, the morphism ω_x is equal to the following composite :

$$\mu(x) \overset{\Pi}{\wedge} A \xrightarrow[p_x \wedge f]{} (\mu(x) \overset{\Pi}{\wedge} \Pi') \overset{\Pi'}{\wedge} A' \xrightarrow[i_x \wedge id_{A'}]{} \mu'(\lambda(x)) \overset{\Pi}{\wedge} A'$$

and p_x is the evident morphism from $\mu(x)$ to $\mu(x) \overset{\Pi}{\wedge} \Pi'$. We shall now define the composite of two such morphisms. Let $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ and $(g,\psi) : (A', \Pi') \longrightarrow (A'', \Pi'')$ be morphisms of sheafs of crossed groups on the site E. Suppose we are given a (f,φ)-morphism (δ,i) : $(G,\mu,j) \longrightarrow (G',\mu',j')$ and a (g,ψ)-morphism $(\lambda,m) : (G',\mu',j') \longrightarrow (G'',\mu'',j'')$. The composite

$$(\lambda,m) \circ (\delta,i)$$

is by definition the $(g,\psi) \circ (f,\varphi)$-morphism $(\lambda \circ \delta, m \,\bar{*}\, i)$. For each object x of G_S, $S \in Ob(E)$, the morphism $(m \,\bar{*}\, i)_x$ is defined by

$$(m \,\bar{*}\, i)_x = m_{\delta(x)} \circ (i_x \wedge \Pi'') \circ c_x$$

where

$$c_x : \mu(x) \overset{\Pi}{\wedge} \Pi'' \xrightarrow{\sim} (\mu(x) \overset{\Pi}{\wedge} \Pi') \overset{\Pi'}{\wedge} \Pi''$$

follows from the associativity of the contracted product and the isomorphism

$$\Pi' \overset{\Pi'}{\wedge} \Pi'' \xrightarrow{\sim} \Pi''.$$

<u>Definition</u> : Let $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ be a morphism of sheafs of crossed groups, (λ_1,i_1) and (λ_2,i_2) two (f,φ)-morphisms of the (A,Π)-gerbe (G,μ,j) to the (A',Π')-gerbe (G',μ',j'). A morphism <u>m from (λ_1,i_1) to (λ_2,i_2)</u> is a morphism of E-functors

$$m : \lambda_1 \longrightarrow \lambda_2$$

such that $(\mu' * m) \circ i_1 = i_2$.

<u>Properties</u> : We shall now give two properties which will allow us to define the new H^2.

(i) Let $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ be a morphism of sheafs of crossed groups on E and $(\lambda,i) : (G,\mu,j) \longrightarrow (G',\mu',j')$ a (f,φ)-morphism of gerbes on E. If (f,φ) is an isomorphism, then λ is an E-equivalence.

<u>Corollary</u> : If $(\lambda,i) : (G,\mu,j) \longrightarrow (G',\mu',j')$ is an $id_{(A,\Pi)}$-morphism, then λ is an E-equivalence.
Then we say that (G,μ,j) and (G',μ',j') are <u>(A,Π)-equivalent gerbes</u> and (λ,i) is called an <u>(A,Π)-equivalence</u>.

(ii) Let (A,Π) be a sheaf of crossed groups on E. The gerbe TORSC(E,A) of A-torsors on E has a canonical structure of (A,Π)-gerbe.
TORSC(E,A) supplied with this canonical structure of (A,Π)-gerbe, will be denoted as follows :

$$(TORSC(E,A),\mu_\Pi,j_\Pi).$$

We shall use the notation $(TORSC(E,A'),\mu'_{\Pi'},j'_{\Pi'})$ to express the fact that we consider the gerbe TORSC(E,A') supplied with its canonical structure of (A',Π')-gerbe.

The cohomology with values in a sheaf of crossed groups $\Phi = (A,\rho,\Pi,\phi)$ on a site E coincides with Giraud's cohomology in dimension 0 and 1.
So we have

$$H^0(E,\Phi) = H^0(A) = \varprojlim(A)$$

$$H^1(E,\Phi) = H^1(A) = \text{the set of classes of A-isomorphic A-torsors on E.}$$

<u>Definition</u>

$H^2(E,\Phi)$ = the set of classes of (A,Π)-equivalent (A,Π)-gerbes on E.

