Reflections generate $O_{n,1}(\mathbb{Z})$ for $n \leq 19$

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Abstract

We extend results of C. T. C. Wall to give a computer-assisted proof of the fact that the orthogonal group $O_{n,1}(\mathbb{Z})$ is generated by reflections for $n \leq 19$ and give explicit finite generating sets.

Keywords:

1 Introduction

Let $\mathbb{Z}^n \oplus \mathbb{Z}$ denote the $n+1$ dimensional lattice with the indefinite symmetric bilinear form $\langle \cdot, \cdot \rangle$ given by the matrix

$$I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

or in formulas for $v = (v_1, v_2, \cdots, v_n, v_{n+1})$ and $w = (w_1, w_2, \cdots, w_n, w_{n+1})$ in $\mathbb{Z}^n \oplus \mathbb{Z}$, which we view as row vectors, we have

$$\langle v, w \rangle = v I_{n,1} w^T = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n - v_{n+1} w_{n+1}.$$

We will refer to the first n coordinates as the positive coordinates and the final coordinate as the negative coordinate. It is often convenient to be able to talk about

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the positive coordinates of a vector v as a whole or the negative coordinate of v, so we will denote these by $v^+ = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ and $v^- = v_{n+1} \in \mathbb{Z}$, respectively. Thus we can write the indefinite form as $\langle v, w\rangle = v^+ \cdot w^+ - v^- w^-$, where $\cdot$ denotes the ordinary dot product on $\mathbb{Z}^n$.

The group $O_{n,1}(\mathbb{Z})$ is the set of all invertible $(n+1) \times (n+1)$ integer matrices that preserve this symmetric bilinear form. The easiest elements of this group to understand are the reflections, where the reflection R_w in the vector w is given by the formula

$$R_w(x) = x - \frac{2\langle x, w\rangle}{\langle w, w\rangle} w,$$

and since we want integral maps we must impose the constraint that $\langle w, w\rangle = \pm 1$ or ± 2. We will refer to such vectors w as reflection vectors. The goal of this paper is to give a computer-assisted proof of the following theorem.

Theorem 1. *The group $O_{n,1}(\mathbb{Z})$ is generated by reflections for $n \leq 19$.*

The proof will do a little more than this, it will give for each $n \leq 19$ a modestly-sized, explicit list of reflections that generate $O_{n,1}(\mathbb{Z})$ (though we have made no effort to check if these lists are minimal). This result for $n \leq 9$ was proved by C. T. C. Wall [9], and our argument can be viewed as a direct extension of Wall's proof, but reworked slightly to allow a computer to perform the key calculations. We will return to this point below. (Wall also proved that reflections generate a subgroup of index at most two in $O_{n,k}(\mathbb{Z})$ for $n, k \geq 2$, so the case $k = 1$ above is the interesting case.)

Theorem 1 above should be contrasted with the remarkable results of Vinberg from the late 1970's. Vinberg considered reflections acting on one sheet of the two-sheeted hyperboloid given by $\langle x, x\rangle = -1$. By careful choices of reflections and building a fundamental (hyperbolic) polytope for the resulting action, Vinberg showed that for $n \leq 19$ the subgroup $R_{n,1}(\mathbb{Z})$ of $O_{n,1}(\mathbb{Z})$ generated by reflections has finite index in $O_{n,1}(\mathbb{Z})$ [6] (with [7] jointly with Kaplinskaja filling in the last two cases). Conversely, Vinberg [5] (with Kneser filling in the last few cases) showed that for $n \geq 20$, the subgroup $R_{n,1}(\mathbb{Z})$ has infinite index in $O_{n,1}(\mathbb{Z})$. In particular, this implies that the condition $n \leq 19$ in Theorem 1 is optimal.

Phrased in this way, Theorem 1 can be viewed as a strengthening of the first part of Vinberg's results, replacing finite-index by exact equality. While this may be technically true, this view does not do justice to Vinberg's work. Since Vinberg was interested in isometries of hyperbolic space, he was working in the index 2 subgroup $O^+_{n,1}(\mathbb{Z})$ of $O_{n,1}(\mathbb{Z})$ of maps preserving causality. (The only reflections in this subgroup are reflections in vectors w with $\langle w, w\rangle = 1$ or 2, so he was also restricting to these.) Therefore if one only cares about $O_{n,1}(\mathbb{Z})$, then equality up to finite index is the most that can be directly extracted from Vinberg's results. One could presumably prove Theorem 1 using Vinberg's results as a starting point. Vinberg builds an explicit finite index normal subgroup of $O_{n,1}(\mathbb{Z})$ generated by reflections. One could then extract from Vinberg's papers an explicit description of the quotient, and check that generators of this quotient are products of reflections. This would however by very time-consuming and detail-oriented work, with many disparate cases to check. In contrast, following Wall will allow us to give a holistic approach that works across dimensions and allows a very simple computer program to do the hard part of the checking.

One odd coincidence of our approach is that, like Vinberg's work, it will rely on building and analyzing a polytope. The reader should however be aware that our polytope has no relation whatsoever to Vinberg's polytope. In Vinberg's work the polytope is hyperbolic and is basically a fundamental domain for a certain action on hyperbolic space. Our polytope is Euclidean and it is merely a way of recording the state of our calculation. It is not a fundamental domain for any obvious action.

2 Wall's argument

Wall's proof for the case $n \leq 9$ runs as follows. We first note that the maps which negate a single coordinate are the reflections in the corresponding coordinate vector, and that the maps which interchange one pair of the positive coordinates are reflections in the difference of the corresponding coordinate vectors. Wall refers to these as trivial elements of $O_{n,1}(\mathbb{Z})$. For readers interested in having a minimal size generating set, note that it suffices to take $n+1$ such reflections. We can take the $n-1$ maps that interchange adjacent positive coordinates and the two maps that negate the last positive coordinate and the negative coordinate. These trivial reflections serve two purposes for us. First, if we are looking at any fixed vector $v = (v_1, \ldots, v_n, v_{n+1})$, then we can always apply a composition of these trivial reflections to standardize v so that

$$v_1 \geq v_2 \geq \cdots \geq v_n \geq 0, \text{ and } v_{n+1} \geq 0.$$

Secondly, the group $O_n(\mathbb{Z})$ is exactly the signed permutation matrices, so the trivial reflections show that $O_n(\mathbb{Z}) \times O_1(\mathbb{Z})$ is generated by reflections.

Suppose we have a set of reflection vectors $\mathcal{W}$. In order to show that $O_{n,1}(\mathbb{Z})$ is generated by the trivial reflections and reflections in vectors from $\mathcal{W}$, it clearly suffices to show that we can reduce any element $T \in O_{n,1}(\mathbb{Z})$ down to the identity matrix by right-multiplying by a sequence of these reflections. However Wall quickly improves on this obvious result, by saying that it suffices to focus on the bottom row v of the matrix T. Since $T \in O_{n,1}(\mathbb{Z})$, its bottom row $v \in \mathbb{Z}^n \oplus \mathbb{Z}$ will be a vector with $\langle v, v\rangle = -1$ whose orthogonal complement is isomorphic to the standard n-dimensional lattice. If we can reduce v by a sequence of trivial reflections and reflections in vectors from $\mathcal{W}$ until $v_{n+1} = \pm 1$, then we will have reduced T to an element of $O_n(\mathbb{Z}) \times O_1(\mathbb{Z})$ and since we have already remarked that this subgroup is generated by the trivial reflections we will be done. We can clearly carry out this reduction inductively. We assume we start with a standardized vector v satisfying the conditions above. We imagine applying a reflection in some vector of $\mathcal{W}$ and restandardizing. If this reduces v_{n+1}, then we have made progress and we say that the reflection has improved our vector. We can clearly keep iterating until we arrive at a vector v which cannot be improved by reflecting in any element of $\mathcal{W}$. We call such a vector $\mathcal{W}$-terminal. Thus we have proven the following lemma.

Lemma 1. *(Wall) If $\mathcal{W}$ is a set of reflection vectors in $\mathbb{Z}^n \oplus \mathbb{Z}$ such that the only standardized $\mathcal{W}$-terminal vector v with $\langle v, v\rangle = -1$ and a standard orthogonal complement is the coordinate vector e_{n+1}, then trivial reflections and reflections in elements of $\mathcal{W}$ generate $O_{n,1}(\mathbb{Z})$.*

For $3 \le n \le 9$, Wall argues that taking $\mathcal{W}$ to consist of the single vector $w = (1, 1, 1, 0, \ldots, 0, 1)$ suffices. We will omit many of the details in describing this argument, since they are easy, in Wall's paper, and will follow from the more general results of later sections. Suppose we have a vector v as in the lemma which is $\mathcal{W}$-terminal. One computes that the last coordinate of $R_w(v)$ is $2v_{n+1} - v_1 - v_2 - v_3$, and hence we improve v if

$$-v_{n+1} < 2v_{n+1} - v_1 - v_2 - v_3 < v_{n+1}.$$

The lower bound cannot fail, so we see that our $\mathcal{W}$-terminal vector must have the upper bound fail, hence $v_{n+1} \ge v_1 + v_2 + v_3$. Thus

$$1 = -\langle v, v\rangle = v_{n+1}^2 - \sum_{k=1}^{n} v_k^2 \ge (v_1 + v_2 + v_3)^2 - \sum_{k=1}^{n} v_k^2 \ge 0.$$

Since we have derived a chain of inequalities involving integers that fit between 1 and 0, one of the two inequalities must be an equality and the other is very close to being an equality. Expanding on these remarks, one can argue that the only vectors that satisfy this chain of inequalities are the last coordinate vector and the vectors

$(1,1,1,1,1,1,1,1,3)$ and $(1,1,1,1,1,1,1,1,0,3)$ for $n = 8$ and 9, respectively. However these last two vectors do not have a standard orthogonal complement. Instead, the orthogonal complements are the E_8 lattice and its stabilization, respectively. Thus the hypotheses of the lemma hold and the proof is complete.

Note that the argument here would be much simpler if rather than looking for terminal vectors of norm -1, we were interested in (primitive) terminal vectors of norm 0. Then the inequality above would read

$$0 = -\langle v, v\rangle = v_{n+1}^2 - \sum_{k=1}^{n} v_k^2 \geq (v_1 + v_2 + v_3)^2 - \sum_{k=1}^{n} v_k^2 \geq 0,$$

and we would immediately conclude that we have equality at every step. From this we would quickly get the vectors $(1,0,0,0,0,0,0,0,0,1)$ and $(1,1,1,1,1,1,1,1,1,3)$ as the only possibilities. Trying to bring this extra homogeneity into the calculation was the original motivation for the approach of the next section.

3 Extending the argument

One could certainly extend Wall's argument to $n > 9$ in the way Wall originally presented it, but dealing with multiple inequalities and their consequences quickly gets finicky. Instead we will present a reformulation that makes the argument amenable to computer checking.

The first step in this reformulation is to focus on a larger class of vectors, specifically standardized vectors $v \in \mathbb{R}^n \oplus \mathbb{R}$ with $\langle v, v\rangle \leq 0$. We still say that reflection in the reflection vector w improves v if the last coordinate of $R_w(v)$ is smaller in absolute value than the last coordinate of v. Since this condition is unchanged by rescaling v, we further define a stereographic projection $\rho : \mathbb{R}^n \oplus \mathbb{R}^* \to \mathbb{R}^n$ by

$$\rho(v) = \frac{v^+}{v^-}.$$

Note that

$$\langle v, w\rangle = v^- w^- (\rho(v) \cdot \rho(w) - 1).$$

Also note that under this projection the vectors with $\langle v, v\rangle \leq 0$ map to the unit ball B^n and the vectors with $\langle v, v\rangle = 0$ map to the unit sphere S^{n-1}. We have restricted to standardized vectors, so we need only consider the part of the unit ball in the sector

$$\{(x_1, x_2, \dots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

Reflection in w improves a vector v if and only if

$$-|v^-| < R_w(v)^- = v^- - \frac{2v^-w^-(\rho(v)\cdot\rho(w)-1)}{(w^-)^2(\|\rho(w)\|^2-1)}w^- < |v^-|,$$

which simplifies down to

$$0 < \frac{\rho(v)\cdot\rho(w)-1}{\|\rho(w)\|^2-1} < 1.$$

In any case, one of these two inequalities is always just a consequence of the Cauchy-Schwarz inequality. If $\langle w,w\rangle = 1$ or 2, then $\|\rho(w)\| > 1 \geq \|\rho(v)\|$, and hence

$$\rho(v)\cdot\rho(w) \leq \|\rho(v)\|\|\rho(w)\| \leq \|\rho(w)\|^2.$$

Thus the upper inequality always holds and we see that reflection in w improves v if and only if $\rho(v)\cdot\rho(w) > 1$. If $\langle w,w\rangle = -1$ or -2, then $\|\rho(w)\|, \|\rho(v)\| < 1$, and hence

$$\rho(v)\cdot\rho(w) \leq \|\rho(v)\|\|\rho(w)\| \leq 1.$$

Thus the lower inequality always holds and we see that reflection in w improves v if and only if $\rho(v)\cdot\rho(w) > \|\rho(w)\|^2$. If we define

$$\rho^*(w) = \begin{cases} \rho(w) & \text{if } \langle w,w\rangle > 0 \\ \frac{1}{\|\rho(w)\|^2}\rho(w) & \text{if } \langle w,w\rangle < 0 \end{cases},$$

for a nontrivial reflection vector w (where nontrivial means that $w^+ \neq 0$ and $w^- \neq 0$, hence $R_w \notin O_n(\mathbb{Z}) \times O_1(\mathbb{Z})$ and hence $\rho(w)$ is defined and nonzero), then we can summarize this calculation by the following lemma.