A class is called neutral if there exists a representative which admits a section. The class containing the (A,Π)-gerbe $(TORSC(E,A),\mu_\Pi,j_\Pi)$ is called the <u>unit element</u>. We shall mostly use the notation $H^2(A,\Pi)$ instead of $H^2(E,\Phi)$

Now we shall show that the new H^2 is functorial. The essential part of the proof consists in showing that it is possible to associate to a given morphism $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ a <u>map</u> from $H^2(A,\Pi)$ to $H^2(A',\Pi')$.

Theorem : Let $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ be a morphism of sheafs of crossed groups and let (G,μ,j) be a (A,Π)-gerbe. Then there exists always a (A',Π')-gerbe (G',μ',j') and also a (f,φ)-morphism $(\lambda,i) : (G,\mu,j) \longrightarrow (G',\mu',j')$.

Proof : Since Giraud has proved that each fibred E-category is E-equivalent to a split E-category, we may assume that G is split.
The construction of G' is carried out in two steps which will be sketched now.

First step : The construction of the split E-prefield G^{*}.

Let S be an object of E. How shall we define the category $G^{*}(S)$?
The objects of $G^{*}(S)$ are, by definition, those of the fibre category G_S. Thus

$$Ob(G^{*}(S)) = Ob(G_S) \text{ for each } S \in Ob(E).$$

Let x and y be objects of $G^{*}(S)$. In what follows the notation $\mu(x,A)$ will be used instead of $\mu(x) \overset{\Pi}{\wedge} A$.
The morphism $\varphi : \Pi \longrightarrow \Pi'$ allows us to consider A' as a left Π-object.
Hence, the contracted product $\mu(x) \overset{\Pi}{\wedge} A'$ exists and we denote it by $\mu(x,A')$.
The sheaf of groups $\underline{Aut}_S(x)$ operates on the sheaf of sets $\underline{Isom}_S(x,y)$ on the right by composition of morphisms.
Then $\underline{Isom}_S(x,y)$, because of the isomorphism $j_x : \underline{Aut}_S(x) \xrightarrow{\sim} \mu(x,A)$, becomes a right $\mu(x,A)$-object. Since the contracted product is functorial we have a morphism $\mu(x,f) : \mu(x,A) \longrightarrow \mu(x,A')$ which allows us to consider $\mu(x,A')$ as a left $\mu(x,A)$-object. We then define :

$$\underline{Isom}^{*}_S(x,y) = \underline{Isom}_S(x,y) \overset{\mu(x,A)}{\wedge} \mu(x,A').$$

The S-morphisms from x to y in $G^{*}(S)$ are defined as the sections of the sheaf $\underline{Isom}^{*}_S(x,y)$. The set of all these morphisms from x to y will be denoted as follows :

$$Isom^{*}_S(x,y).$$

For any three objects x, y and z one has a map

$$Isom^{*}_S(x,y) \times Isom^{*}_S(y,z) \longrightarrow Isom^{*}_S(x,z)$$

which is defined by the following "formula" :

$$([(m,a'_x)],[(n,a'_y)]) \longmapsto [(n \circ m, \mu(m^{-1},A')(a'_y).a'_x)].$$

This defines the composition of morphisms in $G^{*}(S)$.
The reader is referred to [1] for further details about the construction of G^{*}. The construction of G^{*} yields a morphism of split E-categories :

$$\lambda^{*} : G \longrightarrow G^{*} \qquad \lambda^{*}(x) = x,\ x \in Ob(G_S). \tag{1}$$

Second step. The (A',Π')-gerbe (G',μ',j').

We now consider the field associated with G^{*} :

$$G' = A(G^{*}).$$

Moreover, one has a bicovering morphism

$$a : G^{*} \longrightarrow G' = A(G^{*}). \tag{2}$$

The composition of (1) and (2) yields a morphism of split E-categories :

$$\lambda : G \longrightarrow G'.$$

The E-category G' is a gerbe. Besides, we have a cartesian E-functor $\mu' : G' \longrightarrow TORSC(E,\Pi')$ and also an isomorphism of morphisms of fields $j' : Aut(G') \xrightarrow{\sim} (- \overset{\Pi'}{\wedge} A') \circ \mu'$ such that (G',μ',j') is a (A',Π')-gerbe.
Furthermore, for each $x \in Ob(G_S)$, $S \in Ob(E)$ one has a natural isomorphism

$$i_x : \mu(x) \overset{\Pi}{\wedge} \Pi' \longrightarrow \mu'(\lambda(x))$$

such that (λ,i) is a (f,φ)-morphism.

Theorem : Let $(f,\varphi) : (A,\Pi) \longrightarrow (A',\Pi')$ be a morphism of sheafs of crossed groups, $(\lambda_1,i_1) : (G,\mu,j) \longrightarrow (G_1,\mu_1,j_1)$ and $(\lambda_2,i_2) : (G,\mu,j) \longrightarrow (G_2,\mu_2,j$
(f,φ)-morphisms. Then one has an (A',Π')-equivalence.

$$(\delta,\varepsilon) : (G_1,\mu_1,j_1) \xrightarrow{\approx} (G_2,\mu_2,j_2)$$

such that

$$(\delta,\varepsilon) \circ (\lambda_1,i_1) \simeq (\lambda_2,i_2).$$

Proof : Let (G',μ',j') be the (A',Π')-gerbe and (λ,i) the (f,φ)-morphism from (G,μ,j) to (G',μ',j') of the preceding theorem.
Then λ is the following composite :

$$G \xrightarrow{\lambda^{*}} G^{*} \xrightarrow{a} G' = A(G^{*}).$$

It suffices to prove the existence of $id_{(A',\Pi')}$-morphisms

$$(\delta_1,\varepsilon_1) : (G',\mu',j') \longrightarrow (G_1,\mu_1,j_2) \text{ and } (\delta_2,\varepsilon_2) : (G',\mu',j') \longrightarrow (G_2,\mu_2$$

such that

$$(\delta_1,\varepsilon_1) \circ (\lambda,i) \simeq (\lambda_1,i_1) \text{ and } (\delta_2,\varepsilon_2) \circ (\lambda,i) \simeq (\lambda_2,i_2).$$

Indeed, if $(\delta'_1,\varepsilon'_1)$ denotes a quasi-inverse of (δ_1,ε_1), then we have :

$$(\delta_2,\varepsilon_2)\circ(\delta'_1,\varepsilon'_1)\circ(\lambda_1,i_1)\simeq(\lambda_2,i_2).$$

Thus

$$(\delta,\varepsilon)=(\delta_2,\varepsilon_2)\circ(\delta'_1,\varepsilon'_1)$$

is the required equivalence.

In order to prove the existence of (δ_1,ε_1) it suffices to show that there exist a morphism $\delta_1^* : G^* \longrightarrow G_1$ and an isomorphism $\varepsilon_1^* : TORSC(E,1_{\Pi'})\circ\mu^* \longrightarrow \mu_1\circ\delta_1^*$ such that

(i) $\delta_1^*\circ\lambda^* = \lambda_1$.

(ii) For each object x of G^*_S, $S\in Ob(E)$ the following diagram commutes :

$$\begin{array}{ccc}
\underline{Aut}^*_S(x) & \xrightarrow{\;Aut(\delta_1^*)_x\;} & \underline{Aut}_S(\delta_1^*(x)) \\
{\scriptstyle j^*_x}\downarrow\wr & & \wr\downarrow{\scriptstyle j_{1\delta_1^*(x)}} \\
\mu^*(x)\overset{\Pi'}{\wedge}A' & \xrightarrow[\;\omega^*_x\;]{} & \mu_1(\delta_1^*(x))\overset{\Pi'}{\wedge}A'
\end{array}$$

where ω^*_x is the following composite

$$\mu^*(x)\overset{\Pi'}{\wedge}A' \xrightarrow[p^*_x\wedge 1_{A'}]{\sim} (\mu^*(x)\overset{\Pi'}{\wedge}\Pi')\overset{\Pi'}{\wedge}A' \xrightarrow[\varepsilon^*_{1x}\wedge 1_{A'}]{} \mu_1(\delta_1^*(x))\overset{\Pi'}{\wedge}A'.$$

For each $x\in Ob(G^*_S)$ we have $\mu^*(x)=\mu(x)\overset{\Pi}{\wedge}\Pi'$ and the isomorphism j^*_x is defined as the composite of the isomorphisms

$$\mu^*(x)\overset{\Pi'}{\wedge}A' = (\mu(x)\overset{\Pi}{\wedge}\Pi')\overset{\Pi'}{\wedge}A' \simeq \mu(x)\overset{\Pi}{\wedge}A'$$

and

$$\underline{Aut}^*_S(x) = \underline{Aut}_S(x)\overset{\mu(x,A)}{\wedge}\mu(x,A') \simeq \mu(x,A)\overset{\mu(x,A)}{\wedge}\mu(x,A') \simeq \mu(x)\overset{\Pi}{\wedge}A'.$$

Since λ^* is the identity on the objects, we are obliged to define the action of δ_1^* on the objects as follows :

$$\delta_1^*(x)=\lambda_1(x) \text{ for each } x\in Ob(G^*_S),\ S\in Ob(E).$$

Next, for each $x\in Ob(G^*_S)$, we define

$$\varepsilon^*_{1x}=i_{1x}\circ p^{*-1}_x.$$

In order to determine the action of δ_1^* on morphisms it suffices to define its action on the S-automorphisms of an object x of G^*_S. So, let m be an S-automorphisms of x. Then we put

$$\delta_1^*(m)=\left(j^{-1}_{1\delta_1^*(x)}\circ\omega^*_x\circ j^*_x\right)(m).$$

The existence of (δ_2,ε_2) is proved similarly.