Lemma 2. *Suppose $v \in \mathbb{R}^n \oplus \mathbb{R}$ is a standardized vector with $\langle v,v\rangle \leq 0$ and $w \in \mathbb{Z}^n \oplus \mathbb{Z}$ is a nontrivial reflection vector. Then reflection in w does not improve v if and only if*

$$\rho^*(w)\cdot\rho(v) \leq 1.$$

A number of trivial remarks should be made here. First, we have phrased this lemma in the negative way, giving when v is not improved, because that is what we will want most often. Second, note that $\rho^*(w)$ has a nice geometric interpretation. It is either $\rho(w)$ or its dual with respect to the unit sphere, and more specifically it is whichever of these lies outside the unit ball. Third, we have not up to this point assumed that the reflection vectors w are standardized, but now we can. If w is any reflection vector and w_s is its standardization, then it follows from the rearrangement inequality that for any standardized v

$$\rho^*(w_s)\cdot\rho(v) \geq \rho^*(w)\cdot\rho(v).$$

Thus if reflection in w improves a standardized v, then so does reflection in w_s. Thus we may restrict to standardized reflection vectors. Fourth, there is nothing special about reflections in this lemma. If $A \in O_{n,1}(\mathbb{Z})$ is arbitrary, then the condition that A improves v translates into two inequalities $-|v^-| < (Av)^- < |v^-|$ and one of these inequalities always follows from the Cauchy-Schwarz inequality. The only place where we have used that we have a reflection is in the explicit form for $\rho^*(w)$ derived above.

The most significant consequence of this lemma for us will be the following. Suppose $\mathcal{W}$ is a finite set of standardized nontrivial reflection vectors. Then we define a polytope $P_{n,1}(\mathcal{W})$ by

$$P_{n,1}(\mathcal{W}) = \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0, \text{ and } \rho^*(w) \cdot x \leq 1 \; \forall w \in \mathcal{W}\}.$$

Note that as long as $\mathcal{W}$ is nonempty, we will get an upper bound on x_1, and hence $P_{n,1}(\mathcal{W})$ is compact.

Lemma 3. *Suppose $\mathcal{W}$ is a set of standardized nontrivial reflection vectors in $\mathbb{Z}^n \oplus \mathbb{Z}$. Then a vector $v \in \mathbb{Z}^n \oplus \mathbb{Z}$ with $\langle v, v\rangle \leq 0$ is $\mathcal{W}$-terminal if and only if $\rho(v) \in P_{n,1}(\mathcal{W})$.*

Note that given any such set $\mathcal{W}$, the defining inequalities $\rho^*(w) \cdot x \leq 1$ will have rational coefficients, and hence the vertices of the polytope $P_{n,1}(\mathcal{W})$ will be rational vectors in $\mathbb{R}^n$. These vertices can therefore be constructed exactly by standard mathematical software such as Magma [2] or Mathematica [3].

Returning to our digression at the end of the last section, if the goal were to have a finite number of (primitive) $\mathcal{W}$-terminal vectors of norm 0, rather than our real goal of norm -1, then we would be ready to turn the problem over to a computer. We build the vertices of the polytope $P_{n,1}(\mathcal{W})$. If all these vertices lie inside the closed unit ball, then the entire polytope $P_{n,1}(\mathcal{W})$ lies inside this ball and the (primitive) $\mathcal{W}$-terminal vectors of norm 0 correspond to the finite set of vertices that lie on the unit sphere. Things are not quite as easy since we are interested in vectors of norm -1, but as the following lemma shows, they are manageable.

Lemma 4. *If $\mathcal{W}$ is a finite set of standardized nontrivial nontrivial reflection vectors in $\mathbb{Z}^n \oplus \mathbb{Z}$ such that the polytope $P_{n,1}(\mathcal{W})$ lies entirely inside the closed unit ball, then there are only finitely many $\mathcal{W}$-terminal vectors of norm -1.*

Proof. Suppose on the contrary that there are infinitely many such vectors $(v_i)_{i=1}^{\infty}$. Since the unit ball is compact, by passing to a subsequence we may assume that the sequence $\rho(v_i)$ converges. For each $M > 0$, there are only finitely many vectors $v^+ \in \mathbb{Z}^n$ such that $v^+ \cdot v^+ \leq M^2 - 1$, hence there are only finitely many vectors

$v \in \mathbb{Z}^n \oplus \mathbb{Z}$ with norm -1 and $|v^-| \le M$. Thus we must have $\lim_{i\to\infty} v_i^- = \infty$. Since $\|\rho(v_i)\|^2 = 1 - \frac{1}{(v_i^-)^2}$, it follows that $\rho(v_i)$ must converge to some point on the unit sphere. Since $\rho(v_i) \in P_{n,1}(\mathcal{W})$ for all i, this limit must be one of the finitely many vertices of $P_{n,1}(\mathcal{W}) \cap S^{n-1}$, call it $\nu \in S^{n-1}$.