<u>Functoriality of H^2</u>

Let $(f,\varphi) : (A,\Pi)\longrightarrow(A',\Pi')$ be a morphism of sheafs of crossed groups on the site E. With this we associate the <u>mapping</u>

$$(f,\varphi)^{(2)} : H^2(A,\Pi)\longrightarrow H^2(A',\Pi')$$

which assigns to an element $g=[(G,\mu,j)]$ of $H^2(A,\Pi)$ the element $(f,\varphi)^{(2)}(g)=[(F,\nu,k)]$ of $H^2(A',\Pi')$ where (F,ν,k) is a (A',Π')-gerbe for which there exists a (f,φ)-morphism $(\delta,\varepsilon) : (G,\mu,j)\longrightarrow(F,\nu,k)$. This mapping sends the neutral elements of $H^2(A,\Pi)$ over to neutral elements of $H^2(A',\Pi')$. For each couple of composable morphisms of sheafs of crossed groups

$$(A,\Pi)\xrightarrow[(f,\varphi)]{}(A',\Pi')\xrightarrow[(g,\psi)]{}(A'',\Pi'')$$

one has that

$$(g,\psi)^{(2)}\circ(f,\varphi)^{(2)}=(g\circ f,\psi\circ\varphi)^{(2)}.$$

<u>The second coboundary mapping</u>. Let

$$e\to\Phi=(A,\rho,\Pi,\phi)\xrightarrow[(f,1_\Pi)]{}\Phi'=(A',\rho',\Pi,\phi')\xrightarrow[(h,\psi)]{}\Phi''=(A'',\rho'',\Pi'',\phi'')\to e$$

be a short exact sequence of sheafs of crossed groups on the site E.

<u>Theorem</u> : Let $(\lambda,i) : (G'',\mu',j')\longrightarrow(G'',\mu'',j'')$ be a (h,ψ)-morphism of gerbes.

If s is a section of G", then the gerbe K(s) of the relevations of s to G' has a structure of (A,Π)-gerbe. K(s) supplied with this structure will be denoted by $(K(s),\mu_s,j_s)$. Moreover, one has an evident isomorphism

$$i(s) : \mathrm{TORSC}(E,1_\Pi) \circ \mu_s \longrightarrow \mu' \circ k(s)$$

such that

$$(k(s),i(s)) : (K(s),\mu_s,j_s) \longrightarrow (G',\mu',j')$$

is a $(f,1_\Pi)$-morphism.

<u>Proof</u> : First we shall show that K(s) has a structure of (A,Π)-gerbe. So we begin by defining a morphism of gerbes

$$\mu_s : K(s) \longrightarrow \mathrm{TORSC}(E,\Pi).$$

For any S-object (z,m) of K(s) we define

$$\mu_s(z,m) = \mu'(z).$$

Next we have to show that there exists a natural isomorphism

$$j_s(z,m) : \underline{\mathrm{Aut}}_S(z,m) \longrightarrow \mu_s(z,m) \overset{\Pi}{\wedge} A.$$

Since G' is a (A',Π)-gerbe and G" is a (A",Π")-gerbe one has the following diagram :

$$\begin{array}{ccccc}
\underline{\mathrm{Aut}}_S(z,m) & \xrightarrow{\mathrm{Aut}(k(s))_{(z,m)}} & \underline{\mathrm{Aut}}_S(z) & \xrightarrow{\mathrm{Aut}(\lambda)_z} & \underline{\mathrm{Aut}}_S(\lambda(z)) \\
 & & \Big\downarrow{\scriptstyle j'_z \,\wr} & & \Big\downarrow{\scriptstyle \wr\, j''_{\lambda(z)}} \\
\mu'(z) \overset{\Pi}{\wedge} A & \xrightarrow[\mathrm{id}_{\mu'(z)} \wedge f]{} & \mu'(z) \overset{\Pi}{\wedge} A' & \xrightarrow[\beta_z = (i_z \wedge h) \circ (p_z \wedge \mathrm{id}_{A'})]{} & \mu''(\lambda(z)) \overset{\Pi''}{\wedge} A''
\end{array}$$

Since $\underline{\mathrm{Aut}}_S(z,m)$ as well as $\mu'(z) \overset{\Pi}{\wedge} A$ is a kernel of β_z there exists a unique isomorphism

$$j_s(z,m) : \underline{\mathrm{Aut}}_S(z,m) \xrightarrow{\sim} \mu'(z) \overset{\Pi}{\wedge} A$$

such that

$$(\mathrm{id}_{\mu'(z)} \wedge f) \circ j_s(z,m) = j'_z \circ \mathrm{Aut}(k(s))_{(z,m)}.$$

In this way we obtain a (A,Π)-gerbe $(K(s),\mu_s,j_s)$.
Next, for each object (z,m) of $K(s)_S$, one has a natural isomorphism

$$i(s)(z,m) : \mu_s(z,m) \overset{\Pi}{\underset{,}{\wedge}} \Pi \longrightarrow \mu'(k(s)(z,m))$$

because

$$\mu_s(z,m) \overset{\Pi}{\underset{,}{\wedge}} \Pi = \mu'(z) \overset{\Pi}{\wedge} \Pi,\ \mu'(k(s)(z,m)) = \mu'(z) \text{ and } \mu'(z) \overset{\Pi}{\wedge} \Pi \simeq \mu'(z)$$

It is easily checked that (k(s),i(s)) is a $(f,1_\Pi)$-morphism.

<u>Application</u>

The preceding theorem may be applied to the (h,ψ)-morphism

$$(\mathrm{TORSC}(E,h),i_{(h,\psi)}) : (\mathrm{TORSC}(E,A'),\mu'_\Pi,j'_\Pi) \longrightarrow (\mathrm{TORSC}(E,A''),\mu''_{\Pi''},j''_{\Pi''})$$

where, for each A'-torsor Q, the isomorphism $i_{(h,\psi)}(Q)$ is obtained by composing the natural isomorphism

$$\mu'_\Pi(Q) \overset{\Pi}{\wedge} \Pi'' \xrightarrow{\sim} Q \overset{A'}{\wedge} \Pi''$$

with the inverse of the isomorphism

$$\mu''_{\Pi''}(Q \overset{A'}{\wedge} A'') \xrightarrow{\sim} Q \overset{A'}{\wedge} \Pi''.$$

A A"-torsor P determines a section of TORSC(E,A"). It will be denoted by s_P. Because of the preceding theorem, we have a structure of (A,Π)-gerbe on the gerbe K(P) of the relevations of P to A'. The gerbe K(P) supplied with this structure will be denoted by

$$(K(P),\mu_P,j_P)$$

Furthermore, we have an isomorphism

$$i_P : \mathrm{TORSC}(E,1_\Pi) \circ \mu_P \xrightarrow{\sim} \mu_\Pi \circ k(P)$$

such that $(k(P),i_P)$ is a $(f,1_\Pi)$-morphism.

<u>Definition</u>. With a short exact sequence

$$e \to \Phi = (A,\Pi) \xrightarrow[(f,1_\Pi)]{} \Phi' = (A',\Pi) \xrightarrow[(h,\psi)]{} \Phi'' = (A'',\Pi'') \to e \qquad (*)$$

we associate a mapping $d : H^1(E,\Phi'') \longrightarrow H^2(E,\Phi)$ which assigns to any element $p = [P]$ of $H^1(E,\Phi'')$ the element $[(K(P),\mu_P,j_P)]$ of $H^2(E,\Phi)$. This map d is called the <u>second coboundary map.</u>

The short exact sequence (*) gives rise to the cohomology sequence

$$1 \to H^0(E,\Phi) \to \dots \to H^1(E,\Phi'') \xrightarrow{d} H^2(E,\Phi) \xrightarrow{(f,1_\Pi)^{(2)}} H^2(E,\Phi') \xrightarrow{(h,\psi)^{(2)}} H^2(E,\Phi'')$$

which is exact in the sense of the following theorem.

Theorem

(i) An element p of $H^1(E,\Phi'')$ belongs to $\mathrm{Im}(h,\psi)^{(1)}$ if and only if d(p) is neutral

(ii) An element x of $H^2(E,\Phi)$ belongs to Im d if and only if $(f,1_\Pi)^{(2)}(x)$ is the unit element.

(iii) An element x of $H^2(E,\Phi')$ belongs to $\mathrm{Im}(f,1_\Pi)^{(2)}$ if and only if $(h,c)^{(2)}(x)$ is neutral.

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