Since $\nu \in \mathbb{Q}^n$, there will be some common denominator for the entries of ν, call it b. Passing once more to a subsequence, we may assume the differences $\epsilon_i := \nu - \rho(v_i)$ have $\epsilon_i/\|\epsilon_i\|$ converging to some element $\eta \in S^{n-1}$. Since $\nu \in S^{n-1}$ is a vertex of the polytope $P_{n,1}(\mathcal{W}) \subset S^{n-1}$, we see geometrically that η cannot be perpendicular to ν, so $\eta \cdot \nu \neq 0$. However, we have

$$1 - \frac{1}{(v_i^-)^2} = \|\rho(v_i)\|^2 = \|\nu - \epsilon_i\|^2 = \|\nu\|^2 - 2\nu\cdot\epsilon_i + \|\epsilon_i\|^2 = 1 - 2\nu\cdot\epsilon_i + \|\epsilon_i\|^2.$$

Hence

$$\frac{\epsilon_i}{\|\epsilon_i\|}\cdot\nu = \frac{1}{2(v_i^-)^2\|\epsilon_i\|} + \frac{\|\epsilon_i\|}{2}.$$

Note that the right-hand side of this equation is positive. Since $\epsilon_i = \nu - \rho(v_i)$ is a non-zero difference of vectors with denominators b and v_i^-, $\|\epsilon_i\| \ge 1/|bv_i^-|$. Hence

$$\left|\frac{\epsilon_i}{\|\epsilon_i\|}\cdot\nu\right| \le \left|\frac{b}{2v_i^-}\right| + \frac{\|\epsilon_i\|}{2} \to 0,$$

so $\eta\cdot\nu = 0$, which is a contradiction. ■

Though the lemma above as stated is not effective, it should be clear that one can easily adapt the proof to give an effective version, and this is the only piece remaining before we can turn the problem over to a computer. Below is one version of this task.

Suppose v is a vector with norm -1 for which $\rho(v) \in P_{n,1}(\mathcal{W})$. By symmetry, we can assume $v^- > 0$, hence that v is standardized. Since in building $P_{n,1}(\mathcal{W})$ we will automatically produce all its vertices, we may assume $\rho(v)$ is not itself a vertex. (Though in practice most of the vectors v we are looking for arise in this way.)

Visualize $\rho(v)$ as a point on the surface of the ball of radius $\|\rho(v)\| = \sqrt{1-(v^-)^{-2}}$ centered at the origin. Since $\rho(v)$ is inside the compact, convex polytope $P_{n,1}(\mathcal{W})$ and is not itself a vertex, there must be some vertex ν of $P_{n,1}(\mathcal{W})$ that v sees on or above its horizon. In the proof above the focus was on the case where ν lies on the unit sphere, but here that need not be true. Since ν is above the horizon (so $(\nu - \rho(v))\cdot\rho(v) > 0$) and inside the unit ball, we must have

$$1 - \frac{1}{(v^-)^2} + \|\nu - \rho(v)\|^2 = \|\rho(v)\|^2 + \|\nu - \rho(v)\|^2 \le \|\nu\|^2 \le 1.$$

Hence $\epsilon = \nu - \rho(v)$ has length at most $\frac{1}{v^-}$.

First note that if we fix ν and v^-, then this severely restricts the possibilities for v. This says that the vector v^+ must be an integer vector with $\|v^+ - v^-\nu\| \leq 1$. Hence we must have $|v_i - v^-\nu_i| \leq 1$ for each index i. If $v^-\nu$ is not an integer vector, then the inequality must be strict for each i. Thus if $v^-\nu_i$ is an integer, then we must have $v_i = v^-\nu_i$. If $v^-\nu_i$ is not an integer, then v_i must be either the floor or ceiling of $v^-\nu_i$. This gives at most 2^n possibilities for v^+. (For the range $n \leq 19$ considered in this paper, this is already an easily manageable number for a computer, but it should be clear that one could work a little harder here and reduce the number of possibilities substantially.) If $v^-\nu$ is an integer vector, then we must have $|v_i - v^-\nu_i| = 1$ for exactly one index i. In other words, v^+ must differ from $v^-\nu$ by a coordinate vector. Thus in any case, for fixed v^- we get a manageable finite list of possibilities for v. To reduce to a finite search we just need an upper bound on v^-.

If ν is a vertex of $P_{n,1}(\mathcal{W})$ inside the open unit ball and above the horizon of $\rho(v)$, then we must have $\|\nu\| > \|\rho(v)\|$ and hence $v^- \leq \frac{1}{\sqrt{1-\|\nu\|^2}}$, thus there are only finitely many possibilities for v^- and we are done. It is only in the case when ν is on the unit sphere, that we need the effective version of the argument above. As in the proof above, ν has rational coordinates, and we let b be a common denominator for the entries of ν. Let θ be the (necessarily acute) angle formed by going from the origin O to ν and then to $\rho(v)$. Then the computation above gives

$$\cos\theta = \frac{1}{2(v^-)^2\|\epsilon\|} + \frac{\|\epsilon\|}{2}.$$

If we look at all angles formed by going from the origin to ν and then to some point $X \in P_{n,1}(\mathcal{W})$, this angle will always be acute and will be maximized for X some vertex of $P_{n,1}(\mathcal{W})$. Hence we get

$$\min_{X \neq \nu} \cos\angle O\nu X \leq \frac{1}{2(v^-)^2\|\epsilon\|} + \frac{\|\epsilon\|}{2}.$$

Note that the upper bound in this inequality is a decreasing function of $\|\epsilon\|$ on the interval $[0, 1/v^-]$, which we have already seen contains $\|\epsilon\|$. Thus the last piece we need is a lower bound on $\|\epsilon\|$. In the proof above we used the crudest possible lower bound that ϵ is a nonzero rational vector with denominators at most bv^- and hence $\|\epsilon\| \geq \frac{1}{bv^-}$. This would be adequate, but for the expense of a small amount of grunge we can do much better. Suppose $\gcd(b, v^-) = g$ and suppose the i-th coordinate $g\nu_i$ of $g\nu$ is not an integer. Since ν_i can be put over a denominator of b and $\gcd(b, v^-) = g$, it follows that $v^-\nu_i$ is also not an integer. Indeed by Bezout's lemma, there must be some integers α and β with $\alpha v^- + \beta b = g$ and hence $\alpha v^- \nu_i + \beta b \nu_i = g\nu_i$. Since by assumption $\beta b \nu_i$ is an integer and $g\nu_i$ is not, it follows that $v^- \nu_i$ is not an integer. If we define $c_g(\nu)$ to be the maximum of 1 and the number of non-integer entries in $g\nu$, it follows that the vector ϵ has at least $c_g(\nu)$ nonzero entries, each of which is a rational number with denominator at most bv^-/g. Hence we have

$$\|\epsilon\| \geq \frac{g\sqrt{c_g(\nu)}}{bv^-},$$

and hence for all v^- we have

$$\|\epsilon\| \geq \frac{\min_{g|b} g\sqrt{c_g(\nu)}}{bv^-}.$$

(One could do a little better by looking at the specific fractional parts of $g\nu$, but this already seems grungy enough.) Hence we get

$$v^- \leq \frac{b^2 + (\min_{g|b} g\sqrt{c_g(\nu)})^2}{2b \min_{g|b} g\sqrt{c_g(\nu)} \min_{X \neq \nu} \cos \angle O\nu X}. \tag{1}$$

This isn't a pretty bound, but now an algorithm for finding all the $\mathcal{W}$-terminal vectors with norm -1 is straightforward. We compute all the vertices of the polytope $P_{n,1}(\mathcal{W})$ and check that they are all in the closed unit ball. If so, then Lemma 4 shows that there are finitely many such vectors v. We exhaust over all vectors v with norm -1 with $v^- \leq k$ for some bound k, checking whether they are in $P_{n,1}(\mathcal{W})$ by explicitly computing $\rho^*(w) \cdot \rho(v)$ for each $w \in \mathcal{W}$. For each vertex ν of $P_{n,1}(\mathcal{W})$ outside the ball of radius $\sqrt{1 - k^{-2}}$, we compute an upper bound on v^- for a vector v that sees ν over its horizon and exhaust up to that point the list of possibilities v from the calculation above. In our computations we chose the value of k larger than we probably should have for maximal efficiency, to avoid the second step altogether. Only for $n = 19$ did we need the full computation.

As a side note, our argument must break down for $n = 20$ since the desired conclusion is no longer true. Reflections only generate a subgroup of infinite index in $O_{20,1}(\mathbb{Z})$. Therefore it must be impossible to find finitely many reflections for which the resulting polytope is entirely inside the closed unit ball.

4 Computational Results

As a warm-up, let us recover Wall's result using the machinery of the previous section. First suppose $n = 9$ and take $\mathcal{W}$ to consist of the single vector $w_1 =$

$(1,1,1,0,0,0,0,0,0,1)$. Then

$$\rho^*(w_1) = \rho(w_1) = (1,1,1,0,0,0,0,0,0),$$

and

$$P_{9,1}(\mathcal{W}) = \{x \in \mathbb{R}^9 : x_1 \geq x_2 \geq \cdots \geq x_9 \geq 0 \text{ and } x_1 + x_2 + x_3 \leq 1\}.$$

This polytope is defined by 10 linear inequalities in 9 variables, so it is a simplex and the vertices are given by insisting on equality in all but one inequality. Hence they are the origin, the coordinate vector $v_1 = e_1 = (1,0,0,0,0,0,0,0,0)$, $v_2 = (1/2,1/2,0,0,0,0,0,0,0)$, and the seven vectors $v_m = (1/3,1/3,\cdots,1/3,0,\ldots,0)$ with $3 \leq m \leq 9$ and m entries equal to $1/3$. Only v_1 and v_9 lie on the unit sphere (rederiving the digression above). Running through the computations above we see that every case gives $v^- \leq 3$. There are only 5 standardized vectors in $\mathbb{Z}^9 \oplus \mathbb{Z}$ of norm -1 satisfying this bound. Two of them

$$(0,0,0,0,0,0,0,0,0,1) \text{ and } (1,1,1,1,1,1,1,1,0,3)$$

map to $P_{9,1}(\mathcal{W})$ (and indeed have images the vertices at the origin and v_8). The other three, $(1,1,1,0,0,0,0,0,0,2)$, $(2,2,0,0,0,0,0,0,0,3)$, and $(2,1,1,1,1,0,0,0,0,3)$, fail the inequality $v_1 + v_2 + v_3 < v_{10}$ corresponding to $\rho^*(w_1)$, hence have images outside the polytope. As in Wall's analysis, the vector $(1,1,1,1,1,1,1,1,0,3)$ does not have a standard orthogonal complement, hence $O_{9.1}(\mathbb{Z})$ is generated by trivial reflections and reflection in w_1.

The cases $2 \leq n \leq 8$ follow from the case $n = 9$. One way to see this is to view reducing n as equivalent to just restricting to a hyperplane which enforces some terminal zeros in v^+. These hyperplanes are preserved by all the operations used for $n \geq 3$, so the calculation reduces directly. Clearly for $n \leq 7$, one gets only a single vector of norm -1 in $P_{n,1}(\mathcal{W})$ since the other does not have enough trailing zeros. For $n = 2$, reflection in the vector w_1 would send one outside the hyperplane defined by insisting on 7 trailing zeros. Hence one must replace w_1 with $w_1' = (1,1,0,0,0,0,0,0,0,1)$. This is still a valid reflection vector (now with norm 1 rather than 2), and for any vector $v \in \mathbb{Z}^9 \oplus \mathbb{Z}$ with 7 trailing zeros in v^+ we have $\rho^*(w_1) \cdot \rho(v) = \rho^*(w_1') \cdot \rho(v)$, so this replacement does not change whether a vector is shortened. (This is a subtle point. The reflections in w_1 and w_1' are related, but not in a very pretty way. The replacement is handled by the same calculation because $\rho^*(w_1)$ and $\rho^*(w_1')$ are related in a nice way.)

As a further illustration of the calculation, we will sketch how the calculation runs for $10 \leq n \leq 13$, leaving all cases of larger n to the next section. It suffices to look at the case $n = 13$. Reducing n corresponds to intersecting the polytope with the hyperplane where the last $13 - n$ coordinates are zeros. This can only lower the resulting upper bounds. For $n = 13$, we take $\mathcal{W}$ to consist of the two vectors

$$w_1 = (1,1,1,0,0,0,0,0,0,0,0,0,0,1) \text{ and } w_2 = (1,1,1,1,1,1,1,1,1,1,1,0,0,3).$$

The resulting polytope $P_{13,1}(\mathcal{W})$ has 46 vertices, all lying inside the closed unit ball, of which three are ρ applied to the vectors

$$\begin{aligned}&(1,0,0,0,0,0,0,0,0,0,0,0,0,1),\\&(1,1,1,1,1,1,1,1,1,0,0,0,0,3), \text{ and}\\&(2,1,1,1,1,1,1,1,1,1,1,1,1,4)\end{aligned}$$

and lie on the unit sphere. We look for vectors v with norm -1 and $\rho(v)$ inside $P_{13,1}(\mathcal{W})$. The next closest vertex to the sphere has length $\sqrt{1-1/32}$ (so for $v^- \geq 6 > 4\sqrt{2}$ we can ignore these) and the largest of the three bounds coming from equation (1) applied to vertices on the unit sphere gives $v^- \leq 29/\sqrt{26} < 6$. Hence we get $v^- \leq 5$ in all cases. Checking the 21 possibilities, we find only

$$\begin{aligned}&(0,0,0,0,0,0,0,0,0,0,0,0,0,1),\\&(1,1,1,1,1,1,1,1,0,0,0,0,0,3), \text{ and}\\&(2,1,1,1,1,1,1,1,1,1,1,1,0,4)\end{aligned}$$

are standardized $\mathcal{W}$-terminal vectors of norm -1. Only the first of these has a standard orthogonal complement, proving the theorem in these dimensions. One way to see this is to invoke the classification of unimodular forms in dimension 13. There are known to be 3 such forms up to isomorphism [1]. Since odd indefinite forms are all standard (see [4], Theorem II.4.3), it follows that for each such form there must be a vector $v \in \mathbb{Z}^{13} \oplus \mathbb{Z}$ with $\langle v, v\rangle = -1$ whose orthogonal complement represents the form. Hence reducing such a v must produce at least one $\mathcal{W}$-terminal vector whose orthogonal complement represents the form. But we found exactly three $\mathcal{W}$-terminal vectors, so the orthogonal complements of these must correspond one-to-one to the three different forms, and hence only the coordinate vector has standard orthogonal complement.

Note that for $10 \leq n < 13$ we lose any points with fewer than $13-n$ trailing zeros. Also for $n = 10$, since we are restricting to three trailing zeros in the coordinates, the second reflection vector must become $w_2' = (1,1,1,1,1,1,1,1,1,1,3)$, which has norm 1 instead of 2, but is still a valid reflection vector and has the correct effect on vectors with 3 trailing zeros.

5 Higher dimensions

In this section we give some details of the calculations needed to prove the theorem for $14 \leq n \leq 19$.

$14 \leq n \leq 15$. As for the smaller cases, it suffices to focus on the case $n = 15$. For $n = 15$, we take $\mathcal{W}$ to consist of the five vectors

$$\begin{aligned} w_1 &= (1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ w_2 &= (1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,3), \\ w_3 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,4), \\ w_4 &= (2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4), \text{ and} \\ w_5 &= (2,2,2,1,1,1,1,1,1,1,1,1,1,1,0,5). \end{aligned}$$

For $n = 14$ we must drop the last coordinate of the "+" part of these vectors, which for w_4 gives us a vector of norm 1, which is still a valid reflection vector. The resulting polytope $P_{15,1}(\mathcal{W})$ has 197 vertices, all lying inside the closed unit ball, of which four lie on the unit sphere. Running through the computations above, we see that in any case a $\mathcal{W}$-terminal vector v of norm -1 must have $v^- \leq 9$. Checking the 310 possibilities for v, we find only five

$$\begin{aligned} &(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ &(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,3), \\ &(2,1,1,1,1,1,1,1,1,1,1,1,0,0,0,4), \\ &(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4), \text{ and} \\ &(2,2,2,2,2,2,2,2,1,1,1,1,1,1,0,6) \end{aligned}$$

are standardized $\mathcal{W}$-terminal vectors of norm -1. Since this matches the number of unimodular forms in dimension 15, there is exactly one terminal vector for each form, and in particular the coordinate vector is the only one with a standard complement.

The set of vectors $\mathcal{W}$ above was chosen to make the polytope $P_{n,1}(\mathcal{W})$ simple and not to minimize the number of generators, and even so no sophisticated algorithm was used. (We just grabbed "small" reflection vectors until the polytope was entirely inside the closed unit ball.) If one were interested in a minimal generating set of reflections, one could presumably be smarter about choosing $\mathcal{W}$. Still, even using this unsophisticated choice of $\mathcal{W}$, we can argue that a smaller set of reflections suffices to generate $O_{n,1}(\mathbb{Z})$. Letting st denote the standardization map, one can check that

$$\begin{aligned} st \circ R_{w_3}(w_2) &= w_1, \\ R_{w_3}(w_4) &= e_1 + e_{15}, \text{ and} \\ R_{w_1}(w_5) &= w_3. \end{aligned}$$

(For $n = 14$, and using primes to denote the truncated reflection vectors, the second of these becomes $R_{w_3'}(w_4') = e_1$.) Each of these equations says that a product of trivial reflections and the reflections in w_1 and w_3 maps one of the other w_i to either one of w_1, w_3, or a reflection vector corresponding to an element of $O_n(\mathbb{Z}) \times O_1(\mathbb{Z})$. Hence they show that the reflections in w_2, w_4, and w_5 are conjugate to products of trivial reflections and the reflections in w_1 and w_3 (and the conjugating maps are also such products). Thus trivial reflections and the reflections in w_1 and w_3 generate $O_{15,1}(\mathbb{Z})$ and similarly for $n = 14$. It is unclear whether the list of generators could be further reduced.

$n = 16$. We take $\mathcal{W}$ to consist of the eight vectors

$$\begin{aligned} w_1 &= (1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ w_2 &= (1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,3), \\ w_3 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,4), \\ w_4 &= (2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,4), \\ w_5 &= (2,2,2,1,1,1,1,1,1,1,1,1,1,1,0,0,5), \\ w_6 &= (2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,6), \\ w_7 &= (2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,6), \text{ and} \\ w_8 &= (3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,0,9). \end{aligned}$$

The resulting polytope $P_{16,1}(\mathcal{W})$ has 607 vertices, all lying inside the closed unit ball, of which five lie on the unit sphere. Running through the computations above, we see that in any case a $\mathcal{W}$-terminal vector v of norm -1 must have $v^- \leq 13$. Checking the 3040 possibilities, we find only eight

$$\begin{aligned}
&(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1),\\
&(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,3),\\
&(2,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,4),\\
&(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,4),\\
&(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5),\\
&(2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,6),\\
&(3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,7), \text{ and}\\
&(3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,9)
\end{aligned}$$

are standardized $\mathcal{W}$-terminal vectors of norm -1. Since this matches the number of unimodular forms in dimension 16, there is exactly one terminal vector for each form, and in particular the coordinate vector is the only one with a standard complement.

Again one could reduce the list of required generators by computing that

$$\begin{aligned}
st \circ R_{w_3}(w_2) &= w_1,\\
R_{w_3}(w_4) &= e_1 + e_{15},\\
R_{w_1}(w_5) &= w_3 \text{ and}\\
R_{w_6}(w_7) &= e_7,
\end{aligned}$$

from which one can conclude that trivial reflections and reflections in w_1, w_3, w_6, and w_8 generate $O_{16,1}(\mathbb{Z})$.

$n = 17$. We take $\mathcal{W}$ to consist of the ten vectors

$$\begin{aligned}
w_1 &= (1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1),\\
w_2 &= (1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,3),\\
w_3 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,4),\\
w_4 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4),\\
w_5 &= (2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,4),\\
w_6 &= (2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5),\\
w_7 &= (2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,0,6),\\
w_8 &= (2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,6),\\
w_9 &= (2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,6), \text{ and}\\
w_{10} &= (2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,7).
\end{aligned}$$

The resulting polytope $P_{17,1}(\mathcal{W})$ has 6057 vertices, all lying inside the closed unit ball, of which eight lie on the unit sphere. Running through the computations above, we see that in any case a $\mathcal{W}$-terminal vector v of norm -1 must have $v^- \leq 14$. Checking the 5751 possibilities, we find only nine

$$\begin{gathered}
(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\
(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,3), \\
(2,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,4), \\
(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,4), \\
(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,5), \\
(2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,6), \\
(2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,0,6), \\
(3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,7), \text{ and} \\
(3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,0,9)
\end{gathered}$$

are standardized $\mathcal{W}$-terminal vectors of norm -1. Since this matches the number of unimodular forms in dimension 17, there is exactly one terminal vector for each form, and in particular the coordinate vector is the only one with a standard complement.

One can compute that

$$\begin{aligned}
st \circ R_{w_3}(w_2) &= w_1, \\
R_{w_3}(w_5) &= e_1 + e_{15}, \\
R_{w_1} \circ st \circ R_{w_7}(w_5) &= e_3 \text{ and} \\
R_{w_8}(w_9) &= e_7 + e_{17},
\end{aligned}$$

so trivial reflections and reflections in w_1, w_3, w_6, w_7, w_8 and w_{10} generate $O_{17,1}(\mathbb{Z})$.

$n = 18$. We take $\mathcal{W}$ to consist of the 14 vectors

$$
\begin{aligned}
w_1 &= (1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1),\\
w_2 &= (1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,3),\\
w_3 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,4),\\
w_4 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4),\\
w_5 &= (2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,4),\\
w_6 &= (2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,5),\\
w_7 &= (3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5),\\
w_8 &= (2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,0,0,6),\\
w_9 &= (2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,0,6),\\
w_{10} &= (2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,7),\\
w_{11} &= (3,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,7),\\
w_{12} &= (3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,7),\\
w_{13} &= (3,3,3,3,3,2,2,2,2,2,2,2,1,1,1,1,1,1,9), \text{ and}\\
w_{14} &= (3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,1,1,9).
\end{aligned}
$$

The resulting polytope $P_{18,1}(\mathcal{W})$ has 33127 vertices, all lying inside the closed unit ball, of which nine lie on the unit sphere. Running through the computations above, we see that in any case a $\mathcal{W}$-terminal vector v of norm -1 must have $v^- \leq 26$.

Checking the 1144339 possibilities, we find 18

$$\begin{array}{l}
(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1),\\
(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,3),\\
(2,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,4),\\
(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,4),\\
(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,5),\\
(2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5)^A,\\
(2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,0,6),\\
(2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,0,0,6),\\
(3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,7)^B,\\
(2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,7),\\
(4,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,8)^C,\\
(3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,0,0,9),\\
(3,3,3,3,3,3,3,2,2,1,1,1,1,1,1,1,1,1,9)^C,\\
(4,3,3,3,3,3,3,2,2,2,2,2,2,1,1,1,1,1,10)^D,\\
(4,4,4,4,4,3,3,3,3,2,2,2,2,2,2,1,1,1,12)^D,\\
(5,5,4,4,4,4,4,4,4,2,2,2,2,2,2,2,2,1,14)^B,\\
(5,5,5,5,5,3,3,3,3,3,3,3,3,3,3,2,2,1,15)^C, \text{ and}\\
(7,7,7,7,6,6,5,5,5,5,5,3,3,3,3,3,3,3,3,21)^A
\end{array}$$

are standardized $\mathcal{W}$-terminal vectors of norm -1.

For this value of n we first encounter the situation where we have found more $\mathcal{W}$-terminal vectors (18) with norm -1 than there are n-dimensional unimodular lattices (13). Therefore we cannot use simple counts to conclude that the coordinate vector is the only one with standard complement. We could of course analyse the complements of all the other vectors to reach this conclusion, but we chose to resolve the ambiguity by grouping these vectors. The superscripts on some of the vectors above denote groups A, B, C, D which differ by reflections in vectors in $\mathcal{W}$, and hence all vectors within a group have isomorphic orthogonal complements. These were found by building balls around each of the vectors above in the graph whose vertices are standardized vectors of norm -1 and two vectors are adjacent if they differ by a reflection by a vector in $\mathcal{W}$, followed by restandardization. If the balls constructed in this way for two vectors overlap, then the two vectors differ by conjugation. In this case balls of radius two sufficed. Since we have reduced to 13 classes and the coordinate vector is in a singleton class, it is the only one with a standard complement.

One can compute that

$$\begin{aligned} st \circ R_{w_3}(w_2) &= w_1, \\ st \circ R_{w_6}(w_4) &= w_1, \\ R_{w_3}(w_5) &= e_1 + e_{15}, \\ R_{w_8}(w_9) &= e_7 + e_{17}, \\ R_{w_{11}}(w_{12}) &= e_9, \text{ and} \\ st \circ R_{w_{11}}(w_{14}) &= w_7, \end{aligned}$$

so trivial reflections and reflections in w_i for $i \in \{1, 3, 6, 7, 8, 10, 11, 13\}$ generate $O_{18,1}(\mathbb{Z})$.

$n = 19$. We take $\mathcal{W}$ to consist of the 24 vectors

$$\begin{aligned} w_1 &= (1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ w_2 &= (1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,3), \\ w_3 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,4), \\ w_4 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,4), \\ w_5 &= (2,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,4), \\ w_6 &= (2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,5), \\ w_7 &= (3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5), \\ w_8 &= (2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,6), \\ w_9 &= (2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,0,0,0,6), \\ w_{10} &= (2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,0,0,6), \\ w_{11} &= (2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,7), \\ w_{12} &= (3,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,0,7), \\ w_{13} &= (3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,7), \\ w_{14} &= (3,3,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,8), \\ w_{15} &= (3,3,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,8), \\ w_{16} &= (3,3,3,3,3,2,2,2,2,2,2,2,1,1,1,1,1,1,0,9), \\ w_{17} &= (3,3,3,3,3,3,2,2,2,2,1,1,1,1,1,1,1,1,1,9), \\ w_{18} &= (3,3,3,3,3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,9), \end{aligned}$$

$$
\begin{aligned}
w_{19} &= (3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,1,1,1,9),\\
w_{20} &= (3,3,3,3,3,3,3,3,2,2,2,2,2,1,1,1,1,1,1,10),\\
w_{21} &= (4,3,3,3,3,3,3,3,3,2,2,2,2,2,2,2,1,1,1,11),\\
w_{22} &= (4,4,4,4,3,3,3,3,3,3,2,2,2,2,2,1,1,1,1,12),\\
w_{23} &= (5,4,4,4,4,4,4,3,3,3,3,3,2,2,2,2,2,2,2,14), \text{ and}\\
w_{24} &= (6,6,6,6,5,5,4,4,4,4,4,3,3,3,3,3,1,1,1,18).
\end{aligned}
$$

The resulting polytope $P_{19,1}(\mathcal{W})$ has 393088 vertices, all of which are inside the closed unit ball, with 17 on the unit sphere. Running through the computations above, we find that $v^- \leq 58$ suffices for all but one vertex (which requires $v^- \leq 100$). We ran a branching search up to $v^- = 58$, followed by a search of the neighborhood of the last vertex up to $v^- = 100$. This yielded 43 standardized $\mathcal{W}$-terminal norm -1 vectors (of which 42 were vertices of $P_{19,1}(\mathcal{W})$).

$$
\begin{aligned}
&(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1),\\
&(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,3)^A,\\
&(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,4)^B,\\
&(2,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,4)^C,\\
&(2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,5)^D,\\
&(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,5),\\
&(2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,0,0,6)^E,\\
&(2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,0,0,0,6),\\
&(2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,0,7),\\
&(3,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,1,1,1,7)^F,\\
&(3,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,0,0,7)^G,\\
&(3,2,2,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,8)^F,\\
&(4,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,0,8)^I,\\
&(3,3,3,3,3,2,2,2,2,2,2,2,1,1,1,1,1,1,1,9)^H,\\
&(3,3,3,3,3,3,3,2,2,1,1,1,1,1,1,1,1,1,0,9)^I,\\
&(3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,0,0,0,9),\\
&(3,3,3,3,3,3,3,2,2,2,2,2,2,2,2,1,1,1,1,10)^H,\\
&(4,3,3,3,3,3,3,2,2,2,2,2,2,1,1,1,1,1,0,10)^J,\\
&(4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,1,1,1,12)^H,
\end{aligned}
$$

$(4, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 12)^H,$
$(4, 4, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 1, 1, 1, 0, 12)^J,$
$(4, 4, 4, 4, 4, 4, 3, 3, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 12)^K,$
$(4, 4, 4, 4, 4, 4, 4, 4, 3, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 13)^F,$
$(5, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 14)^H,$
$(5, 5, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 2, 2, 2, 1, 0, 14)^G,$
$(6, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 1, 14)^K,$
$(5, 5, 5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 15)^H,$
$(5, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 1, 0, 15)^I,$
$(6, 5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 15)^F,$
$(6, 5, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 2, 1, 16)^E,$
$(6, 5, 5, 5, 4, 4, 4, 4, 4, 4, 3, 3, 3, 2, 2, 2, 2, 2, 1, 16)^H,$
$(6, 6, 6, 6, 5, 5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 1, 18)^H,$
$(6, 6, 6, 6, 5, 5, 5, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2, 18)^H,$
$(7, 7, 6, 6, 6, 6, 5, 5, 4, 4, 4, 4, 3, 3, 3, 2, 2, 2, 2, 20)^H,$
$(7, 7, 7, 7, 6, 6, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 0, 21)^D,$
$(8, 7, 7, 7, 7, 6, 6, 6, 6, 6, 6, 6, 3, 3, 3, 3, 3, 3, 3, 24)^B,$
$(8, 8, 8, 8, 6, 6, 6, 6, 6, 5, 5, 4, 4, 4, 4, 4, 2, 2, 1, 24)^E,$
$(10, 10, 10, 9, 8, 8, 8, 8, 6, 6, 6, 6, 5, 5, 5, 4, 3, 3, 3, 30)^F,$
$(12, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 4, 4, 4, 4, 4, 4, 4, 31)^A,$
$(12, 12, 12, 11, 10, 9, 9, 9, 8, 8, 8, 6, 6, 6, 6, 6, 3, 3, 3, 36)^G,$
$(12, 12, 12, 12, 9, 9, 9, 8, 8, 8, 8, 7, 6, 6, 6, 6, 3, 3, 3, 36)^B,$
$(12, 12, 12, 12, 9, 9, 9, 9, 8, 7, 7, 7, 7, 6, 6, 6, 3, 3, 3, 36)^B,$ and
$(14, 14, 14, 14, 10, 10, 10, 10, 10, 10, 10, 7, 7, 7, 7, 7, 4, 3, 3, 42)^C.$

As for $n = 18$, this gives more $\mathcal{W}$-terminal norm -1 vectors than there are unimodular lattices in dimension 19 (there are 16), but again we found classes with the same orthogonal complement to reduce to 16 cases (denoted by the letters A-K above). In this case balls of radius three sufficed. Since we have reduced to 16 classes and the coordinate vector is in a singleton class, it is the only one with a standard complement.

One can compute that

$$\begin{aligned} st \circ R_{w_3}(w_2) &= w_1, \\ st \circ R_{w_6}(w_4) &= w_1, \\ R_{w_3}(w_5) &= e_1 + e_{15}, \\ st \circ R_{w_{11}}(w_{10}) &= w_1, \\ R_{w_{12}}(w_{13}) &= e_9 + e_{19}, \\ R_{w_{14}}(w_{15}) &= e_{12}, \\ st \circ R_{w_3}(w_{18}) &= w_{13}, \\ st \circ R_{w_{12}}(w_{19}) &= w_7, \\ st \circ R_{w_3} \circ st \circ R_{w_1} \circ st \circ R_{w_6} \circ st \circ R_{w_1} \circ st \circ R_{w_8}(w_{23}) &= w_{20}, \text{ and} \\ st \circ R_{w_1} \circ st \circ R_{w_1} \circ st \circ R_{w_9}(w_{24}) &= w_{14}, \end{aligned}$$

so trivial reflections and reflections in w_i with $i \in \{1, 3, 6, 7, 8, 9, 11, 12, 14, 16, 17, 20, 21, 22\}$ generate $O_{19,1}(\mathbb{Z})$.

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